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Completely almost periodic functionals

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Abstract

Using the notion of complete compactness introduced by H. Saar, we define completely almost periodic functionals on completely contractive Banach algebras. We show that, if (M, Γ) is a Hopf–von Neumann algebra with M injective, then the space of completely almost periodic functionals on M_* is a C^*-subalgebra of M.

Keywords: completely compact map; completely almost periodic functional; Hopf–von Neumann algebra.

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Introduction

The almost periodic and weakly almost periodic continuous functions on a locally compact group G form C^*-subalgebras of $L^\infty(G)$, usually denoted by $\mathcal{AP}(G)$ and $\mathcal{WAP}(G)$, respectively: this is fairly elementary to prove and well known (see [Bur] for instance). In a more abstract setting, one can define, for a general Banach algebra $\mathfrak{A}$ the spaces $\mathcal{AP}(\mathfrak{A})$ and $\mathcal{WAP}(\mathfrak{A})$ of almost periodic and weakly almost periodic functionals on $\mathfrak{A}$; if $\mathfrak{A} = L^1(G)$, we have $\mathcal{AP}(\mathfrak{A}) = \mathcal{AP}(G)$ and $\mathcal{WAP}(\mathfrak{A}) = \mathcal{WAP}(G)$.

For $\mathfrak{A} = A(G)$, Eymard's Fourier algebra ([Eym]), the spaces $\mathcal{AP}(\mathfrak{A})$ and $\mathcal{WAP}(\mathfrak{A})$ are usually denoted by $\mathcal{AP}(\hat{G})$ and $\mathcal{WAP}(\hat{G})$: they were first considered by C. F. Dunkl and D. E. Ramirez ([D–R]) and further studied by E. E. Granirer ([Gra 1] and [Gra 2]), A. T.-M. Lau ([Lau]), and others. Except in a few special cases, e.g., if G is abelian or discrete and amenable, it is unknown whether $\mathcal{AP}(\hat{G})$ and $\mathcal{WAP}(\hat{G})$ are C^*-subalgebras of the group von Neumann algebra $\mathrm{VN}(G)$.

Recently, M. Daws considered the almost periodic and weakly almost periodic functionals on the predual of a Hopf–von Neumann algebra with underlying von Neumann algebra M ([Daw]). He proved: *If M is abelian*, then both $\mathcal{AP}(M_*)$ and $\mathcal{WAP}(M_*)$ are C^*-subalgebras of M ([Daw, Theorems 1 and 4]). Unfortunately, the demand that M be abelian is crucial for Daws' proofs to work (see [Run] for a discussion).

Over the past two decades, it has become apparent that purely Banach algebraic notions aren't well suited for the study of $A(G)$: one often has to tweak these notions in a way that takes the canonical operator space structure of $A(G)$—as the predual of the group von Neumann algebra—into account. For instance, Banach algebraic amenability of $A(G)$ forces G to be finite-by-abelian ([F–R]) whereas $A(G)$ is *operator* amenable if and only if G is amenable ([Rua]), a much more satisfactory result.

We apply this philosophy to almost periodicity. A functional ϕ on a Banach algebra $\mathfrak{A}$ is called almost periodic if the maps

$$\mathfrak{A} \to \mathfrak{A}^*, \quad a \mapsto \begin{cases} a \cdot \phi \\ \phi \cdot a \end{cases} \tag{$*$}$$

are compact. Suppose now that $\mathfrak{A}$ is a completely contractive Banach algebra. Then the maps $(*)$ are completely bounded. There are various definitions that attempt to fit the notion of a compact operator to a completely bounded context (see [Saa], [Web], [Oik], and [Yew], for instance). We focus on the definition of a completely compact map from [Saa], and define $\phi \in \mathfrak{A}^*$ to be completely almost periodic if the maps $(*)$ are completely compact.

Our main result is that, if (M, Γ) is a Hopf–von Neumann algebra such that M is injective, then the completely almost periodic functionals on M_* form a C^*-subalgebra of M. This applies, in particular, to $A(G)$ in the cases where G is amenable or connected.

1 Completely compact maps

There are various ways to adapt the notion of a compact operator to an operator space setting: operator compactness ([Web] and [Yew]), complete compactness ([Saa]), and Gelfand complete compactness ([Oik]).

The notion of a completely compact map between C^*-algebras was introduced by H. Saar in his Diplomarbeit [Saa] under G. Wittstock's supervision. The starting point of his definition is the following observation: if E and F are Banach spaces, and $T \colon E \to F$ is a bounded linear map, then T is compact if and only if, for each $\epsilon > 0$, there is a finite-dimensional subspace Y_ϵ of F such that $\|Q_{Y_\epsilon} T\| < \epsilon$, where $Q_{Y_\epsilon} \colon F \to F/Y_\epsilon$ is the quotient map.

This can be used to define an operator space analog of compactness, namely complete compactness. Saar didn't define complete compactness for maps between general, abstract operator spaces—simply because these objects hadn't been defined yet at that time—, but his definition obviously extends to general operator spaces. (Our reference for operator space theory is [E–R], the notation and terminology of which we adopt.) In modern language, Saar's definition reads:

Definition 1.1. Let E and F be operator spaces. Then $T \in \mathcal{CB}(E, F)$ is called *completely compact* if, for each $\epsilon > 0$, there is a finite-dimensional subspace Y_ϵ of F such that $\|Q_{Y_\epsilon} T\|_{\mathrm{cb}} < \epsilon$, where $Q_{Y_\epsilon} \colon F \to F/Y_\epsilon$ is the quotient map.

Remarks. 1. Trivially, completely compact maps are compact.

2. It is obvious that, if E is a Banach space and F is an operator space, then $T \in \mathcal{B}(E,F) = \mathcal{CB}(\max E, F)$ is completely compact if and only if it is compact.

3. On [Saa, pp. 32–34], Saar constructs an example of a compact, completely bounded map on $\mathcal{K}(\ell^2)$ that fails to be completely compact.

4. Complete compactness may not be stable under co-restrictions, i.e., if $T \in \mathcal{CB}(E,F)$ be completely compact, and let Y be a closed subspace of F containing TE, then it is not clear why T viewed as an element of $\mathcal{CB}(E,Y)$ should be completely compact.

5. Very recently, complete compactness was put to use for the study of so-called operator multipliers (see [J–L–T–T] and [T–T]).

Given two operator spaces E and F, we write $\mathcal{CK}(E,F)$ for the completely compact operators in $\mathcal{CB}(E,F)$.

The following proposition is essentially [Saa, Lemma 1 a) and Lemma 2]. (Of course, Saar only considers maps between C^*-algebras, but his proofs carry over more or less verbatim.)

Proposition 1.2. *Let E and F be operator spaces. Then:*

(i) *$\mathcal{CK}(E,F)$ is a closed subspace of $\mathcal{CB}(E,F)$ containing all finite rank operators;*

(ii) *if $T \in \mathcal{CK}(E,F)$, X is another operator space, and $R \in \mathcal{CB}(X,E)$, then $TR \in \mathcal{CK}(X,F)$;*

(iii) *if $T \in \mathcal{CK}(E,F)$, Y is another operator space, and $S \in \mathcal{CB}(F,Y)$, then $ST \in \mathcal{CK}(E,Y)$.*

From Schauder's theorem and Saar's characterization of compact operators, it follows immediately that a bounded linear operator T from a Banach space E into a Banach space F is compact if and only if, for each $\epsilon > 0$, there is a closed subspace X_ϵ of E with finite co-dimension such that $\|T|_{X_\epsilon}\| < \epsilon$ (compare also [Lac]).

Following [Oik], we define:

Definition 1.3. Let E and F be operator spaces. Then $T \in \mathcal{CB}(E,F)$ is called *Gelfand completely compact* if, for each $\epsilon > 0$, there is a closed subspace X_ϵ of E with finite co-dimension such that $\|T|_{X_\epsilon}\|_{\mathrm{cb}} < \epsilon$.

Remarks. 1. Obviously, $T \in \mathcal{CB}(E,F)$ is Gelfand completely compact if and only if $T^* \in \mathcal{CB}(F^*,E^*)$ is completely compact (and vice versa).

2. A result analogous to Proposition 1.2 holds for Gelfand completely compact maps.

3. Based on results from the unpublished paper [Web], T. Oikhberg gives examples of completely compact maps that fail to be Gelfand completely compact and vice versa ([Oik, pp. 155-156]). Hence, an analog for Schauder's theorem fails for complete compactness.

Under certain circumstances, every completely compact map is Gelfand completely compact ([Oik, Theorem 3.1]). For our purposes, the following is important (see [E–R, p. 70] for the notion of an injective operator space):

Proposition 1.4. *Let E and F be operator spaces such that E^* and F^* are injective. Then the following are equivalent for $T \in \mathcal{CB}(E, F^*)$:*

(i) *T is completely compact;*

(ii) *T is Gelfand completely compact;*

(iii) *T is a* cb*-norm limit of finite rank operators.*

Proof. Obviously, (iii) implies both (i) and (ii).

(ii) $\Longrightarrow$ (iii): Let T be Gelfand completely compact, and let $\epsilon > 0$. Then there is a closed subspace X_ϵ of E with finite co-dimension such that $\|T|_{X_\epsilon}\|_{\mathrm{cb}} < \epsilon$. Due to the injectivity of F^*, there is $\tilde{T} \in \mathcal{CB}(E, F^*)$ such that $\tilde{T}|_{X_\epsilon} = T|_{X_\epsilon}$ and $\|\tilde{T}\|_{\mathrm{cb}} = \|T|_{X_\epsilon}\|_{\mathrm{cb}} < \epsilon$. Then $S := T - \tilde{T}$ is a finite rank operator such that $\|S - T\|_{\mathrm{cb}} < \epsilon$.

(i) $\Longrightarrow$ (iii): As $T^* \in \mathcal{CB}(F^{**}, E^*)$ is Gelfand completely compact, it is a cb-norm limit of finite rank operators—by the argument used for (ii) $\Longrightarrow$ (iii)—as is its adjoint $T^{**} \in \mathcal{CB}(E^{**}, F^{***})$. Hence, given $\epsilon > 0$, there is a finite rank operator $S \colon E^{**} \to F^{***}$ such that $\|S - T^{**}\|_{\mathrm{cb}} < \epsilon$. Let $Q \colon F^{***} \to F^*$ be the Dixmier projection. Then $S_0 := QS|_E$ is a finite rank operator from E to F^* such that $\|S_0 - T\|_{\mathrm{cb}} < \epsilon$. □

2 Completely almost periodic functionals

If $\mathfrak{A}$ is a Banach algebra, then its dual spaces becomes a Banach $\mathfrak{A}$-bimodule via

$$\langle x, a \cdot \phi \rangle := \langle xa, \phi \rangle \quad \text{and} \quad \langle x, \phi \cdot a \rangle := \langle ax, \phi \rangle \qquad (a, x \in \mathfrak{A},\ \phi \in \mathfrak{A}^*).$$

For $\phi \in \mathfrak{A}^*$, define $L_\phi, R_\phi \colon \mathfrak{A} \to \mathfrak{A}^*$ via

$$L_\phi a := \phi \cdot a \quad \text{and} \quad R_\phi a := a \cdot \phi \qquad (a \in \mathfrak{A}).$$

A functional $\phi \in \mathfrak{A}^*$ is commonly called *almost periodic* if L_ϕ and R_ϕ are compact operators. (As $L_\phi = R_\phi^*|_{\mathfrak{A}}$ and $R_\phi = L_\phi^*|_{\mathfrak{A}}$, it is sufficient to require that only one of L_ϕ and R_ϕ be compact.) We denote the space of all almost periodic functionals in $\mathfrak{A}^*$ by $\mathcal{AP}(\mathfrak{A})$.

An operator space that is also an algebra such that multiplication is completely contractive, is called a *completely contractive Banach algebra*.

We define:

Definition 2.1. Let $\mathfrak{A}$ be a completely contractive Banach algebra. We call $\phi \in \mathfrak{A}^*$ *completely almost periodic* if $L_\phi, R_\phi \in \mathcal{CK}(\mathfrak{A}, \mathfrak{A}^*)$ and denote the collection of completely almost periodic functionals in $\mathfrak{A}^*$ by $\mathcal{CAP}(\mathfrak{A})$.

Remarks. 1. Obviously, $\mathcal{CAP}(\mathfrak{A})$ is a closed linear subspace of $\mathfrak{A}^*$.

2. Since Schauder's theorem is no longer true for complete compactness, we do need the requirement that both L_ϕ and R_ϕ are completely compact.

We shall now look at a special class of completely contractive Banach algebras which arise naturally in abstract harmonic analysis.

Recall that a *Hopf–von Neumann algebra* is a pair (M, Γ) where M is a von Neumann algebra and $\Gamma \colon M \to M \bar{\otimes} M$ is a *co-multiplication*, i.e., a normal, unital, *-homomorphism satisfying

$$(\mathrm{id} \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \mathrm{id}) \circ \Gamma.$$

The co-multiplication induces a product $*$ on M_* via

$$\langle f * g, x \rangle := \langle f \otimes g, \Gamma x \rangle \qquad (f, g \in M_*,\ x \in M),$$

which turns M_* into a completely contractive Banach algebra.

Examples. 1. Let G be a locally compact group, and define a co-multiplication Γ : $L^\infty(G) \to L^\infty(G) \bar{\otimes} L^\infty(G)$—noting that $L^\infty(G) \bar{\otimes} L^\infty(G) \cong L^\infty(G \times G)$—by letting

$$(\Gamma \phi)(x, y) := \phi(xy) \qquad (x, y \in G,\ \phi \in L^\infty(G)).$$

Then $L^\infty(G)_* = L^1(G)$ with the product induced by Γ is just the usual convolution algebra $L^1(G)$. Since the operator space structure on $L^1(G)$ is maximal, $\mathcal{CAP}(L^1(G))$ equals $\mathcal{AP}(L^1(G))$; it consists precisely of the almost periodic continuous functions on G in the classical sense (see [Bur], for instance) and thus, in particular, is a C^*-subalgebra of $L^\infty(G)$.

2. Let again G be a locally compact group, let $\mathrm{VN}(G)$ be its group von Neumann algebra, i.e., the von Neumann algebra generated by $\lambda(G)$, where λ is the left regular representation of G on $L^2(G)$, and let $\Gamma \colon \mathrm{VN}(G) \to \mathrm{VN}(G) \bar{\otimes} \mathrm{VN}(G)$ be given by

$$\Gamma \lambda(x) = \lambda(x) \otimes \lambda(x) \qquad (x \in G).$$

Then $\mathrm{VN}(G)_*$ is Eymard's Fourier algebra $A(G)$ ([Eym]), and the product induced by Γ is pointwise multiplication. The space $\mathcal{AP}(A(G))$—often denoted by $\mathcal{AP}(\hat{G})$—was first considered in [D–R] and further studied in [Gra 1] and [Lau], for instance.

Except in a few special cases, e.g., if G is abelian or discrete and amenable (as a consequence of [Gra 1, Theorem 12] and [Gra 2, Proposition 2]), is unknown whether or not $\mathcal{AP}(\hat{G})$ is a C^*-subalgebra of $\mathrm{VN}(G)$.

The picture changes once we replace almost periodicity with complete almost periodicity:

Theorem 2.2. *Let (M, Γ) be a Hopf–von Neumann algebra such that M is injective. Then $\mathcal{CAP}(M_*)$ is a C^*-subalgebra of M.*

Proof. Let $x \in M$. Then, by Proposition 1.4, L_x is completely compact if and only if it is a cb-norm limit of finite rank operators. In view of the completely isometric identifications ([E–R, Corollary 7.1.5] and [E–R, Theorem 7.2.4])

$$M \bar{\otimes} M \cong (M_* \hat{\otimes} M_*)^* \cong \mathcal{CB}(M_*, M)$$

and of [E–R, Proposition 8.1.2], this means that L_x is completely compact if and only if $\Gamma x \in M \check{\otimes} M$. A similar assertion holds for R_x.

All in all, we have that

$$\mathcal{CAP}(M_*) = \{x \in M : \Gamma x \in M \check{\otimes} M\}.$$

Since $M \check{\otimes} M$ is a C^*-subalgebra of $M \bar{\otimes} M$, and since Γ is a *-homomorphism, this proves the claim. $\square$

If G is an amenable or connected locally compact group, then $\mathrm{VN}(G)$ is well known to be injective. Writing $\mathcal{CAP}(\hat{G})$ for $\mathcal{CAP}(A(G))$, we thus have:

Corollary 2.3. *Let G be an amenable or connected locally compact group. Then $\mathcal{CAP}(\hat{G})$ is a C^*-subalgebra of $\mathrm{VN}(G)$.*

Remarks.

1. If G is abelian—or, more generally, finite-by-abelian—, then the canonical operator space structure on $A(G)$ is equivalent to $\max A(G)$, so that $\mathcal{CAP}(\hat{G}) = \mathcal{AP}(\hat{G})$.

2. Suppose that G is discrete and abelian. Then $\mathcal{AP}(\hat{G})$ is all of $C^*_r(G)$, the (reduced) group C^*-algebra of G: this follows from [Gra 1, Theorem 12] and [Gra 2, Proposition 2(b)]. Since $\Gamma C^*_r(G) \subset C^*_r(G) \check{\otimes} C^*_r(G) \subset \mathrm{VN}(G) \check{\otimes} \mathrm{VN}(G)$, an inspection of the proof of Theorem 2.2 shows that therefore $\mathcal{AP}(\hat{G}) \subset \mathcal{CAP}(\hat{G})$ and thus, trivially, $\mathcal{AP}(\hat{G}) = \mathcal{CAP}(\hat{G})$.

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