On parabolic Whittaker functions

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Abstract

We derive a Mellin-Barnes integral representation for solution to generalized (parabolic) quantum Toda lattice introduced in [GLO], which presumably describes the $S^1 \times U_N$-equivariant Gromov-Witten invariants of Grassmann variety.

Introduction

The $\mathfrak{gl}_N$-Whittaker functions, being solutions to the quantum cohomology D-module $QH^*(\mathrm{Fl}_N)$ of the complete flag variety $\mathrm{Fl}_N = GL_N(\mathbb{C})/B$, describe the corresponding equivariant Gromov-Witten invariants of Fl_N (see [Giv1], [Giv2] and references therein). However, the Givental's approach to representation theory description of quantum cohomology of homogeneous spaces is inapplicable to generic incomplete flag variety $\mathrm{Fl}_{m_1,\ldots,m_k}$, since no relevant Whittaker model (Toda lattice) associated with an incomplete flag variety was known.

From the other hand, in [HV] it was conjectured a description of quantum cohomology of Grassmannians in terms of (non-Abelian) gauged topological theories, together with a period-type integral representation for the corresponding generating function.

Recently, in [GLO] a generalization of the $\mathfrak{gl}_N$-Whittaker function to the case of the Grassmann variety $\mathrm{Gr}_{m,\,N} = GL_N(\mathbb{C})/P_m$, $1 \leq m < N$ is proposed. Namely, in [GLO] it is defined a Toda-type D-module and its solution $\Psi^{(m,N)}_{\lambda_1,\ldots,\lambda_N}(x_1,\ldots,x_N)$ (referred to as $\mathrm{Gr}_{m,\,N}$-Whittaker function), such that after specialization $x_2 = \ldots = x_N = 0$ the symbols of this D-module reproduce the small quantum cohomology algebra $qH^*(\mathrm{Gr}_{m,\,N})$ according to [AS] and [K]. Conjecturally, the constructed generalized Whittaker function describe the equivariant Gromov-Witten invariants of $\mathrm{Gr}_{m,N}$, and in [GLO] this conjecture is verified in particular case of projective space $\mathbb{P}^{N-1} = \mathrm{Gr}_{1,N}$.

In this note we construct Mellin-Barnes type integral representation of the specialized $\mathrm{Gr}_{m,N}$-Whittaker function, following an original generalization of Whittaker models to incomplete flag manifolds from [GLO]; this integral formula has been announced in [GLO]. Our derivation involves a generalization of the Gelfand-Zetlin realization to infinite-dimensional $\mathcal{U}(\mathfrak{gl}_N)$-modules introduced in [GKL]. Our main result (Theorem 2.1) generalizes Theorem 1.1 of [GLO] to arbitrary Grassmannian $\mathrm{Gr}_{m,\,N}$. Moreover, our integral representation verifies the conjectural integral formula from [HV], although we construct another solution to the D-module with a different asymptotic behavior.

The paper is organized as follows. In Section 1 we review on parabolic Whittaker functions introduced in [GLO], and formulate our main results: the Mellin-Barnes integral representation for the specialized $\mathrm{Gr}_{m,\,N}$-Whittaker function (Theorem 1.1), and its asymptotic behavior (Theorem 1.2). The second part of the text contains a detailed proof of the main results. In particular, we

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recall the generalized Gelfand-Zetlin realization of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_N)$ from [GKL], and then we find out the Whittaker vectors (Proposition 2.1). In Section 3 we prove Theorems 1.1 and 1.2.

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1 The $\mathrm{Gr}_{m,N}$-Whittaker function and its integral representation

Let $\mathfrak{gl}_N$ be the Lie algebra of $(N\times N)$ real matrices with the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{gl}_N$ of diagonal matrices, and let $\mathfrak{b}_\pm\subset\mathfrak{gl}_N$ be a pair of opposed Borel subalgebras containing $\mathfrak{h}$. Then one has the triangular decomposition $\mathfrak{gl}_N=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$, where $\mathfrak{n}_\pm\subset\mathfrak{b}_\pm$ are the nilpotent radicals given by strictly lower- and upper-triangular matrices. In this way, the set of roots $R\subset\mathfrak{h}^*$ decomposes into $R_+\sqcup R_-$, where $R_+\subset R\subset\mathfrak{h}^*$ is the set of positive roots. Identifying $\mathfrak{h}\simeq\mathbb{R}^N$ with coordinates $\underline{x}=(x_1,\ldots,x_N)$ one may write $R=\{\alpha\in\mathfrak{h}^*|\,\alpha(\underline{x})=x_i-x_j,\,i\neq j\}$ and $R_+=\{\alpha\in\mathfrak{h}^*|\,\alpha(\underline{x})=x_i-x_j,\,i<j\}$. Clearly, positive roots span the Borel subalgebra $\mathfrak{b}_+$, and R_- span $\mathfrak{b}_-\subset\mathfrak{gl}_N$. Let $\Delta\subset R_+$ be the set of simple roots $\alpha_i(\underline{x})=x_i-x_{i+1}\in\mathfrak{h}^*$, $1\leq i\leq N-1$, and let $\{\omega_m,\,1\leq m\leq N\}$ be the non-reduced set of fundamental weights given by $\omega_m(\underline{x})=x_1+\ldots+x_m$. The Weyl group $\mathfrak{S}_N$ is generated by simple reflections $s_i=s_{\alpha_i}$, and acts in $\mathfrak{h}^*$ by linear transformations:

$$s_i(\beta)\,=\,\beta\,-\,(\alpha_i,\beta)\alpha_i\,,\qquad\qquad \beta\in\mathfrak{h}^*.$$

In particular one has $\mathfrak{S}_N\cdot R_+=R_-$. Let $I=\{1,2,\ldots,N-1\}$ be the set of vertices of Dynkin diagram, then given a subset $J\subseteq I$, let $\overline{J}=I\setminus J$, and let us consider the subgroup $\mathfrak{W}_J\subset\mathfrak{S}_N$ generated by $\{s_j,\,j\in\overline{J}\}$. Then let $R_J\subseteq R_+$ be a subset of positive roots defined by $\mathfrak{W}_J{\cdot}R_J=-R_J$, and let $\overline{R}_J=R_+\setminus R_J$. Then the corresponding parabolic subalgbera is spanned by R_- and R_J, and the corresponding parabolic subgroup is denoted by P_J. In this paper we restrict ourselves to the case $J=\{m\}\subset\{1,2,\ldots,N-1\}$ with $\mathfrak{W}_m=\mathfrak{S}_m\times\mathfrak{S}_{N-m}$, and $GL_N(\mathbb{C})/P_m$ being isomorphic to the Grassmannian $\mathrm{Gr}_{m,N}$. In this case we have $I=I'\sqcup I''$ with $I'=\{1,\ldots,m\}$ and $I''=\{m+1,\ldots,N\}$, then $\overline{R}_m$ is spanned by positive roots α of the form $\alpha(\underline{x})=x_i-x_j;\,i\in I',\,j\in I''$.

Next, let us recall an original construction of $\mathrm{Gr}_{m,N}$-Whittaker functions from [GLO]. Let $B=B_-\subset GL_N(\mathbb{C})$ be the Borel subgroup of lower-triangular matrices, and let us pick a character $\chi_{\underline{\lambda}}:B_-\to\mathbb{C}$ defined by $\underline{\lambda}=(\lambda_1,\ldots,\lambda_N)\in\mathbb{C}^N$. Then the associated Whittaker function is defined as a certain matrix element of a principle series representation $\mathcal{V}_{\underline{\lambda}}=\mathrm{Ind}_B^{GL_N}\chi_{\underline{\lambda}}$.

Let us associate with P_m a decomposition of the Borel subalgebra $\mathfrak{b}_+\subset\mathfrak{gl}_N$

$$\mathfrak{b}_+\;=\;\mathfrak{h}^{(m)}\;\oplus\;\mathfrak{n}_+^{(m)}\,,$$

into the commutative subalgebra $\mathfrak{h}^{(m)}\subset\mathfrak{b}_+$ spanned by

$$\begin{gathered} H_1\,=\,E_{11}+\ldots+E_{mm}\,;\qquad\qquad H_k\,=\,E_{1,k},\quad 2\leq k\leq m\,;\\ H_{m+k}\,=\,E_{m+k,\,\ell+m},\quad 1\leq k\leq N-m-1\,;\\ H_{\ell+m}\,=\,E_{m+1,\,m+1}+\ldots+E_{\ell+m,\,\ell+m}\,, \end{gathered}\tag{1.1}$$

and the Lie subalgebra $\mathfrak{n}_+^{(m)}\subset\mathfrak{b}_+$ generated by

$$\begin{aligned} \mathfrak{n}_+^{(m)}\,=\,\big\langle &E_{1,\,\ell+m};\,E_{1,\,m+1};\,E_{m,\,\ell+m};\\ &E_{kk}\,,2\leq k\leq N-1\,;\,E_{j,\,j+1},\,2\leq j\leq N-2\big\rangle\,. \end{aligned}\tag{1.2}$$

Note that $\dim \mathfrak{h}^{(m)} = \mathrm{rank}\ \mathfrak{gl}_N = N$ and $\dim \mathfrak{n}_+^{(m)} = N(N-1)/2$. Let $H^{(m)}$ and $N_+^{(m)}$ be the Lie groups corresponding to the Lie algebras $\mathfrak{h}^{(m)}$ and $\mathfrak{n}_+^{(m)}$. An open part GL_N° (the big Bruhat cell) of GL_N allows the following analog of the Gauss decomposition:

$$GL_N^\circ \,=\, N_-\, H^{(m)}\, N_+^{(m)}\,. \tag{1.3}$$

Let $\mathcal{U} = \mathcal{U}(\mathfrak{gl}_N)$ be the universal enveloping algebra of $\mathfrak{gl}_N$. The principal series representation $\mathcal{V}_{\underline{\lambda}}$ admits a natural structure of $\mathcal{U}$-module, as well, as a module over the opposite algebra $\mathcal{U}^{\mathrm{opp}}$. Let us assume that the action of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_N$ in $\mathcal{V}_{\underline{\lambda}}$ is integrable to the action of the Cartan torus $H \subset GL_N(\mathbb{R})$. Below we introduce a pair of elements, $\langle\psi_L|$, $|\psi_R\rangle \in \mathcal{V}_{\underline{\lambda}}$, generating a pair of dual submodules, $\mathcal{W}_L = \langle\psi_L|\mathcal{U}^{\mathrm{opp}}$ and $\mathcal{W}_R = \mathcal{U}|\psi_R\rangle$, in $\mathcal{V}_{\underline{\lambda}}$ (we adopt the bra- and ket-vector notations to distinguish $\mathcal{U}$- and $\mathcal{U}^{\mathrm{opp}}$-modules structures on $\mathcal{W}_R$ and $\mathcal{W}_L$ respectively).

Definition 1.1 *[GLO] The* $\mathrm{Gr}_{m,N}$*-Whittaker vectors* $\langle\psi_L| \in \mathcal{V}'_{\underline{\lambda}}$ *and* $|\psi_R\rangle \in \mathcal{V}_{\underline{\lambda}}$ *are defined by the following conditions:*

$$\langle\psi_L|E_{n+1,\,n} \,=\, \hbar^{-1}\langle\psi_L|\,, \qquad\qquad 1\leq n\leq N-1\,, \tag{1.4}$$

$$\begin{cases} E_{kk}|\psi_R\rangle \,=\, 0\,, & 2\leq k\leq N-1\,;\\ E_{k,\,k+1}|\psi_R\rangle \,=\, 0, & 2\leq k\leq N-2\,;\\ E_{1,\,m+1}|\psi_R\rangle \,=\, E_{m,\,N}|\psi_R\rangle \,=\, 0\,; & \\ E_{1,\,N}|\psi_R\rangle \,=\, (-1)^{\epsilon(m,N)}\dfrac{1}{\hbar}|\psi_R\rangle & \end{cases} \tag{1.5}$$

where $\epsilon(m,N)$ *is an integer number and* $\hbar\in\mathbb{R}$.

Note that the equations (1.4) define a one-dimensional representation $\langle\psi_L|$ of the Lie algebra $\mathfrak{n}_-$ of strictly lower-triangular matrices, and the equations (1.5) define a one-dimensional representation of $\mathfrak{n}_+^{(m)}$.

Definition 1.2 *[GLO] The* $\mathrm{Gr}_{m,N}$*-Whittaker function associated with the principal series representation* $(\pi_{\underline{\lambda}},\, \mathcal{V}_{\underline{\lambda}})$ *is defined as the following matrix element:*

$$\Psi^{(m,N)}_{\underline{\lambda}}(\underline{x}) \,=\, e^{-x_1\frac{m(N-m)}{2}}\,\big\langle\psi_L|\pi_{\underline{\lambda}}\big(g(x_1,\ldots,x_N)\big)|\psi_R\big\rangle\,, \tag{1.6}$$

where the left and right vectors solve the equations (1.4) *and* (1.5) *respectively. Here* $g(x)$ *is a Cartan group valued function given by*

$$g(\underline{x}) \,=\, \exp\Big\{-\sum_{i=1}^N x_iH_i\Big\}\,, \tag{1.7}$$

where $\underline{x}=(x_1,\ldots,x_N)$ *and the generators* H_i*,* $i=1,\ldots,N$ *are defined by* (1.1).

In [GLO] (Theorem 1.1) an integral representation of the $\mathrm{Gr}_{m,N}$-Whittaker function (1.6) was constructed in the case of projective space $\mathbb{P}^{N-1}$, corresponding to $m=1$. We generalize this construction to generic Grassmannians $\mathrm{Gr}_{m,N}$.

Theorem 1.1 *The specialized* $\mathrm{Gr}_{m,N}$*-Whittaker function possesses the following integral representation:*

$$\Psi^{(m,\,N)}_{\underline{\lambda}}(x,0,\ldots,0) \;=\; \int\limits_{\mathcal{C}} d\underline{\gamma}\; e^{-\frac{x}{\hbar}\sum\limits_{i=1}^{m}\gamma_i}\;\frac{\prod\limits_{i=1}^{m}\prod\limits_{j=1}^{N}\Gamma_1\big(\gamma_i-\lambda_j|\hbar\big)}{\prod\limits_{\substack{i,k=1\\k\neq i}}^{m}\Gamma_1\big(\gamma_i-\gamma_k|\hbar\big)}\,, \tag{1.8}$$

with $\underline{\gamma}=(\gamma_1,\ldots,\gamma_m)$ *and* $\underline{\lambda}=(\lambda_1,\ldots,\lambda_N)\in\mathbb{R}^N$*. The integration contour is given by* $\mathcal{C}=\big(\imath\mathbb{R}+\epsilon\big)^m$*, where* $\epsilon>\max\limits_{1\leq j\leq N}\{\lambda_j\}$*.*

Here we use the following normalization of classical Gamma-function:

$$\Gamma_1\big(z\big|\hbar\big) \;=\; \hbar^{\frac{z}{\hbar}}\,\Gamma\Big(\frac{z}{\hbar}\Big)\,.$$

We prove Theorem 1.1 in Section 3.

Clearly, the integral (1.8) converges absolutely due to the Stirling formula:

$$\Gamma\big(z+\lambda\big) \;=\; \sqrt{2\pi}\,z^{z+\lambda-\frac{1}{2}}\,e^{-z}\Big[1\,+\,O(z^{-1})\Big]\,, \qquad\qquad z\to\infty\,,$$

when $|\arg(z)|<\pi$.

Integral representation (1.8) coincides with the expected one (5.1) in [GLO]. Besides, a similar integral formula was conjectured in [HV] (see formulas (A.1) and (A.2) in Appendix), but with different integration measure $\widetilde{\mu}=\prod\limits_{i<j}(\gamma_i-\gamma_j)$. Our choice of measure (2.3) is provided by the generalized Gelfand-Zetlin realization [GKL], and it is crucial in our representation theory framework. Actually, the two solutions given by integral formulas (1.8) and the one from [HV], have different asymptotic behavior, and below we derive asymptotic of our solution.

Theorem 1.2 *When* $x\to-\infty$*, the specialized* $\mathrm{Gr}_{m,N}$*-Whittaker function has the following asymptotic behavior:*

$$\Psi^{(m,\,N)}_{\underline{\gamma}_N}(x,0,\ldots,0) \;\sim\; m!\sum_{\sigma\in\,\mathfrak{S}_N/\mathfrak{W}_m} e^{-x\big(\sigma\cdot\,\omega_m(\underline{\lambda})\big)}\big(\sigma\cdot c_m\big)(\underline{\lambda})\,. \tag{1.9}$$

with $\omega_m(\underline{\lambda})=\lambda_1+\ldots+\lambda_m$*, and*

$$c_m(\underline{\lambda}) \;=\; \prod_{\alpha\in\overline{R}_m}\Gamma_1\big(\alpha(\underline{\lambda})\big|\hbar\big)\,, \qquad\qquad \big(\sigma\cdot c_m\big)(\underline{\lambda}) \;=\; \prod_{\alpha\in\overline{R}_m}\Gamma_1\big(\sigma\cdot\alpha(\underline{\lambda})\big|\hbar\big)\,. \tag{1.10}$$

A proof of Theorem 1.2 is given in Section 3.1.

2 Construction of $\mathrm{Gr}_{m,N}$-Whittaker vectors

In this Section we construct explicit solutions to (1.4) and (1.5) using the generalized Gelfand-Zetlin realization of principal series $U(\mathfrak{gl}_N)$-modules from [GKL]. Namely, let $\underline{\gamma}_1,\ldots,\underline{\gamma}_N$ be a triangular array consisting of $N(N-1)/2$ variables $\underline{\gamma}_n=(\gamma_{n1},\ldots,\gamma_{nn})\in\mathbb{C}^n, n=1,\ldots,N$. The following operators define a representation π of $\mathcal{U}$ in the space $\mathcal{M}_N$ of meromorphic functions in $N(N-1)/2$ variables $(\underline{\gamma}_1,\ldots,\underline{\gamma}_{N-1})$:

$$
\begin{aligned}
E_{kk} &= \frac{1}{\hbar}\Big(\sum_{j=1}^{n}\gamma_{n,j} - \sum_{i=1}^{n-1}\gamma_{n-1,\,i}\Big)\,, && 1\le k\le N\,;\\
E_{n,\,n+1} &= -\frac{1}{\hbar}\sum_{i=1}^{n}\frac{\prod\limits_{j=1}^{n+1}(\gamma_{n,i}-\gamma_{n+1,\,j}-\frac{\hbar}{2})}{\prod\limits_{s\neq i}(\gamma_{n,i}-\gamma_{n,s})}\,e^{-\hbar\partial_{n,i}}\,, && 1\le n\le N-1\,;\\
E_{n+1,\,n} &= \frac{1}{\hbar}\sum_{i=1}^{n}\frac{\prod\limits_{j=1}^{n-1}(\gamma_{n,i}-\gamma_{n-1,\,j}+\frac{\hbar}{2})}{\prod\limits_{s\neq i}(\gamma_{n,i}-\gamma_{n,s})}\,e^{\hbar\partial_{n,i}}\,, && 1\le n\le N-1\,,
\end{aligned}
\tag{2.1}
$$

where $E_{ij}=\pi(e_{ij})$, $1\le i,j=1\le N$ for the standard elementary matrix units $e_{ij}\in\mathfrak{gl}_N$. This realization of universal enveloping algebra $\mathcal{U}$ is referred to as generalized Gelfand-Zetlin realization.

Remark 2.1 *Evidently the Weyl group $\mathfrak{S}_N$ acts on E_{ij} in (2.1) by permutations of indices (i,j). This provides $N!$ different realizations of $\mathcal{U}$, and we use certain $\mathfrak{S}_N$-twisted generalized Gelfand-Zetlin realizations of $\mathcal{U}$ in the next Section for derivation of Whittaker vectors (1.4), (1.5).*

The the universal enveloping algebra $\mathcal{U}$ acts in $\mathcal{W}_R\subseteq\mathcal{M}_N$ by differential operators(2.1), and the opposite algebra $\mathcal{U}^{\rm opp}$ acts in $\mathcal{W}_L\subseteq\mathcal{M}_N$ via the adjoint operators:

$$
E^{\dagger}_{ij} = \mu(\gamma)^{-1}\,E_{ij}\,\mu(\gamma)\,,\qquad 1\le i,j\le N\,,
\tag{2.2}
$$

with

$$
\mu(\gamma) = \prod_{n=2}^{N-1}\mu_n(\underline{\gamma}_n) = \prod_{n=2}^{N-1}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\frac{1}{\Gamma\big(\frac{\gamma_{ni}-\gamma_{nj}}{\hbar}\big)}\,.
\tag{2.3}
$$

It was shown in [GKL] that there exist a non-degenerate pairing between the modules $\mathcal{W}_L$ and $\mathcal{W}_R$ with measure (2.3):

$$
\langle\phi_1,\,\phi_2\rangle = \int\limits_{\mathbb{R}^{N(N-1)/2}}\phi_1(\gamma)\phi_2(\gamma)\;\mu(\gamma)\prod_{\substack{k,n=1\\ k\le n}}^{N}d\gamma_{nk}\,,
\tag{2.4}
$$

where $\phi_1\in\mathcal{W}_L$ and $\phi_2\in\mathcal{W}_R$.

Proposition 2.1 *For $1<m<N$ the Whittaker vectors have the following expressions.*

1. *A solution to (1.4) is given by*

$$
\psi^{(m)}_L = e^{\imath\pi\gamma_{11}}\prod_{i=1}^{m-1}\prod_{j=1}^{m}\frac{1}{\Gamma_1(\gamma_{m-1,\,i}-\gamma_{m,j}+\frac{\hbar}{2}|\hbar)}\,.
\tag{2.5}
$$

2. A solution to (1.5) *is given by*

$$\psi_R^{(m)} \;=\; \delta(\gamma_{11}) \prod_{i=1}^{m}\prod_{j=1}^{N} \Gamma_1\Big(\gamma_{N-1,\,i}-\gamma_{Nj}+\frac{\hbar}{2}\Big|\hbar\Big)\cdot \prod_{a=1}^{m-1}\prod_{b=1}^{m} \Gamma_1\Big(\gamma_{m-1,\,a}-\gamma_{mb}+\frac{\hbar}{2}\Big|\hbar\Big)$$
$$\times \prod_{\substack{n=2\\ n\neq m}}^{N-1} \Big[\delta\Big(\sum_{j=1}^{n}\gamma_{nj}-\sum_{i=1}^{n-1}\gamma_{n-1,\,i}\Big)\prod_{k=1}^{n-1}\delta\Big(\gamma_{n-1,\,k}-\gamma_{nk}+\frac{\hbar}{2}\Big)\prod_{\substack{i,j=1\\ i\neq j}}^{n}\Gamma_1\big(\gamma_{ni}-\gamma_{nj}\big|\hbar\big)\Big]\,, \tag{2.6}$$

Further, substituting the Whittaker vectors (2.5) and (2.6) into the pairing (2.4) we arrive to the following integral representation.

2.1 Proof of Proposition

Due to the action of the Weyl group $\mathfrak{S}_N$, actually one has $N!$ realizations of $U(\mathfrak{gl}_N)$ defined by

$$E^w_{ij} \;:=\; wE_{ij}w^{-1}\,, \qquad\qquad w\in\mathfrak{S}_N\,, \qquad i,j=1,\ldots,N\,. \tag{2.7}$$

Let us call these realizations of $U(\mathfrak{gl}_N)$ the w-twisted Gelfand-Zetlin realizations.

Given the simple reflections $s_i\in\mathfrak{S}_N$, $i\in I$, let us introduce the Coxeter elements $c_n = s_n\cdot\ldots\cdot s_1$, $n\in I$. In particular, one has $c_1=s_1$, and for the longest element $w_0\in\mathfrak{S}_N$ the following decomposition holds:

$$w_0 \;=\; c_1c_2\cdot\ldots\cdot c_{N-1}\,.$$

Proof of Proposition 2.1. Given $m>1$ let us solve the defining relations (1.4), (1.5), using the c_{m-1}-twisted Gelfand-Zetlin realization (2.7).

2.1.1 Let us start from solving the equations (1.4) for the left $\mathrm{Gr}_{m,N}$-Whittaker vector. Namely, one has to check that (2.5) satisfies (1.4):

$$\big(E^{c_{m-1}}_{k+1,\,k}\big)^{\dagger}\psi_L^{(m)} \;=\; \hbar^{-1}\psi_L^{(m)}\,, \qquad\qquad 1\leq k<N\,. \tag{2.8}$$

Actually, one has to check only two relations:

$$-\big(E^{c_{m-1}}_{21}\big)^{\dagger}\psi_L^{(m)} \;=\; \big(E^{c_{m-1}}_{m+1,\,m}\big)^{\dagger}\psi_L^{(m)} \;=\; \hbar^{-1}\psi_L^{(m)}\,, \tag{2.9}$$

and the other relations are evidently true, since the difference operators $E^{c_{m-1}}_{k+1,\,k}$, $k\neq 1,m$ act trivially on $\psi_L^{(m)}=\psi_L^{(m)}\big(\underline{\gamma}_{m-1},\,\underline{\gamma}_m\big)$. The relations (2.9) can be verified using the following combinatorial formulas.

Lemma 2.1 *Given a set of variables* $\underline{\gamma}=(\gamma_1,\ldots,\gamma_n)$ *the following identities hold:*

1.

$$\sum_{i=1}^{n}\gamma_i^m\prod_{i\neq k}\frac{1}{\gamma_i-\gamma_k} \;=\; \delta_{m,\,n-1}\,, \qquad\qquad m<n\,; \tag{2.10}$$

More generally, one has

$$\sum_{i=1}^{n}\gamma_i^m\prod_{i\neq k}\frac{1}{\gamma_i-\gamma_k} \;=\; \sum_{k_1+\cdots+k_n=n+1-m}\gamma_1^{k_1}\cdot\ldots\cdot\gamma_n^{k_n}\,.$$

2.

$$\sum_{i=1}^{n}\prod_{i\neq k}\frac{c-\gamma_k}{\gamma_i-\gamma_k}\;=\;1\,, \tag{2.11}$$

for any constant c.

Proof. We have

$$\sum_{i=1}^{n}\gamma_i^m\prod_{i\neq k}\frac{1}{\gamma_i-\gamma_k}\;=\oint\limits_{\gamma_i} d\lambda\frac{\lambda^m}{A_n(\lambda)}\,, \qquad\qquad A_n(\lambda)=\prod_{i=1}^{n}(\lambda-\gamma_i)\,.$$

Taking the residue at infinity we get

$$\oint_\infty d\lambda\,\frac{\lambda^{m-n}}{1+\sum\limits_{k=1}^{n}(-1)^k\sigma_k(\underline{\gamma})\lambda^{-k}}\;=\;\chi_{n+1-m}(\underline{\gamma})\,\Theta(m+1-n)\,,$$

where

$$\sigma_k(\underline{\gamma})\;=\;\sum_{i_1<\ldots<i_k}\gamma_{i_1}\cdot\ldots\cdot\gamma_{i_k}\,,\qquad\qquad \chi_k(\underline{\gamma})\;=\;\sum_{i_1+\ldots+i_n=k}\gamma_1^{k_1}\cdot\ldots\cdot\gamma_n^{k_n}$$

are the characters of finite-dimensional representations $\bigwedge^k\mathbb{C}^n$ and $\mathrm{Sym}^k\mathbb{C}^n$ respectively.

The other identity can be proved similarly. □

Then for the second relation in (2.9) one readily derives the following:

$$\begin{aligned}\big(E^{c_{m-1}}_{m+1,\,m}\big)^\dagger\psi_L^{(m)}\;&=\;E^\dagger_{m+1,\,m-1}\psi_L^{(m)}\;=\;\frac{1}{\hbar}\sum_{i_1=1}^{m}\prod_{\substack{k_1=1\\k_1\neq i_1}}^{m}\frac{1}{\gamma_{m,\,i_1}-\gamma_{m,\,k_1}}\\ &\times\sum_{i_2=1}^{m-1}\prod_{\substack{k_2=1\\k_2\neq i_2}}^{m-1}\frac{\big(\gamma_{m,\,i_1}-\gamma_{m-1,\,k_2}-\frac{\hbar}{2}\big)}{\gamma_{m-1,\,i_2}-\gamma_{m-1,\,k_2}}\prod_{j=1}^{m-2}\big(\gamma_{m-1,\,i_2}-\gamma_{m-2,\,j}-\frac{\hbar}{2}\big)\,e^{-\hbar(\partial_{m,\,i_1}+\partial_{m-1,\,i_2})}\cdot\psi_L^{(m)}\end{aligned} \tag{2.12}$$

$$=\frac{1}{\hbar}\sum_{i_1=1}^{m}\prod_{\substack{k_1=1\\k_1\neq i_1}}^{m}\frac{1}{\gamma_{m,\,i_1}-\gamma_{m,\,k_1}}\sum_{i_2=1}^{m-1}\frac{\prod\limits_{\substack{j_1=1\\j_1\neq i_1}}^{m}\big(\gamma_{m-1,\,i_1}-\gamma_{m,\,j_1}-\frac{\hbar}{2}\big)}{\prod\limits_{k_2\neq i_2}\big(\gamma_{m-1,\,i_2}-\gamma_{m-1,\,k_2}\big)}\prod_{j_2=1}^{m-2}\big(\gamma_{m-1,\,i_2}-\gamma_{m-2,\,j}-\frac{\hbar}{2}\big)\psi_L^{(m)}\;=\ldots$$

Using (2.10) we have

$$\ldots=\frac{1}{\hbar}\sum_{i_1=1}^{m}\prod_{\substack{k_1=1\\k_1\neq i_1}}^{m}\frac{\gamma_{m,\,k_1}}{\gamma_{m,\,i_1}-\gamma_{m,\,k_1}}\sum_{i_2=1}^{m-1}\frac{-\gamma_{m-1,\,i_2}^{m-2}}{\prod\limits_{k_2\neq i_2}\big(\gamma_{m-1,\,i_2}-\gamma_{m-1,\,k_2}\big)}\psi_L^{(m)}\;=\ldots$$

and finally we apply (2.11) and obtain

$$=\frac{1}{\hbar}\sum_{i_1=1}^{m}\prod_{\substack{k_1=1\\k_1\neq i_1}}^{m}\frac{-\gamma_{m,\,k_1}}{\gamma_{m,\,i_1}-\gamma_{m,\,k_1}}\;=\;\frac{1}{\hbar}\psi_L^{(m)}\,.$$

The first relation in (2.9) reads as follows:

$$
\begin{aligned}
\big(E_{21}^{c_{m-1}}\big)^\dagger \psi_L^{(m)} \;=\; E_{1m}^\dagger \psi_L^{(m)} \;=\; -\frac{1}{\hbar}\sum_{i_1=1}^{m-1} \frac{\prod\limits_{j_1=1}^{m+1}\big(\gamma_{m-1,\,i_1}-\gamma_{m,\,j_1}+\frac{\hbar}{2}\big)}{\prod\limits_{k_1\neq i_1}\big(\gamma_{m-1,\,i_1}-\gamma_{m-1,\,k_1}\big)} \sum_{i_2=1}^{m-2} \frac{\prod\limits_{\substack{j_2=1\\ j_2\neq i_1}}^{m}\big(\gamma_{m-2,\,i_2}-\gamma_{m-1,\,j_2}+\frac{\hbar}{2}\big)}{\prod\limits_{k_2\neq i_2}\big(\gamma_{m-2,\,i_2}-\gamma_{m-2,\,k_2}\big)} \times \ldots \\
\times \sum_{i_{m-2}=1}^{2} \frac{\prod\limits_{\substack{j_{m-2}=1\\ j_{m-2}\neq i_{m-1}}}^{3}\big(\gamma_{2,\,i_{m-2}}-\gamma_{3,\,j_{m-2}}+\frac{\hbar}{2}\big)}{\prod\limits_{k_{m-2}\neq i_{m-2}}\big(\gamma_{2,\,i_{m-2}}-\gamma_{2,\,k_{m-2}}\big)} \prod_{\substack{j_{m-1}=1\\ j_{m-1}\neq i_{m-2}}}^{2}\Big(\gamma_{11}-\gamma_{2,\,j_{m-1}}+\frac{\hbar}{2}\Big)\, e^{\hbar\big(\partial_{11}+\sum\limits_{n=2}^{m-1}\partial_{n,\,m-n}\big)} \cdot \psi_L^{(m)} \;=\; -\frac{1}{\hbar}\psi_L^{(m)}
\end{aligned}
\tag{2.13}
$$

Thus we have to check the following identity:

$$
\begin{aligned}
&\frac{1}{\hbar}\sum_{i_1=1}^{m-1}\prod_{k_1\neq i_1}\frac{1}{\gamma_{m-1,\,i_1}-\gamma_{m-1,\,k_1}} \sum_{i_2=1}^{m-2} \frac{\prod\limits_{\substack{j_2=1\\ j_2\neq i_1}}^{m}\big(\gamma_{m-2,\,i_2}-\gamma_{m-1,\,j_2}+\frac{\hbar}{2}\big)}{\prod\limits_{k_2\neq i_2}\big(\gamma_{m-2,\,i_2}-\gamma_{m-2,\,k_2}\big)} \times \ldots \\
&\times \sum_{i_{m-2}=1}^{2} \frac{\prod\limits_{\substack{j_{m-2}=1\\ j_{m-2}\neq i_{m-1}}}^{3}\big(\gamma_{2,\,i_{m-2}}-\gamma_{3,\,j_{m-2}}+\frac{\hbar}{2}\big)}{\prod\limits_{k_{m-2}\neq i_{m-2}}\big(\gamma_{2,\,i_{m-2}}-\gamma_{2,\,k_{m-2}}\big)} \prod_{\substack{j_{m-1}=1\\ j_{m-1}\neq i_{m-2}}}^{2}\Big(\gamma_{11}-\gamma_{2,\,j_{m-1}}+\frac{\hbar}{2}\Big) \;=\; \frac{1}{\hbar}\,.
\end{aligned}
\tag{2.14}
$$

This identity can be verified by induction over m. Indeed, for $m=2$ (2.13) reads

$$
\begin{aligned}
\big(E_{21}^{c_2}\big)^\dagger \psi_L^{(2)} \;&=\; E_{12}^\dagger \psi_L^{(2)} \\
=\; -\frac{1}{\hbar}\Big(\gamma_{11}-\gamma_{21}+\frac{\hbar}{2}\Big)\Big(\gamma_{11}-\gamma_{22}+\frac{\hbar}{2}\Big)&e^{\hbar\partial_{11}}\cdot e^{\imath\pi\gamma_{11}}\prod_{i=1}^{2}\frac{1}{\Gamma_1\big(\gamma_{11}-\gamma_{2i}+\frac{1}{2}\big|\hbar\big)} \\
=\; \frac{1}{\hbar}\psi_L^{(2)}\,.
\end{aligned}
\tag{2.15}
$$

The inductive step directly follows the reasoning from (2.12), using the combinatorial identities (2.10) and (2.11).

2.1.2 Let us check that the expression (2.6) satisfies the relations the c_{m-1}-twisted relations (1.5):

$$
\begin{aligned}
&E_{nn}^{c_{m-1}}\psi_R^{(m)} \;=\; 0, \quad n=2,\ldots,N-1\,; \qquad && E_{k,\,k+1}^{c_{m-1}}\psi_R^{(m)} \;=\; 0, \quad k=2,\ldots,N-2\,; \\
&E_{1,\,m+1}^{c_{m-1}}\psi_R^{(m)} \;=\; E_{m,\,N}^{c_{m-1}}\psi_R^{(m)} \;=\; 0\,; && E_{1,\,N}^{c_{m-1}}\psi_R^{(m)} \;=\; (-1)^{\epsilon(m,N)}\hbar^{-1}\psi_R^{(m)}\,.
\end{aligned}
\tag{2.16}
$$

The relations corresponding to Cartan generators $E_{nn}^{c_{m-1}}\psi_R^{(m)}=0\,, n=2,\ldots,N-1$ hold due to the delta-factors

$$
\delta(\gamma_{11})\prod_{\substack{n=2\\ n\neq m}}^{N-1}\delta\Big(\sum_{j=1}^{n}\gamma_{nj}-\sum_{i=1}^{n-1}\gamma_{n-1,\,i}\Big)
$$

in (2.6), and since

$$
E_{kk}^{c_m} \;=\; E_{k-1,\,k-1}, \quad k=2,\ldots,m\,; \qquad E_{kk}^{c_m} \;=\; E_{kk}, \quad k=m+1,\ldots,N-1\,.
$$

Similarly, the relations $E^{c_{m-1}}_{k,\,k+1}\psi^{(m)}_R = 0\,, k = 2, \ldots , N - 2$ hold due to the delta-factors

$$\prod_{\substack{n=2\\ n\neq m}}^{N-1}\prod_{k=1}^{n-1}\delta\Big(\gamma_{n-1,\,k}-\gamma_{nk}+\frac{\hbar}{2}\Big) \tag{2.17}$$

in (2.6), and since

$$E^{c_{m-1}}_{k,\,k+1} \;=\; E_{k-1,\,k}\,, \qquad k=2,\ldots,m\,; \qquad\qquad E^{c_{m-1}}_{k,\,k+1} \;=\; E_{k,\,k+1}\,, \qquad k=m+1,\ldots,N-2\,.$$

Due to the same delta-factor (2.17) one has $E^{c_{m-1}}_{1,\,m+1}\psi^{(m)}_R \;=\; E_{m,\,m+1}\psi^{(m)}_R = 0$.

Thus we have to check the remaining two relations:

$$E^{c_{m-1}}_{m,\,N}\,\psi^{(m)}_R \;=\; E_{m-1,\,N}\psi^{(m)}_R \;=\; 0\,, \qquad E^{c_{m-1}}_{1,\,N}\,\psi^{(m)}_R \;=\; E_{m,\,N}\psi^{(m)}_R \;=\; \frac{(-1)^m}{\hbar}\,\psi^{(m)}_R\,. \tag{2.18}$$

Lemma 2.2 *For $n = 1, \ldots , N - 1$ the following holds:*

$$\begin{aligned} E_{n,\,N} \;=\; -\frac{1}{\hbar}\sum_{i_1=1}^{N-1}\frac{\prod\limits_{j_1=1}^{N}\Big(\gamma_{N-1,\,i_1}-\gamma_{N,\,j_1}-\frac{\hbar}{2}\Big)}{\prod\limits_{k_1\neq i_1}\Big(\gamma_{N-1,\,i_1}-\gamma_{N-1,\,k_1}\Big)}\;\sum_{i_2=1}^{N-2}\frac{\prod\limits_{\substack{j_2=1\\ j_2\neq i_1}}^{N-1}\Big(\gamma_{N-2,\,i_2}-\gamma_{N-1,\,j_2}-\frac{\hbar}{2}\Big)}{\prod\limits_{k_2\neq i_2}\Big(\gamma_{N-2,\,i_2}-\gamma_{N-2,\,k_2}\Big)}\times\ldots \\ \times\sum_{i_{N-n}=1}^{n}\frac{\prod\limits_{\substack{j_{N-n}=1\\ j_{N-n}\neq i_{N+1-n}}}^{N+1-n}\Big(\gamma_{n,\,i_{N-n}}-\gamma_{n+1,\,j_{N-n}}-\frac{\hbar}{2}\Big)}{\prod\limits_{k_{N-n}\neq i_{N-n}}\Big(\gamma_{n,\,i_{N-n}}-\gamma_{n,\,k_{N-n}}\Big)}\;e^{-\hbar\sum\limits_{a=n}^{N-1}\partial_{a,\,i_{N-a}}}\,. \end{aligned} \tag{2.19}$$

Proof. Direct calculation using (2.1). □

At first let us note that due to the delta-factors

$$\prod_{n=m+1}^{N-1}\prod_{k=1}^{n}\delta\Big(\gamma_{n-1,i}-\gamma_{n,i}+\frac{\hbar}{2}\Big)$$

in (2.6) one gets only m non-vanishing terms in (2.19) (for $n=m$):

$$
\begin{aligned}
&E_{1,N}^{c_{m-1}}\psi_R^{(m)} \;=\; E_{m,N}\psi_R^{(m)}\\
&= -\frac{1}{\hbar}\Big\{\sum_{i_1=1}^{N-1}\frac{\prod\limits_{j_1=1}^{N}\big(\gamma_{N-1,i_1}-\gamma_{N,j_1}-\frac{\hbar}{2}\big)}{\prod\limits_{k_1\neq i_1}\big(\gamma_{N-1,i_1}-\gamma_{N-1,k_1}\big)}\;\sum_{i_2=1}^{N-2}\frac{\prod\limits_{\substack{j_2=1\\ j_2\neq i_1}}^{N-1}\big(\gamma_{N-2,i_2}-\gamma_{N-1,j_2}-\frac{\hbar}{2}\big)}{\prod\limits_{k_2\neq i_2}\big(\gamma_{N-2,i_2}-\gamma_{N-2,k_2}\big)}\times\ldots\\
&\times\sum_{i_{N-m}=1}^{m}\frac{\prod\limits_{\substack{j_{N-m}=1\\ j_{N-m}\neq i_{N-m-1}}}^{m}\big(\gamma_{m,i_{N-m}}-\gamma_{m+1,j_{N-m}}-\frac{\hbar}{2}\big)}{\prod\limits_{k_{N-m}\neq i_{N-m}}\big(\gamma_{m,i_{N-m}}-\gamma_{m,k_{N-m}}\big)}\;e^{-\hbar\sum\limits_{a=m}^{N-1}\partial_{a,i_{N-a}}}\Big\}\cdot\psi_R^{(m)}
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
&= -\frac{1}{\hbar}\Big\{\sum_{i_1=1}^{m}\frac{\prod\limits_{j_1=1}^{N}\big(\gamma_{N-1,i_1}-\gamma_{N,j_1}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_1=1\\ k_1\neq i_1}}^{N-1}\big(\gamma_{N-1,i_1}-\gamma_{N-1,k_1}\big)}\frac{\prod\limits_{\substack{j_2=1\\ j_2\neq i_1}}^{N-1}\big(\gamma_{N-2,i_1}-\gamma_{N-1,j_2}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_2=1\\ k_2\neq i_1}}^{N-1}\big(\gamma_{N-2,i_1}-\gamma_{N-2,k_2}\big)}\times\ldots\\
&\ldots\times\frac{\prod\limits_{\substack{j_{N-m}=1\\ j_{N-m}\neq i_1}}^{m+1}\big(\gamma_{m,i_1}-\gamma_{m+1,j_{N-m}}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_{N-m}=1\\ k_{N-m}\neq i_1}}^{m}\big(\gamma_{m,i_1}-\gamma_{m,k_{N-m}}\big)}\;e^{-\hbar\sum\limits_{a=m}^{N-1}\partial_{a,i_1}}\Big\}\cdot\psi_R^{(m)}\;=\ldots
\end{aligned}
$$

Secondly, taking into account that

$$
\begin{aligned}
&e^{-\hbar\partial_{N-1,i}}\cdot\prod_{a=1}^{m}\prod_{b=1}^{N}\Gamma_1\Big(\gamma_{N-1,a}-\gamma_{N,b}+\frac{\hbar}{2}\Big|\,\hbar\Big)\\
&=\prod_{j=1}^{N}\frac{1}{\gamma_{N-1,i}-\gamma_{N,j}-\frac{\hbar}{2}}\;\prod_{a=1}^{m}\prod_{b=1}^{N}\Gamma_1\Big(\gamma_{N-1,a}-\gamma_{N,b}+\frac{\hbar}{2}\Big|\,\hbar\Big)\cdot e^{-\hbar\partial_{N-1,i}}\,;
\end{aligned}
\tag{2.21}
$$

and due to the factors $\prod\limits_{n=m+1}^{N-1}\;\prod\limits_{\substack{i,j=1\\ i\neq j}}^{n}\Gamma_1\big(\gamma_{ni}-\gamma_{nj}\big|\hbar\big)$ in (2.6) one has

$$
\begin{aligned}
&\ldots= -\frac{1}{\hbar}\Big\{\sum_{i_1=1}^{m}\prod_{\substack{k_1=1\\ k_1\neq i_1}}^{N-1}\frac{1}{\big(\gamma_{N-1,i_1}-\gamma_{N-1,k_1}-\hbar\big)}\frac{\prod\limits_{\substack{j_2=1\\ j_2\neq i_1}}^{N-1}\big(\gamma_{N-2,i_1}-\gamma_{N-1,j_2}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_2=1\\ k_2\neq i_1}}^{N-1}\big(\gamma_{N-2,i_q}-\gamma_{N-2,k_2}-\hbar\big)}\times\ldots\\
&\ldots\times\frac{\prod\limits_{\substack{j_{N-m-1}=1\\ j_{N-m-1}\neq i_1}}^{m+2}\big(\gamma_{m+1,i_1}-\gamma_{m+2,j_{N-m-1}}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_{N-m-1}=1\\ k_{N-m-1}\neq i_1}}^{m+1}\big(\gamma_{m+1,i_1}-\gamma_{m+1,k_{N-m-1}}-\hbar\big)}\;\frac{\prod\limits_{\substack{j_{N-m}=1\\ j_{N-m}\neq i_1}}^{m+1}\big(\gamma_{m,i_1}-\gamma_{m+1,j_{N-m}}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_{N-m}=1\\ k_{N-m}\neq i_1}}^{m}\big(\gamma_{m,i_1}-\gamma_{m,k_{N-m}}\big)}\\
&\times\,e^{-\hbar\sum\limits_{a=m}^{N-1}\partial_{a,i_1}}\Big\}\cdot\psi_R^{(m)}\;=\ldots
\end{aligned}
\tag{2.22}
$$

Next, since

$$\prod_{\substack{j_{a+1}=1\\ j_{a+1}\neq i_{a+1}}}^{N-a}\Big(\gamma_{N-a-1,\,i_{a+1}}-\gamma_{N-a,\,j_{a+1}}-\frac{\hbar}{2}\Big)\equiv$$
$$\equiv\prod_{k_a\neq i_a}\big(\gamma_{N-a,\,i_a}-\gamma_{N-a,\,k_a}-\hbar\big)\quad \mathrm{mod}\ \prod_{n=m+1}^{N-1}\prod_{k=1}^{n-1}\delta\Big(\gamma_{n-1,\,k}-\gamma_{nk}-\frac{\hbar}{2}\Big) \tag{2.23}$$

and since the factor $\prod\limits_{a=1}^{m-1}\prod\limits_{b=1}^{m}\Gamma_1\Big(\gamma_{m-1,\,a}-\gamma_{mb}+\frac{\hbar}{2}\Big|\hbar\Big)$ in (2.6) produces

$$e^{-\hbar\partial_{m,\,i}}\cdot\prod_{a=1}^{m-1}\prod_{b=1}^{m}\Gamma_1\Big(\gamma_{m-1,\,a}-\gamma_{n,\,b}+\frac{\hbar}{2}\Big|\,\hbar\Big)$$
$$=\prod_{r=1}^{m-1}\big(\gamma_{m-1,\,r}-\gamma_{m,\,i}-\frac{\hbar}{2}\big)\ \prod_{a=1}^{m-1}\prod_{b=1}^{m}\Gamma_1\Big(\gamma_{m-1,\,a}-\gamma_{m,\,b}+\frac{\hbar}{2}\Big|\,\hbar\Big)\cdot e^{-\hbar\partial_{m,\,i}}\,, \tag{2.24}$$

one arrives to the following:

$$\ldots=-\frac{1}{\hbar}\sum_{i_1=1}^{m}\frac{\prod\limits_{r=1}^{m-1}\big(\gamma_{m-1,\,r}-\gamma_{m,\,i_1}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_{N-m}=1\\ k_{N-m}\neq i_1}}^{m}\big(\gamma_{m,\,i_1}-\gamma_{m,\,k_{N-m}}\big)}\,\psi_R^{(m)}\ =\ \frac{(-1)^m}{\hbar}\psi_R^{(m)}\,, \tag{2.25}$$

where the last equality follows from (2.10).

At last we have to verify the remaining first relation in (2.18). Following the same reasoning as in (2.20)-(2.22) above, we obtain the following:

$$E_{m,\,N}^{c_{m-1}}\psi_R^{(m)}\ =\ E_{m-1,\,N}\psi_R^{(m)}$$
$$=-\frac{1}{\hbar}\Big\{\sum_{i_1=1}^{m}\prod_{\substack{k_1=1\\ k_1\neq i_1}}^{N-1}\frac{1}{\big(\gamma_{N-1,\,i_1}-\gamma_{N-1,\,k_1}-\hbar\big)}\,\frac{\prod\limits_{\substack{j_2=1\\ j_2\neq i_1}}^{N-1}\big(\gamma_{N-2,\,i_1}-\gamma_{N-1,\,j_2}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_2=1\\ k_2\neq i_1}}^{N-1}\big(\gamma_{N-2,\,i_1}-\gamma_{N-2,\,k_2}-\hbar\big)}\times\ldots$$
$$\ldots\times\frac{\prod\limits_{\substack{j_{N-m-1}=1\\ j_{N-m-1}\neq i_1}}^{m+2}\big(\gamma_{m+1,\,i_1}-\gamma_{m+2,\,j_{N-m-1}}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_{N-m-1}=1\\ k_{N-m-1}\neq i_1}}^{m}\big(\gamma_{m+1,\,i_1}-\gamma_{m+1,\,k_{N-m-1}}-\hbar\big)}\ \frac{\prod\limits_{\substack{j_{N-m}=1\\ j_{N-m}\neq i_1}}^{m+1}\big(\gamma_{m,\,i_1}-\gamma_{m+1,\,j_{N-m}}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_{N-m}=1\\ k_{N-m}\neq i_1}}^{m}\big(\gamma_{m,\,i_1}-\gamma_{m,\,k_{N-m}}\big)} \tag{2.26}$$
$$\times\frac{\prod\limits_{\substack{j_{N+1-m}=1\\ j_{N+1-m}\neq i_1}}^{m}\big(\gamma_{m-1,\,i_1}-\gamma_{m,\,j_{N-m}}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_{N+1-m}=1\\ k_{N+1-m}\neq i_1}}^{m-1}\big(\gamma_{m-1,\,i_1}-\gamma_{m-1,\,k_{N+1-m}}-\hbar\big)}\ e^{-\hbar\sum\limits_{a=m-1}^{N-1}\partial_{a,\,i_1}}\Big\}\cdot\psi_R^{(m)}\ =\ldots$$

Then we make cancelations due to the factors $\prod_{n=m+1}^{N-1}\prod_{\substack{i,j\\ i\neq j}}^{n}\Gamma_1\Big(\gamma_{ni}-\gamma_{nj}\Big|\hbar\Big)$ in (2.6) and relation (2.23), take into account the factor $\prod_{a=1}^{m-1}\prod_{b=1}^{m}\Gamma_1\Big(\gamma_{m-1,\,a}-\gamma_{mb}+\frac{\hbar}{2}\Big|\hbar\Big)$ in (2.6) satisfying the relation:

$$
\begin{gathered}
e^{-\hbar\left(\partial_{m-1,\,j}+\partial_{m,\,i}\right)}\cdot\prod_{a=1}^{m-1}\prod_{b=1}^{m}\Gamma_1\Big(\gamma_{m-1,\,a}-\gamma_{n,\,b}+\frac{\hbar}{2}\Big|\,\hbar\Big)\\
=\frac{\prod\limits_{\substack{r=1\\ r\neq j}}^{m-1}\big(\gamma_{m-1,\,r}-\gamma_{m,\,i}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{p=1\\ p\neq i}}^{m}\big(\gamma_{m-1,\,j}-\gamma_{m,\,p}-\frac{\hbar}{2}\big)}\;\prod_{a=1}^{m-1}\prod_{b=1}^{m}\Gamma_1\Big(\gamma_{m-1,\,a}-\gamma_{m,\,b}+\frac{\hbar}{2}\Big|\,\hbar\Big)\cdot e^{-\hbar\left(\partial_{m-1,\,j}+\partial_{m,\,i}\right)}\,,
\end{gathered}
\tag{2.27}
$$

and then arrive to the following:

$$
\begin{gathered}
\ldots=-\frac{1}{\hbar}\sum_{i_1=1}^{m}\prod_{\substack{k_{N-m}=1\\ k_{N-m}\neq i_1}}^{m}\frac{1}{\gamma_{m,\,i_1}-\gamma_{m,\,k_{N-m}}}\\
\times\sum_{i_{N+1-m}=1}^{m-1}\frac{\prod\limits_{\substack{j=1\\ j\neq i_{N+1-m}}}^{m-1}\big(\gamma_{m-1,\,j}-\gamma_{m,\,i_1}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_{N+1-m}=1\\ k_{N+1-m}\neq i_{N=1-m}}}^{m-1}\big(\gamma_{m-1,\,i_{N+1-m}}-\gamma_{m-1,\,k_{N+1-m}}-\hbar\big)}\;e^{-\hbar\sum\limits_{a=m}^{N-1}\partial_{a,\,i_1}-\partial_{m-1,\,i_{N=1-m}}}\cdot\psi_R^{(m)}
\end{gathered}
\tag{2.28}
$$

$$
\begin{gathered}
=-\frac{1}{\hbar}\sum_{i_{N+1-m}=1}^{m-1}\prod_{\substack{k_{N+1-m}=1\\ k_{N+1-m}\neq i_{N+1-m}}}^{m-1}\frac{1}{\gamma_{m-1,\,i_{N+1-m}}-\gamma_{m-1,\,k_{N+1-m}}-\hbar}\\
\times\underbrace{\sum_{i_1=1}^{m}\frac{\prod\limits_{\substack{j=1\\ j\neq i_{N=1-m}}}^{m-1}\big(\gamma_{m-1,\,j}-\gamma_{m,\,i_1}-\frac{\hbar}{2}\big)}{\prod\limits_{\substack{k_{N-m}=1\\ k_{N-m}\neq i_1}}^{m}\big(\gamma_{m,\,i_1}-\gamma_{m,\,k_{N-m}}\big)}}_{=0}\;e^{-\hbar\sum\limits_{a=m}^{N-1}\partial_{a,\,i_1}-\partial_{m-1,\,i_{N=1-m}}}\cdot\psi_R^{(m)}\;=\;0\,,
\end{gathered}
\tag{2.29}
$$

where the sum above vanishes due to (2.10). This completes the proof of Proposition.

3 Mellin-Barnes integral and its asymptotic

Now we are ready to construct the integral representation for the $\mathrm{Gr}_{m,N}$-Whittaker function (1.6). Let us substitute (2.5) and (2.6) into (1.6) and then obtain:

$$\begin{aligned}
\Psi^{(m,\,N)}_{\underline{\gamma}_N}(x) \;&=\; e^{-x\frac{m(N-m)}{2}}\int\prod_{n=1}^{N-1}d\underline{\gamma}_n\;\prod_{n=2}^{N-1}\prod_{\substack{i,j=1\\ i\neq j}}^{n}\frac{1}{\Gamma_1\big(\gamma_{ni}-\gamma_{nj}\big|\hbar\big)}\;e^{-\frac{x}{\hbar}\sum\limits_{k=1}^{m}\gamma_{mk}}\\
&\times\delta(\gamma_{11})\prod_{i=1}^{m}\prod_{j=1}^{N}\Gamma_1\Big(\gamma_{N-1,\,i}-\gamma_{Nj}+\frac{\hbar}{2}\Big|\hbar\Big)\cdot\prod_{a=1}^{m-1}\prod_{b=1}^{m}\Gamma_1\Big(\gamma_{m-1,\,a}-\gamma_{mb}+\frac{\hbar}{2}\Big|\hbar\Big)\\
&\times\prod_{\substack{n=2\\ n\neq m}}^{N-1}\Big[\delta\Big(\sum_{j=1}^{n}\gamma_{nj}-\sum_{i=1}^{n-1}\gamma_{n-1,\,i}\Big)\prod_{k=1}^{n-1}\delta\big(\gamma_{n-1,\,k}-\gamma_{nk}+\frac{\hbar}{2}\big)\prod_{\substack{i,j=1\\ i\neq j}}^{n}\Gamma_1\big(\gamma_{ni}-\gamma_{nj}\big|\hbar\big)\Big]\\
&\qquad\times e^{\imath\pi\gamma_{11}}\prod_{i=1}^{m-1}\prod_{j=1}^{m}\frac{1}{\Gamma_1\big(\gamma_{m-1,\,i}-\gamma_{m,j}+\frac{\hbar}{2}\big|\hbar\big)}
\end{aligned}\tag{3.1}$$

$$\begin{aligned}
&=\;e^{-x\frac{m(N-m)}{2}}\int\limits_{\mathcal{S}}\prod_{n=1}^{N-1}d\underline{\gamma}_n\;e^{-\frac{x}{\hbar}\sum\limits_{k=1}^{m}\gamma_{mk}}\;\frac{\prod\limits_{i=1}^{m}\prod\limits_{j=1}^{N}\Gamma_1\Big(\gamma_{N-1,\,i}-\gamma_{Nj}+\frac{\hbar}{2}\Big|\hbar\Big)}{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{m}\Gamma_1\big(\gamma_{ni}-\gamma_{nj}\big|\hbar\big)}\\
&\times\delta(\gamma_{11})e^{\imath\pi\gamma_{11}}\prod_{\substack{n=2\\ n\neq m}}^{N-1}\Big[\delta\Big(\sum_{j=1}^{n}\gamma_{nj}-\sum_{i=1}^{n-1}\gamma_{n-1,\,i}\Big)\prod_{k=1}^{n-1}\delta\big(\gamma_{n-1,\,k}-\gamma_{nk}+\frac{\hbar}{2}\big)\Big]\;=\;\ldots
\end{aligned}$$

Making integration over $\prod\limits_{n=1}^{N-2}d\underline{\gamma}_n\prod\limits_{k=m+1}^{N}d\gamma_{N-1,\,k}$ we integrate out the delta-functions and arrive to

$$\ldots=e^{-x\frac{m(N-m)}{2}}\int\prod_{n=1}^{N-1}d\underline{\gamma}_n\;e^{-\frac{x}{\hbar}\sum\limits_{k=1}^{m}\left(\gamma_{N-1,k}-\frac{N-1-m}{2}\hbar\right)}\;\frac{\prod\limits_{i=1}^{m}\prod\limits_{j=1}^{N}\Gamma_1\Big(\gamma_{N-1,\,i}-\gamma_{Nj}+\frac{\hbar}{2}\Big|\hbar\Big)}{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{m}\Gamma_1\big(\gamma_{ni}-\gamma_{nj}\big|\hbar\big)}\,.\tag{3.2}$$

Finally, we shift the integration contour by $\gamma_k=\gamma_{N-1,\,k}+\frac{\hbar}{2}\,,k=1,\ldots,m\,,$ and readily get (1.8). □

3.1 Asymptotic of $\Psi^{(m,N)}(x,0,\ldots,0)$

To complete the analysis of integral representation (1.8) let us derive its asymptotic when $x\to-\infty$.

Proof of Theorem 2.2. The contour of integration $\mathcal{C}=\mathcal{C}_m$ is a product of m copies of contour $\mathcal{C}_1$, corresponding to integration over γ_k, $k=1,\ldots,m$, going from $\epsilon-\imath\infty$ to $\epsilon+\imath\infty$ with $\epsilon>\max\{\lambda_1,\ldots,\lambda_N\}$. Let us enclose $\mathcal{C}_1$ by a half-circle of infinitely large radius in the left half-plane, then the closed contour $\tilde{\mathcal{C}}_1$ embraces all the poles of gamma-factors

$$\prod_{j=1}^{N}\Gamma_1\big(\gamma_k-\lambda_j\big|\hbar\big)$$

in (1.8). Actually, we can replace the integration over $\mathcal{C}_1$ by integration over $\tilde{\mathcal{C}}_1$, since the contribution over the half-circle of infinitely large radius vanishes due to the exponent $e^{-\hbar^{-1}x\gamma_k}$ in the integrand of (1.8). Then this transformation of integration contour allows to calculate the integral as the sum over the residues at poles of Gamma-factors. Namely, each Gamma-function $\Gamma_1(\gamma_k-\lambda_j|\hbar)$ has the poles at $\gamma_k=\lambda_j-n\hbar$, $n=0,1,2,\ldots$, and therefore integration over γ_k implies the following:

$$e^{-x\gamma_k}\prod_{j=1}^N\Gamma_1\big(\gamma_k-\lambda_j\big|\hbar\big)\;=\;\sum_{i=1}^N\sum_{n=0}^\infty n!\;e^{-x(\lambda_j-n\hbar)}\prod_{\substack{j=1\\ j\neq i}}^N\Gamma_1\big(\lambda_i-\lambda_k-n\hbar\big|\hbar\big)\,.$$

When $x\to-\infty$ only the terms with $n=0$ give contributions into the asymptotic:

$$e^{-x\gamma_k}\prod_{j=1}^N\Gamma_1\big(\gamma_k-\lambda_j\big|\hbar\big)\;\sim\;\sum_{i=1}^N e^{-x\lambda_j}\prod_{\substack{j=1\\ j\neq i}}^N\Gamma_1\big(\lambda_i-\lambda_j\big|\hbar\big)\,,\qquad\qquad x\to-\infty\,.$$

At the next step we obtain:

$$\begin{aligned}&e^{-x(\gamma_1+\gamma_2)}\prod_{j=1}^N\Gamma_1\big(\gamma_1-\lambda_j\big|\hbar\big)\Gamma_1\big(\gamma_2-\lambda_j\big|\hbar\big)\\ \sim\;&\sum_{i_1=1}^N\prod_{\substack{j=1\\ j\neq i_1}}^N\Gamma_1\big(\lambda_{i_1}-\lambda_j\big|\hbar\big)\sum_{\substack{i_2=1\\ i_2\neq i_1}}^N e^{-x(\lambda_{i_1}+\lambda_{i_2})}\frac{\prod\limits_{\substack{j=1\\ j\neq i_2}}^N\Gamma_1\big(\lambda_{i_2}-\lambda_j\big|\hbar\big)}{\Gamma_1\big(\lambda_{i_1}-\lambda_{i_2}\big|\hbar\big)\Gamma_1\big(\lambda_{i_2}-\lambda_{i_1}\big|\hbar\big)}\\ =\;&\sum_{i_1=1}^N\prod_{\substack{j=1\\ j\neq i_1}}^N\Gamma_1\big(\lambda_{i_1}-\lambda_j\big|\hbar\big)\sum_{\substack{i_2=1\\ i_2\neq i_1}}^N e^{-x(\lambda_{i_1}+\lambda_{i_2})}\frac{\prod\limits_{\substack{j=1\\ j\neq i_1,\,i_2}}^N\Gamma_1\big(\lambda_{i_2}-\lambda_j\big|\hbar\big)}{\Gamma_1\big(\lambda_{i_1}-\lambda_{i_2}\big|\hbar\big)}\,.\end{aligned}\tag{3.3}$$

In this way we proceed step by step over k, making cancelations of Gamma-factors in measure $\mu_m(\underline{\gamma})$, and finally we arrive to $\frac{N!}{m!(N-m)!}$ terms (with multiplicities $m!$) that can be arranged into the $\mathfrak{S}_N/\mathfrak{W}_m$-orbit of the term

$$m!\,e^{-x(\gamma_1+\ldots+\gamma_m)}\prod_{i=1}^m\prod_{k=1}^{N-m}\Gamma_1\big(\lambda_i-\lambda_{m+k}\big|\hbar\big)\,.$$

Thus we obtain (1.9). □

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