The distribution functions of $\sigma(n)/n$ and $n/\varphi(n)$, II

Andreas Weingartner

1. Introduction

Let

$$A(t) := \lim_{N\to\infty} \frac{1}{N} \left|\{n \le N : \sigma(n)/n \ge t\}\right|,$$

where $\sigma(n)$ is the sum of the positive divisors of n, and

$$B(t) := \lim_{N\to\infty} \frac{1}{N} \left|\{n \le N : n/\varphi(n) \ge t\}\right|,$$

where φ denotes Euler's totient function. Both of these limits exist and are continuous functions of t [**1, 3**].

We are interested in the size of $A(t)$ and $B(t)$ as t tends to infinity. From the work of Erdős [**2**] it follows that

$$B(t) = \exp\left\{-e^{t\,e^{-\gamma}}\,(1+o(1))\right\} \qquad (t\to\infty),$$

which was sharpened and extended to $A(t)$ by the author [**6**] with the result

$$A(t), B(t) = \exp\left\{-e^{t\,e^{-\gamma}}\left(1+O\left(t^{-2}\right)\right)\right\} \qquad (t\to\infty) \tag{1}$$

where $\gamma = 0.5772...$ is Euler's constant.

The purpose of this note is to make further improvements to the error term.

Theorem 1. *We have*

$$A(t), B(t) = \exp\left\{-e^{t\,e^{-\gamma}}\left(1+\sum_{j=2}^{m}\frac{a_j}{t^j}+O_m\left(\frac{1}{t^{m+1}}\right)\right)\right\},$$

where

$$a_2 = -\frac{\pi^2}{6}\,e^{2\gamma}, \quad a_3 = \frac{\pi^2}{6}\,e^{3\gamma}, \quad a_4 = -\left(\frac{\pi^2}{6}+\frac{37\pi^4}{360}\right)e^{4\gamma}.$$

Additional coefficients a_i can be determined without major difficulties by following the proofs of Lemma 5, Lemma 6 and Section 5, starting with the coefficients b_i from Lemma 5.

Throughout we will use the notation

$$y = y(t) := e^{t\,e^{-\gamma}}. \tag{2}$$

We can further decrease the size of the error term in Theorem 1 in exchange for a more complex main term. Let

$$I(y,s) := \int_e^y \log\left(1 + x\, e^{-s/x}\right) \frac{dx}{\log x} + \int_y^{y \log y} \log\left(1 + x^{-1}\, e^{s/x}\right) \frac{dx}{\log x}, \tag{3}$$

and

$$L(y) := \exp\left\{ \frac{(\log y)^{3/5}}{(\log\log y)^{1/5}} \right\}.$$

Theorem 2. *There exists a positive constant c such that*

$$A(t), B(t) = \exp\left\{ -y + \min_{s\in J} I(y,s) + R(y) \right\},$$

where $J = [y\log y - y, y \log y + y]$ and

$$R(y) = O\left(\frac{y}{L(y)^c} \right).$$

Assuming the Riemann hypothesis we have

$$R(y) = O\left(\sqrt{y}\, (\log y)^2 \right).$$

The behavior of $B(t)$ near $t = 1$ is described by Tenenbaum and Toulmonde [**4**, Thm. 1.2], who show that

$$1 - B(1 + 1/(\sigma - 1)) = \sum_{j=1}^m \frac{g_j}{(\log \sigma)^j} + O\left(\frac{|g_{m+1}|}{(\log\sigma)^{m+1}} + \frac{1}{L(\sigma)^c} \right), \tag{4}$$

for some $c > 0$, where

$$g_1 = e^{-\gamma}, \quad g_2 = 0, \quad g_3 = -\frac{1}{12}\, \pi^2\, e^{-\gamma},$$

and

$$g_j = \{1 + O(j^{-1})\}\, e^{-\gamma} (-1)^{j+1} (j-3)! \qquad (j \geq 3).$$

A classic result (see e.g. [**3**]) states that for all $s \in \mathbb{C}$ we have

$$W(s) := \lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \left(\frac{n}{\varphi(n)} \right)^s = \prod_p \left(1 + \frac{(1-p^{-1})^{-s} - 1}{p} \right) \tag{5}$$

and thus

$$\int_0^\infty B(x)\, x^{s-1} dx = 0 - \frac{1}{s} \int_0^\infty x^s\, dB(x) = \frac{W(s)}{s}, \quad (\Re(s) > 0). \tag{6}$$

Hence $\frac{W(s)}{s}$ is the Mellin transform of $B(t)$. The method used in [**4**] to establish (4) is essentially that of inversion of the Mellin transform with the abscissa of integration moved to $-\sigma$. For large t on the other hand, we find that $W(s)\, t^{-s}$ is small when $\Re(s)$ is close to $y \log y$. It turns out that the minimum of $W(s)\, t^{-s}$ with respect to s along the positive real axis is already an excellent estimate for $B(t)$ (see Lemma 2), and it appears that inversion is not a natural choice in this case because of the slower convergence of the product in (5) when $\Re(s) > 0$. Therefore we will restrict our investigation to $s \in [0, \infty)$.

The following result shows that $A(t)$ and $B(t)$ are close enough so that it suffices to show that Theorems 1 and 2 hold for $B(t)$, which is the simpler object since $\varphi(n)$ does not depend on the multiplicities of the prime factors of n.

Theorem 3. *For $t \ge t_0$ we have*

$$A(t) \le B(t) < e^{3\sqrt{y}}\, A\left(t - \frac{5e^\gamma}{\sqrt{y}}\right)$$

Another arithmetic function closely related to φ and σ is Dedekind's ψ function, defined by

$$\psi(n) = n \prod_{p|n} (1 + p^{-1}).$$

With

$$D(t) := \lim_{N\to\infty} \frac{1}{N} \left|\{n \le N : \psi(n)/n \ge t\}\right|,$$

one can show that $D(t/\zeta(2))$ also satisfies Theorems 1 and 2. It is easy to see that $D(t/\zeta(2)) \ge B(t)$ using the definition of ψ and φ. For the upper bound of $D(t/\zeta(2))$ one can consider the analog of Lemma 2 (i) below.

2. Proof of Theorem 3

The inequality $A(t) \le B(t)$, valid for all t, follows from

$$\frac{\sigma(n)}{n} = \prod_{p^\nu || n} \frac{1 + p + \ldots + p^\nu}{p^\nu} = \prod_{p^\nu || n} \frac{1 - p^{-\nu-1}}{1 - p^{-1}} < \prod_{p|n} \frac{1}{1 - p^{-1}} = \frac{n}{\varphi(n)}.$$

To establish the second inequality of Theorem 3 we let

$$m = m(t) = \prod_{p \le \sqrt{y}} p^{h_p}, \quad \text{where } h_p = \left\lfloor \frac{\log y}{\log p} \right\rfloor.$$

For every n that satisfies

$$\frac{n}{\varphi(n)} = \prod_{p|n} \frac{1}{1 - p^{-1}} \ge t,$$

nm will satisfy

$$\frac{\sigma(nm)}{nm} = \prod_{p^k || nm} \frac{1 - p^{-k-1}}{1 - p^{-1}} = \prod_{p || nm} \frac{1}{1 - p^{-1}} \prod_{p^k || nm} (1 - p^{-k-1}) \ge tP,$$

where

$$P = \prod_{p^k || nm} (1 - p^{-k-1}) \ge \prod_{p \le \sqrt{y}} \left(1 - \frac{1}{y}\right) \prod_{p > \sqrt{y}} \left(1 - \frac{1}{p^2}\right) \ge 1 - \frac{5}{\sqrt{y}\log y},$$

for $t \ge t_0$, by a standard application of the prime number theorem. Thus

$$\frac{\sigma(nm)}{nm} \ge t\left(1 - \frac{5}{\sqrt{y}\log y}\right) = t - \frac{5e^\gamma}{\sqrt{y}},$$

which implies

$$A\left(t - \frac{5e^\gamma}{\sqrt{y}}\right) \ge \frac{1}{m}\, B(t).$$

The result now follows since, for $t \ge t_0$,

$$\log m = \sum_{p \le \sqrt{y}} h_p \log p \le \sum_{p \le \sqrt{y}} \log y < 3\sqrt{y}.$$

3. The relation between $B(t)$ and $W(s)$.

Lemma 1. *Let* $s \ge 1$. *If*
$$B(t)t^{s-1} = \max_{x\ge 0} B(x)x^{s-1}$$
then
$$s = y\log y + O(y).$$

Proof. Assume $B(t)t^{s-1} \ge B(t+h)(t+h)^{s-1}$ for $|h| \le 1$. After taking logarithms we use (1) to obtain
$$y\,(e^{h\,e^{-\gamma}} - 1) + O(y\,t^{-2}) \ge (s-1)\log(1 + h\,t^{-1}),$$
and hence
$$y\,h\,e^{-\gamma} \ge (s-1)h\,t^{-1} + O(s\,h^2\,t^{-2} + yh^2 + y\,t^{-2}).$$
The result now follows if we first let $h = t^{-1}$, and then $h = -t^{-1}$, and multiply the last inequality by $h^{-1}t$ in each case. □

Lemma 2. (i) *For all* $s \ge 0$, $t > 0$ *we have*
$$B(t) \le \frac{W(s)}{t^s}.$$
(ii) *Let* $s \ge 1$ *and* $t \ge t_0$. *If* $B(t)t^{s-1} = \max_{x\ge 0} B(x)x^{s-1}$, *then*
$$\frac{W(s)}{3s\,t^s} \le B(t)$$
and
$$\log B(t) = O(t) + \min_{u\ge 0}\log\frac{W(u)}{t^u}.$$

Proof. (i) For all $s \ge 0$,
$$B(t) = \lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n\le N\\ n\ge t\,\varphi(n)}} 1 \le \lim_{N\to\infty}\frac{1}{N}\sum_{n\le N}\left(\frac{n}{t\,\varphi(n)}\right)^s = \frac{W(s)}{t^s}.$$

(ii) From (6) we have
$$\frac{W(s)}{s} = \int_0^t B(x)\,x^{s-1}dx + \int_t^{2t} B(x)\,x^{s-1}dx + \int_{2t}^{\infty} B(x)\,x^{s-1}dx =: I_1 + I_2 + I_3$$
Since $\max_{x\ge 0}(B(x)\,x^{s-1}) = B(t)\,t^{s-1}$, we have $I_1, I_2 \le t\,B(t)\,t^{s-1} = B(t)\,t^s$. If c is the implied constant in the error term of (1), then for $x \ge t$
$$\begin{aligned}
B(2x) &\le \exp\left\{-e^{2xe^{-\gamma}}(1 - cx^{-2})\right\} \le \exp\left\{-e^{xe^{-\gamma}}(y/2)(1 + cx^{-2})\right\}\\
&\le B(x)^{y/2} \le B(x)^{1+\log y} \le B(x)B(t)^{\log y}\\
&\le B(x)\exp\{-y\log y + O(y/\log y)\} = B(x)\exp\{-s + O(y)\}\\
&\le \frac{B(x)}{2^{s+1}},
\end{aligned}$$
since $s = y\log y + O(y)$ by Lemma 1. We conclude that for $k \ge 1$
$$\int_{t2^k}^{t2^{k+1}} B(x)\,x^{s-1}dx = 2^s\int_{t2^{k-1}}^{t2^k} B(2x)\,x^{s-1}dx \le \frac{1}{2}\int_{t2^{k-1}}^{t2^k} B(x)\,x^{s-1}dx,$$
and thus $I_3 \le I_2 \le B(t)\,t^s$.

The second assertion in (ii) follows from the first and (i), since $s = y\log y + O(y)$. □

4. The study of the product $W(s)$.

Let

$$t_u := \prod_{p\le u} \frac{1}{1-p^{-1}}, \quad P_u := \prod_{p\le u} p.$$

Lemma 3. *Let $2 \le u \le v$. For $s \ll v$ we have*

$$\frac{W(s)}{t_u^s} = \frac{t_u}{t_v\, P_u}\left(1+O\left(\frac{s}{v\log v}\right)\right)\prod_{p\le u}\left(1+p\left(1-p^{-1}\right)^{s+1}\right)\prod_{u<p\le v}\left(1+p^{-1}(1-p^{-1})^{-s-1}\right)$$

Proof. The contribution from primes $p>v$ to the product (5) is

$$\begin{aligned}\prod_{p>v}\left(1+\frac{(1-p^{-1})^{-s}-1}{p}\right) &= \prod_{p>v}\left(1+\frac{1}{p}\left(e^{O\left(\frac{s}{p}\right)}-1\right)\right)\\ &= \prod_{p>v}\left(1+O\left(\frac{s}{p^2}\right)\right) = 1+O\left(\frac{s}{v\log v}\right).\end{aligned}$$

For primes p in the range $u<p\le v$ we write

$$\prod_{u<p\le v}\left(1+\frac{(1-p^{-1})^{-s}-1}{p}\right) = \prod_{u<p\le v}(1-p^{-1})\prod_{u<p\le v}\left(1+p^{-1}(1-p^{-1})^{-s-1}\right).$$

Finally, the product over small primes is

$$\prod_{p\le u}\frac{(1-p^{-1})^{-s}}{p}\prod_{p\le u}\left(1+p\left(1-p^{-1}\right)^{s+1}\right) = \frac{t_u^s}{P_u}\prod_{p\le u}\left(1+p\left(1-p^{-1}\right)^{s+1}\right).$$

□

Lemma 4. *Let $2\le u\le v$. For $v \gg s = u\log u + O(u)$ we have*

$$\frac{W(s)}{t_u^s} = \frac{t_u}{t_v\, P_u}\left(1+O\left(\frac{1}{\log u}\right)\right)\prod_{p\le u}\left(1+p\, e^{-s/p}\right)\prod_{u<p\le v}\left(1+p^{-1}\, e^{s/p}\right).$$

Proof. We write

$$\begin{aligned}\prod_{u<p\le v}\left(1+p^{-1}(1-p^{-1})^{-s-1}\right) &= \prod_{u<p\le v}\left(1+p^{-1}\exp\left(\frac{s}{p}+O\left(\frac{s}{p^2}\right)\right)\right)\\ &= \prod_{u<p\le v}\left(1+p^{-1}e^{s/p}\right)\left(1+O\left(\frac{s}{p^3}e^{s/p}\right)\right).\end{aligned}$$

After taking the logarithm of the last expression, the contribution from the error term is

$$\begin{aligned}&\ll \sum_{p>u}\frac{s}{p^3}e^{s/p} \asymp \int_u^\infty \frac{s}{x^3}e^{s/x}\frac{dx}{\log x}\\ &\asymp \frac{1}{u\log u}\int_u^\infty \frac{s}{x^2}e^{s/x}dx \qquad (7)\\ &= \frac{1}{u\log u}e^{s/u} \asymp \frac{1}{\log u}.\end{aligned}$$

Thus

$$\prod_{u<p\le v}\left(1+p^{-1}(1-p^{-1})^{-s-1}\right) = \left(1+O\left(\frac{1}{\log u}\right)\right)\prod_{u<p\le v}\left(1+p^{-1}e^{s/p}\right).$$

Similarily,

$$\prod_{p\le u}\left(1+p\,(1-p^{-1})^{s+1}\right)=\prod_{p\le u}\left(1+p\,\exp\left(-\frac{s}{p}+O\left(\frac{s}{p^2}\right)\right)\right)$$
$$=\prod_{p\le u}\left(1+p\,e^{-s/p}\right)\left(1+O\left(\frac{s}{p}\,e^{-s/p}\right)\right).$$

The contribution from the error term to the logarithm of the last expression is

$$\begin{aligned}&\asymp\sum_{p\le u}\frac{s}{p}\,e^{-s/p}\asymp\int_2^u\frac{s}{x}\,e^{-s/x}\frac{dx}{\log x}\\ &\asymp\frac{u}{\log u}\int_2^u\frac{s}{x^2}\,e^{-s/x}dx\\ &\asymp\frac{u}{\log u}\,e^{-s/u}\asymp\frac{1}{\log u}.\end{aligned}\tag{8}$$

Thus

$$\prod_{p\le u}\left(1+p\,(1-p^{-1})^{s+1}\right)=\left(1+O\left(\frac{1}{\log u}\right)\right)\prod_{p\le u}\left(1+p\,e^{-s/p}\right).$$

The result now follows from Lemma 3. □

Lemma 5. *Let $s\ge e$ and define z by $s=z\log z$. For $m\ge 2$ we have*

$$W(s)=\exp\left(z\log z\log(e^\gamma\log z)-z+z\sum_{j=2}^m\frac{b_j}{(\log z)^j}+O_m\left(\frac{z}{(\log z)^{m+1}}\right)\right),$$

where

$$b_2=\frac{\pi^2}{6},\quad b_3=-\frac{\pi^2}{6},\quad b_4=\frac{\pi^2}{6}+\frac{7\pi^4}{60}.$$

Proof. We apply Lemma 4 with $u=z$ and $v=s$ to obtain

$$\begin{aligned}\log W(s)=&-s\sum_{p\le z}\log(1-p^{-1})-\sum_{p\le z}\log p+O\left(\frac{\log_2 z}{\log z}\right)\\ &+\sum_{p\le z}\log\left(1+p\,e^{-s/p}\right)+\sum_{z<p\le s}\log\left(1+p^{-1}\,e^{s/p}\right)\\ =&z\log z\log(e^\gamma\log z)-z+O\left(\frac{z}{\exp(\sqrt{\log z})}\right)\\ &+\int_e^z\log\left(1+x\,e^{-s/x}\right)\frac{dx}{\log x}+\int_z^s\log\left(1+x^{-1}\,e^{s/x}\right)\frac{dx}{\log x},\end{aligned}\tag{9}$$

by a strong form of Mertens' Theorem [**5**] and a standard application of the prime number theorem. We need to estimate the two integrals in (9). The first integral is

$$\sum_{k\ge1}\frac{(-1)^{k+1}}{k}\int_e^z x^k\,e^{-sk/x}\frac{dx}{\log x}=\sum_{k\ge1}\frac{(-1)^{k+1}}{k}I_k(k,1),\tag{10}$$

where

$$I_k(a,b) := \int_e^z x^a\, e^{-sk/x} \frac{dx}{(\log x)^b} = \frac{1}{sk}\int_e^z \left(\frac{sk}{x^2}\, e^{-sk/x}\right) \frac{x^{a+2}}{(\log x)^b}\, dx$$
$$\le \frac{z^{a+2}}{sk(\log z)^b}\int_e^z \frac{sk}{x^2}\, e^{-sk/x} dx \le \frac{z^{1+a-k}}{k(\log z)^{b+1}}, \tag{11}$$

for $a \ge b$, since $x/\log x$ is increasing for $x \ge e$. Integration by parts applied to the second integral in (11) shows that

$$I_k(a,b) = \frac{z^{1+a-k}}{k(\log z)^{b+1}} - \frac{a+2}{sk} I_k(a+1,b) + \frac{b}{sk} I_k(a+1,b+1) + O_m(1/(sk)), \tag{12}$$

for $a \le k+m$. After $m-1$ iterations of (12), starting with $I_k(k,1)$, we find that

$$I_k(k,1) = \sum_{j=2}^{m} \frac{z}{(\log z)^j}\, q_j(k) + O_m\left(\frac{z}{k(\log z)^{m+1}}\right), \tag{13}$$

where $q_j(k)$ is a rational function of k with $q_j(k) = O(1/k)$. In particular,

$$q_2(k) = \frac{1}{k}, \quad q_3(k) = -\frac{k+2}{k^2}, \quad q_4(k) = \frac{1}{k^2} + \frac{(k+2)(k+3)}{k^3}.$$

Inserting (13) into (10) gives

$$\int_e^z \log\left(1 + x\, e^{-s/x}\right) \frac{dx}{\log x} = z \sum_{j=2}^{m} \frac{\theta_j}{(\log z)^j} + O_m\left(\frac{z}{(\log z)^{m+1}}\right), \tag{14}$$

where

$$\theta_j = \sum_{k\ge 1} (-1)^{k+1} \frac{q_j(k)}{k}.$$

Similarily, the second integral in (9) is

$$\sum_{k\ge 1} \frac{(-1)^{k+1}}{k} \int_z^s x^{-k}\, e^{sk/x} \frac{dx}{\log x} = \sum_{k\ge 1} \frac{(-1)^{k+1}}{k} J_k(k,1), \tag{15}$$

where

$$J_k(a,b) := \int_z^s x^{-a}\, e^{sk/x} \frac{dx}{(\log x)^b} = \frac{1}{sk}\int_z^s \left(\frac{sk}{x^2}\, e^{sk/x}\right) \frac{x^{-a+2}}{(\log x)^b}\, dx$$
$$\asymp \frac{1}{sk(\log z)^b}\int_k^{k\log z} e^w \left(\frac{sk}{w}\right)^{2-a} dw = O_m\left(\frac{z^{1+k-a}}{k(\log z)^{b+1}}\right), \tag{16}$$

for $a \ge -m$. Integration by parts applied to the second integral in (16) shows that

$$J_k(a,b) = \frac{z^{1+k-a}}{k(\log z)^{b+1}} + \frac{2-a}{sk} J_k(a-1,b) - \frac{b}{sk} J_k(a-1,b+1) + O_m\left(k^{-1}(e/s)^{a-1}\right), \tag{17}$$

for $k - a \le m$. After $m-1$ iterations of (17), starting with $J_k(k,1)$, we find that

$$J_k(k,1) = \sum_{j=2}^{m} \frac{z}{(\log z)^j}\, r_j(k) + O_m\left(\frac{z}{k(\log z)^{m+1}}\right), \tag{18}$$

where $r_j(k)$ is a rational function of k with $r_j(k) = O(1/k)$. In particular,

$$r_2(k) = \frac{1}{k}, \quad r_3(k) = \frac{2-k}{k^2}, \quad r_4(k) = \frac{(2-k)(3-k)}{k^3} - \frac{1}{k^2}.$$

Inserting (18) into (15) gives

$$\int_z^s \log\left(1+x^{-1}\,e^{s/x}\right)\frac{dx}{\log x} = z\sum_{j=2}^{m}\frac{\rho_j}{(\log z)^j} + O_m\left(\frac{z}{(\log z)^{m+1}}\right), \tag{19}$$

where

$$\rho_j = \sum_{k\geq 1}(-1)^{k+1}\frac{r_j(k)}{k}.$$

Let $b_j = \theta_j + \rho_j$, then

$$b_2 = \sum_{k\geq 1}\frac{(-1)^{k+1}}{k}(q_2(k)+r_2(k)) = 2\sum_{k\geq 1}\frac{(-1)^{k+1}}{k^2} = 2\sum_{k\geq 1}\frac{1}{k^2} - 4\sum_{k\geq 1}\frac{1}{(2k)^2} = \frac{\pi^2}{6},$$

$$b_3 = \sum_{k\geq 1}\frac{(-1)^{k+1}}{k}(q_3(k)+r_3(k)) = -2\sum_{k\geq 1}\frac{(-1)^{k+1}}{k^2} = -\frac{\pi^2}{6},$$

and

$$b_4 = \sum_{k\geq 1}\frac{(-1)^{k+1}}{k}(q_4(k)+r_4(k)) = \sum_{k\geq 1}\frac{(-1)^{k+1}}{k}\left(\frac{2}{k}+\frac{12}{k^3}\right) = \frac{\pi^2}{6}+\frac{7\pi^4}{60}.$$

The result now follows from combining (9), (14) and (19). □

Lemma 6. *For $t\geq 1$ and $y = e^{t\,e^{-\gamma}}$ we have*

$$\min_{s\geq e}\frac{W(s)}{t^s} = \exp\left(-y + y\sum_{k=2}^{m}\frac{c_k}{(\log y)^k} + O_m\left(\frac{y}{(\log y)^{m+1}}\right)\right),$$

where

$$c_2 = \frac{\pi^2}{6},\quad c_3 = -\frac{\pi^2}{6},\quad c_4 = \frac{\pi^2}{6}+\frac{37\pi^4}{360}.$$

Proof. Let $t\geq 1$ be given. From Lemma 5 we have

$$\log\frac{W(s)}{t^s} = z\left(\log z\log(\log z/\log y) - 1 + \sum_{k=2}^{m}\frac{b_k}{(\log z)^k} + O_m\left(\frac{1}{(\log z)^{m+1}}\right)\right) =: h(z), \tag{20}$$

where $s = z\log z$. We see that $h(y)\sim -y$ and $h(z)>0$ for $z\geq ey$, so that the minimum of $h(z)$ occurs at some $z\in[e,ey]$, where the error term of (20) is uniformly $O_m\left(y/(\log y)^{m+1}\right)$. Therefore we only need to minimize

$$f(z) := z\left(\log z\log(\log z/\log y) - 1 + \sum_{k=2}^{m}\frac{b_k}{(\log z)^k}\right). \tag{21}$$

To that end we set $f'(z)=0$, which is equivalent to

$$\log y = \log z\exp\left(\sum_{k=2}^{m+1}\frac{\alpha_k}{(1+\log z)\log^k z}\right), \tag{22}$$

where $\alpha_2 = b_2$, $\alpha_k = b_k-(k-1)b_{k-1}$ for $k=3,\dots,m-1$, and $\alpha_{m+1} = -mb_m$. Thus

$$\alpha_2 = b_2 = \frac{\pi^2}{6},\quad \alpha_3 = b_3 - 2b_2 = -\frac{\pi^2}{2},\quad \alpha_4 = b_4 - 3b_3 = \frac{2\pi^2}{3}+\frac{7\pi^4}{60}.$$

Since $f(e) \sim -e\log\log y$, $f(y) \sim -y$, and $f(ey) > 0$, the unique solution to (22) is the minimizer of $f(z)$. We rewrite (22) as

$$\log z \log\left(\frac{\log z}{\log y}\right) = -\sum_{k=2}^{m} \frac{\beta_k}{(\log z)^k} + O\left(\frac{1}{(\log z)^{m+1}}\right), \tag{23}$$

where $\beta_2 = \alpha_2$ and $\beta_k = \alpha_k - \beta_{k-1}$ for $k = 3, \ldots, m$. Thus

$$\beta_2 = \alpha_2 = \frac{\pi^2}{6}, \quad \beta_3 = \alpha_3 - \beta_2 = -\frac{2\pi^2}{3}, \quad \beta_4 = \alpha_4 - \beta_3 = \frac{4\pi^2}{3} + \frac{7\pi^4}{60}.$$

To express z in terms of y we first write (23) as

$$\begin{aligned} \log y &= \log z \exp\left(\sum_{k=2}^{m} \frac{\beta_k}{(\log z)^{k+1}} + O\left(\frac{1}{(\log z)^{m+2}}\right)\right) \\ &= \log z \left(1 + \sum_{k=2}^{m} \frac{\delta_k}{(\log z)^{k+1}} + O\left(\frac{1}{(\log z)^{m+2}}\right)\right), \end{aligned} \tag{24}$$

where

$$\delta_2 = \beta_2 = \frac{\pi^2}{6}, \quad \delta_3 = \beta_3 = -\frac{2\pi^2}{3}, \quad \delta_4 = \beta_4 = \frac{4\pi^2}{3} + \frac{7\pi^4}{60}.$$

Using series inversion on (24) we obtain

$$\log z = \log y \left(1 + \sum_{k=2}^{m} \frac{\eta_k}{(\log y)^{k+1}} + O\left(\frac{1}{(\log y)^{m+2}}\right)\right), \tag{25}$$

where

$$\eta_2 = -\delta_2 = -\frac{\pi^2}{6}, \quad \eta_3 = -\delta_3 = \frac{2\pi^2}{3}, \quad \eta_4 = -\delta_4 = -\frac{4\pi^2}{3} - \frac{7\pi^4}{60}.$$

We exponentiate (25) to get

$$\begin{aligned} z &= y \exp\left(\sum_{k=2}^{m} \frac{\eta_k}{(\log y)^{k}} + O\left(\frac{1}{(\log y)^{m+1}}\right)\right) \\ &= y \left(1 + \sum_{k=2}^{m} \frac{\lambda_k}{(\log y)^{k}} + O\left(\frac{1}{(\log y)^{m+1}}\right)\right), \end{aligned} \tag{26}$$

where

$$\lambda_2 = \eta_2 = -\frac{\pi^2}{6}, \quad \lambda_3 = \eta_3 = \frac{2\pi^2}{3}, \quad \lambda_4 = \eta_4 + \frac{\eta_2^2}{2} = -\frac{4\pi^2}{3} - \frac{37\pi^4}{360}.$$

Combining (21), (23) and (26) we see that $\min_z f(z)$ is

$$\begin{aligned} &y\left(1 + \sum_{k=2}^{m} \frac{\lambda_k}{\log^k y} + O\left(\frac{1}{\log^{m+1} y}\right)\right)\left(-1 + \sum_{k=2}^{m} \frac{b_k - \beta_k}{\log^k z} + O\left(\frac{1}{\log^{m+1} z}\right)\right) \\ &\quad = y\left(1 + \sum_{k=2}^{m} \frac{\lambda_k}{(\log y)^k} + O\left(\frac{1}{(\log y)^{m+1}}\right)\right)\left(-1 + \sum_{k=2}^{m} \frac{\mu_k}{(\log y)^k} + O\left(\frac{1}{(\log y)^{m+1}}\right)\right), \end{aligned}$$

where (24) implies

$$\mu_2 = b_2 - \beta_2 = 0, \quad \mu_3 = b_3 - \beta_3 = \frac{\pi^2}{2}, \quad \mu_4 = b_4 - \beta_4 = -\frac{7\pi^2}{6}.$$

Thus

$$\min_z f(z) = -y + y \sum_{k=2}^m \frac{c_k}{(\log y)^k} + O_m\left(\frac{y}{(\log y)^{m+1}}\right),$$

where

$$c_2 = \mu_2 - \lambda_2 = \frac{\pi^2}{6}, \quad c_3 = \mu_3 - \lambda_3 = -\frac{\pi^2}{6}, \quad c_4 = \mu_4 + \mu_2\lambda_2 - \lambda_4 = \frac{\pi^2}{6} + \frac{37\pi^4}{360}.$$

□

Lemma 7. *Let $g(s) = \log W(s)$. We have*

$$g''(s) \asymp \frac{1}{s \log s} \qquad (s \geq 2).$$

Proof. From (5) we find that

$$\begin{aligned} g''(s) &= \sum_p p^{-1}(1-p^{-1}) \log^2(1-p^{-1}) \frac{(1-p^{-1})^{-s}}{(1+p^{-1}((1-p^{-1})^{-s}-1))^2} \\ &\asymp \sum_p p^{-3} \frac{(1-p^{-1})^{-s}}{(1+p^{-1}(1-p^{-1})^{-s})^2} \end{aligned}$$

Let u be given by $u^{-1}(1-u^{-1})^{-s} = 1$, so that $s = u \log u + O(\log u)$. Then

$$\begin{aligned} g''(s) &\asymp \sum_{p \leq u} p^{-1}(1-p^{-1})^s + \sum_{p > u} p^{-3}(1-p^{-1})^{-s} \\ &\asymp \sum_{p \leq u} p^{-1} e^{-s/p} + \sum_{p > u} p^{-3} e^{s/p} \asymp \frac{1}{s \log s}, \end{aligned}$$

where the last two sums are estimated just like in (7) and (8). □

5. Proof of Theorems 1 and 2

Define the set of maximizers

$$M := \{t \geq 1 : \exists s > 1 \text{ with } \max_{x \geq 0} B(x) x^{s-1} = B(t) t^{s-1}\}.$$

Lemma 8. *There is a constant $c > 0$ such that for every $t \geq 1$ there is a $t_1 \in M$ with*

$$|t - t_1| \leq c\sqrt{t/y}.$$

Proof. Let $t \geq 1$ and let s be given by $\min_u \frac{W(u)}{t^u} = \frac{W(s)}{t^s}$. Let $t_1 \in M$ satisfy $\max_{x \geq 1} B(x) x^{s-1} = B(t_1) t_1^{s-1}$. Finally, define s_1 by $\min_u \frac{W(u)}{t_1^u} = \frac{W(s_1)}{t_1^{s_1}}$. From Lemma 2 we find

$$\frac{W(s)}{3 s t_1^s} \leq B(t_1) \leq \frac{W(s_1)}{t_1^{s_1}} \leq \frac{W(s)}{t_1^s},$$

so

$$\log \frac{W(s)}{t_1^s} = \log \frac{W(s_1)}{t_1^{s_1}} + O(\log s).$$

By Taylor's theorem there is an s_0 between s and s_1 with

$$\log \frac{W(s)}{t_1^s} = \log \frac{W(s_1)}{t_1^{s_1}} + \frac{g''(s_0)}{2}(s - s_1)^2,$$

where $g(u) = \log W(u)$. Combining the last two equations with Lemma 7 we obtain

$$|s - s_1| = O(\sqrt{s} \log s).$$

Let $f(u) = \exp(g'(u))$. From the definition of s and s_1 we have $t = f(s)$ and $t_1 = f(s_1)$. Thus $|t - t_1| \le |s - s_1| \max_I f'(u)$, where I is the interval with endpoints s, s_1. Now $f'(u) = f(u)\, g''(u)$, so Lemma 7 yields

$$|t - t_1| = O\left(\frac{t\sqrt{s}\log s}{s \log s}\right) = O\left(\sqrt{t/y}\right),$$

by Lemma 1. □

Proof of Theorem 1. If $t \in M$ then the result follows from Lemma 2 (ii) and Lemma 6 with $a_j = -c_j e^{j\gamma}$. If $t \notin M$, the result follows from Lemma 8 and the monotonicity of $B(t)$. □

Proof of Theorem 2. We apply Lemma 4 with $u = y$ and $v = y \log y$. For $s = y\log y + O(y)$,

$$\begin{aligned} \log \frac{W(s)}{t^s} &= -\, s \log t - s \sum_{p \le y} \log(1 - p^{-1}) - \sum_{p \le y} \log p + O\left(\frac{\log_2 y}{\log y}\right) \\ &\quad + \sum_{p\le y} \log\left(1 + p\, e^{-s/p}\right) + \sum_{y < p \le y \log y} \log\left(1 + p^{-1}\, e^{s/p}\right) \\ &= -\, y + I(y, s) + O\left(\frac{y}{L(y)^c}\right), \end{aligned}$$

by a strong form of Mertens' Theorem [**5**] and a standard application of the prime number theorem. Under the assumption of the Riemann hypothesis, the error term can be replaced by $O(\sqrt{y}\,(\log y)^2)$. If $t \in M$ and $\max_{x\ge 0} B(x) x^{s-1} = B(t) t^{s-1}$, then $s = y \log y + o(y)$, by an argument like in the proof of Lemma 1, but this time using Theorem 1 with $m = 2$ instead of (1). Therefore Lemma 2 implies

$$\log B(t) = O(t) + \min_{u \in J} \frac{W(u)}{t^u} = O(t) - y + \min_{u\in J} I(y, u) + O\left(\frac{y}{L(y)^c}\right),$$

where $J = [y \log y - y, y \log y + y]$. If $t \notin M$, there is a $t_1 \in M$ with $|t - t_1| = O(\sqrt{t/y})$ by Lemma 8. For $s = y \log y + O(y)$ and $y_1 := e^{t_1 e^{-\gamma}} = y + O(\sqrt{t\, y})$ we have $I(y_1, s) = I(y, s) + O(\sqrt{t\, y})$. Thus the result follows again from the monotonicity of $B(t)$. □

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Department of Mathematics, Southern Utah University, Cedar City, Utah 84720
E-mail address: weingartner@suu.edu