HOMOMORPHISMS OF QUANTUM GROUPS

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Abstract. We introduce some equivalent notions of homomorphisms between quantum groups that behave well with respect to duality of quantum groups. Our equivalent definitions are based on bicharacters, coactions, and universal quantum groups, respectively.

1. Introduction

Throughout this article, a *morphism* between two C^*-algebras A and B is a non-degenerate *-homomorphism from A to the multiplier algebra $\mathcal{M}(B)$ or, equivalently, a strictly continuous unital *-homomorphism $\mathcal{M}(A) \to \mathcal{M}(B)$. Thus C^*-algebras with the above morphisms form a category. All tensor products in the following are spatial tensor products.

Let (C, Δ_C) and (A, Δ_A) be two locally compact quantum groups. A *strong quantum group homomorphism* from (C, Δ_C) to (A, Δ_A) is a morphism $\varphi\colon C \to A$ that intertwines the comultiplications, that is, the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\;f\;} & A \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_A} \\ C \otimes C & \xrightarrow[f\otimes f]{} & A \otimes A. \end{array}$$

It is easy to see that a group homomorphism $f\colon G \to H$ induces strong quantum group homomorphisms from $\mathrm{C}_0(H)$ to $\mathrm{C}_0(G)$ and from $\mathrm{C}^*(G)$ to $\mathrm{C}^*(H)$. But the latter does not always descend to the reduced group C^*-algebras. For instance, the constant map from G to the trivial group $\{1\}$ induces a strong quantum group homomorphism $\mathrm{C}^*_\mathrm{r}(G) \to \mathrm{C}^*_\mathrm{r}(\{1\}) = \mathbb{C}$ if and only if G is amenable.

Thus strong quantum group homomorphisms are not compatible with duality, unless we use full duals everywhere: a strong quantum homomorphism from C to A need not induce a strong quantum group homomorphism from $\hat{A}$ to $\hat{C}$.

We will introduce a less restrictive notion of homomorphism for quantum groups for which duality is a contravariant functor.

The definition of quantum group that we adopt here is the one by Piotr Sołtan and the third author based on modular multiplicative unitaries (see [5]). Multiplicative unitaries were introduced by Baaj and Skandalis in [1], and manageable multiplicative unitaries were introduced in [8]. We follow the conventions of [5] throughout. In particular, the reduced bicharacter of a locally compact quantum group is a unitary multiplier of $\hat{C} \otimes C$, not of $C \otimes \hat{C}$ as in [3].

We write $\mathcal{UM}(C)$ for the group of unitary multipliers of a C^*-algebra C, and $\mathcal{U}(\mathcal{H})$ for the group of unitary operators on the Hilbert space $\mathcal{H}$.

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The locally compact quantum groups considered in [5] are constructed from modular multiplicative unitaries. Hence they are concrete C^*-algebras, represented on some Hilbert space by definition. This representation is not canonical because several non-equivalent multiplicative unitaries may give the same locally compact quantum group, that is, isomorphic pairs (C,Δ_C). Therefore, we distinguish between elements of C^*-algebras such as $\hat{C}\otimes C$ and the Hilbert space operators they generate in the given representation of $\hat{C}\otimes C$. For a unitary multiplier U of $\hat{C}\otimes C$, we write $\mathbb{U}$ for U considered as an operator on the Hilbert space $\mathcal{H}_C\otimes\mathcal{H}_C$, where $\hat{C}, C\subseteq\mathbb{B}(\mathcal{H}_C)$. In particular, the reduced bicharacter $\mathrm{W}^C\in\mathcal{UM}(\hat{C}\otimes C)$ is represented by the modular multiplicative unitary $\mathbb{W}^C\in\mathcal{U}(\mathcal{H}_C\otimes\mathcal{H}_C)$ that gives rise to the quantum group (C,Δ_C).

A strong quantum group homomorphism $f\colon C\to A$ yields a unitary multiplier

$$V_f := (\mathrm{id}_{\hat{C}}\otimes f)(\mathrm{W}^C)\in\mathcal{UM}(\hat{C}\otimes A).$$

Since the elements of the form $(\omega\otimes\mathrm{id}_C)(\mathrm{W}^C)$ for linear functionals ω on $\hat{C}$ are dense in C and $f\big((\omega\otimes\mathrm{id}_C)(\mathrm{W}^C)\big) = (\omega\otimes\mathrm{id}_A)(V_f)$, the unitary V_f determines f uniquely. Furthermore, if f admits a dual quantum group homomorphism $\hat{f}\colon\hat{A}\to\hat{C}$, then these two are related by

$$(\hat{f}\otimes\mathrm{id}_A)(\mathrm{W}^A) = V_f = (\mathrm{id}_{\hat{C}}\otimes f)(\mathrm{W}^C). \tag{1.1}$$

The unitary operator $\mathbb{V}=\mathbb{V}_f$ on $\mathcal{H}_C\otimes\mathcal{H}_A$ satisfies two pentagon equations involving the multiplicative unitaries $\mathbb{W}^C$ and $\mathbb{W}^A$ of C and A:

$$\mathbb{V}_{23}\mathbb{W}^C_{12} = \mathbb{W}^C_{12}\mathbb{V}_{13}\mathbb{V}_{23}. \tag{1.2}$$

$$\mathbb{W}^A_{23}\mathbb{V}_{12} = \mathbb{V}_{12}\mathbb{V}_{13}\mathbb{W}^A_{23}. \tag{1.3}$$

A unitary multiplier of $\hat{C}\otimes A$ satisfying these two conditions is called a *linking unitary* between C and A. We propose these linking unitaries as the correct notion of quantum group homomorphism. Linking unitaries are special types of adapted unitaries introduced by the third author in [8].

Linking unitaries form a category, as expected. The composition of a linking unitary $\mathrm{V}^{C\to A}\in\mathcal{UM}(\hat{C}\otimes A)$ with the linking unitary V_f of a strong quantum group homomorphism $f\colon A\to B$ yields $(\mathrm{id}_{\hat{C}}\otimes f)(\mathrm{V}^{C\to A})$. More generally, the composition of two linking unitaries $\mathrm{V}^{C\to A}\in\mathcal{UM}(\hat{C}\otimes A)$ and $\mathrm{V}^{A\to B}\in\mathcal{UM}(\hat{A}\otimes B)$ is the unique $\mathrm{V}^{C\to B}\in\mathcal{UM}(\hat{C}\otimes B)$ that satisfies

$$\mathbb{V}^{A\to B}_{23}\mathbb{V}^{C\to A}_{12} = \mathbb{V}^{C\to A}_{12}\mathbb{V}^{C\to B}_{13}\mathbb{V}^{A\to B}_{23}.$$

The identity linking unitary for a quantum group C is its multiplicative unitary W^C. The unit element of $\mathcal{UM}(\hat{C}\otimes\mathbb{C})$ is also a linking unitary, the corresponding quantum group homomorphism from C to the trivial quantum group is the counit of $\hat{C}$.

As expected, a linking unitary $V\in\mathcal{UM}(\hat{C}\otimes A)$ induces linking unitaries

$$V^{\mathrm{op}}\in\mathcal{UM}(\widehat{C^{\mathrm{op}}}\otimes A^{\mathrm{op}}),\qquad V^{\mathrm{cop}}\in\mathcal{UM}(\widehat{C^{\mathrm{cop}}}\otimes A^{\mathrm{cop}}),\qquad \hat{V}\in\mathcal{UM}(\hat{\hat{A}}\otimes\hat{C}),$$

that is, taking opposite and coopposite quantum groups yields covariant functors on our new quantum group category, and taking duals yields a contravariant functor. With the usual identifications, $\hat{\hat{A}}\cong A$ as quantum groups, and $A = A^{\mathrm{op}} = A^{\mathrm{cop}}$ as vector spaces, we have

$$V^{\mathrm{op}} = V,\qquad V^{\mathrm{cop}} = V,\qquad \hat{V} = \sigma(V^*),$$

where $\sigma\colon\hat{C}\otimes A\to A\otimes\hat{C}$ is the flip automorphism.

The pentagon equations (1.2) and (1.3) involve the modular multiplicative unitaries that give rise to C and A. We may, however, rewrite (1.2) and (1.3) using the comultiplications on $\hat{C}$ and A: they mean that V is a bicharacter of $\hat{C} \otimes A$. Thus we interpret bicharacters of $\hat{C} \otimes A$ as quantum group homomorphisms from C to A.

We may also replace linking unitaries by certain coactions. A *right quantum group homomorphism* from C to A is a morphism $\Delta_R \colon C \to C \otimes A$ such that the following two diagrams commute:

$$\begin{CD} C @>{\Delta_R}>> C \otimes A \\ @V{\Delta_C}VV @VV{\Delta_C \otimes \mathrm{id}_A}V \\ C \otimes C @>>{\mathrm{id}_C \otimes \Delta_R}> C \otimes C \otimes A, \end{CD} \qquad \begin{CD} C @>{\Delta_R}>> C \otimes A \\ @V{\Delta_R}VV @VV{\mathrm{id}_C \otimes \Delta_A}V \\ C \otimes A @>>{\Delta_R \otimes \mathrm{id}_A}> C \otimes A \otimes A. \end{CD}$$

A strong quantum group homomorphism $f \colon C \to A$ gives a right quantum group homomorphism $(\mathrm{id}_C \otimes f) \circ \Delta_C$.

There is a bijective correspondence between right quantum group homomorphisms and linking unitaries. Given a linking unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$, we get a right quantum group homomorphism $\Delta_R \colon C \to C \otimes A$ by

$$\Delta_R(x) := \mathbb{V}(x \otimes 1)\mathbb{V}^* \qquad \text{for all } x \in C.$$

Conversely, given Δ_R, the corresponding linking unitary is the unique unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ with

$$(\mathrm{id}_{\hat{C}} \otimes \Delta_R)(\mathrm{W}) = \mathrm{W}_{12} V_{13}.$$

Left quantum group homomorphisms are defined similarly. Since left and right quantum group homomorphisms are exchanged by taking coopposite quantum groups, they are essentially equivalent.

Previously, it was suggested to define quantum group homomorphisms by passing to universal quantum groups [7]. The universal quantum group $(C^{\mathrm{u}}, \Delta^{\mathrm{u}})$ of a quantum group (C, Δ) is defined by the universal property that morphisms $C^{\mathrm{u}} \to D$ correspond to unitary left corepresentations of the dual quantum group $\hat{C}$ on D, where a unitary left corepresentation of $\hat{C}$ is a unitary $U \in \mathcal{UM}(\hat{C} \otimes D)$ that satisfies $(\Delta_{\hat{C}} \otimes \mathrm{id}_D)(U) = U_{23} U_{13}$. This universal property immediately implies that linking unitaries between C and A correspond bijectively to strong quantum group homomorphisms $C^{\mathrm{u}} \to A$.

Moreover, we show that any strong quantum group homomorphisms $C^{\mathrm{u}} \to A$ lifts uniquely to a strong quantum group homomorphisms $C^{\mathrm{u}} \to A^{\mathrm{u}}$. Thus our new quantum group morphisms $C \to A$ are in bijection with strong quantum group homomorphisms $C^{\mathrm{u}} \to A^{\mathrm{u}}$. We also show that the composition of quantum group morphisms described above corresponds to the usual composition of strong quantum group homomorphisms on the level of universal quantum groups. The duality on the level of linking unitaries implies that duality is an anti-equivalence on the level of universal quantum groups. The dual $\hat{f} \colon \hat{A}^{\mathrm{u}} \to \hat{C}^{\mathrm{u}}$ of a strong quantum group homomorphisms $f \colon C^{\mathrm{u}} \to A^{\mathrm{u}}$ is characterised by a variant of (1.1).

Finally, we mention two more technical results that are known to hold in the presence of Haar measures and which we establish in the framework of [5], without using Haar measures. First, if $a \in \mathcal{M}(\mathcal{H}_A)$ satisfies $\mathbb{W}(a \otimes 1) = (1 \otimes a)\mathbb{W}$, then already $a \in \mathbb{C} \cdot 1$. This implies that a multiplier of A is constant if it is left or right invariant. This single idea is used in almost all our arguments. Secondly, we lift the reduced bicharacter in $\mathcal{UM}(\hat{C} \otimes C)$ to a universal bicharacter in $\mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes C^{\mathrm{u}})$.

2. Invariants are constant

Let (C, Δ_C) and (A, Δ_A) be two quantum groups in the sense of [5]. That is, they are obtained from modular multiplicative unitaries $\mathbb{W}^C \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_C)$ and $\mathbb{W}^A \in \mathcal{U}(\mathcal{H}_A \otimes \mathcal{H}_A)$ for certain Hilbert spaces $\mathcal{H}_C$ and $\mathcal{H}_A$. Let $\mathrm{W}^C \in \mathcal{UM}(\hat{C} \otimes C)$ and

$\mathrm{W}^A \in \mathcal{UM}(\hat{A}\otimes A)$ be their reduced bicharacters. While the same quantum groups may be obtained from different modular multiplicative unitaries, these bicharacters are uniquely determined. Recall the following properties:

$$\mathbb{W}^C_{23}\mathbb{W}^C_{12} = \mathbb{W}^C_{12}\mathbb{W}^C_{13}\mathbb{W}^C_{23} \qquad \text{in } \mathcal{U}(\mathcal{H}_C\otimes\mathcal{H}_C\otimes\mathcal{H}_C), \tag{2.1}$$

$$\Delta_C(x) = \mathbb{W}^C(x\otimes 1)(\mathbb{W}^C)^* \qquad \text{in } \mathbb{B}(\mathcal{H}_C\otimes\mathcal{H}_C) \text{ for all } x\in C, \tag{2.2}$$

$$\Delta_{\hat{C}}(y) = \Sigma(\mathbb{W}^C)^*(1\otimes y)\mathbb{W}^C\Sigma \qquad \text{in } \mathbb{B}(\mathcal{H}_C\otimes\mathcal{H}_C) \text{ for all } y\in \hat{C}, \tag{2.3}$$

$$(\mathrm{id}_{\hat{C}}\otimes\Delta_C)(\mathrm{W}^C) = \mathrm{W}^C_{12}\mathrm{W}^C_{13} \qquad \text{in } \hat{C}\otimes C\otimes C, \tag{2.4}$$

$$(\Delta_{\hat{C}}\otimes\mathrm{id}_C)(\mathrm{W}^C) = \mathrm{W}^C_{23}\mathrm{W}^C_{13} \qquad \text{in } \hat{C}\otimes\hat{C}\otimes C. \tag{2.5}$$

Similar equations hold for $\mathbb{W}^A$, of course.

Theorem 2.6. *Let $\mathcal{H}$ be a Hilbert space and let $\mathbb{W}\in\mathbb{B}(\mathcal{H}\otimes\mathcal{H})$ be a modular multiplicative unitary. If $a,b\in\mathbb{B}(\mathcal{H})$ satisfy $\mathbb{W}(a\otimes 1) = (1\otimes b)\mathbb{W}$, then $a=b=\lambda 1$ for some $\lambda\in\mathbb{C}$. More generally, if $a,b\in\mathcal{M}(\mathbb{K}(\mathcal{H})\otimes D)$ for some C^*-algebra D satisfy $\mathbb{W}(a\otimes 1) = (1\otimes b)\mathbb{W}$, then $a=b\in\mathbb{C}\cdot 1_{\mathcal{H}}\otimes\mathcal{M}(D)$.*

Proof. Define the operators $\hat{Q}$, Q, and $\widetilde{\mathbb{W}}$ as in [4, Definition 2.1]. First we prove the assertion under the additional assumption $b^*\mathcal{D}(Q)\subseteq\mathcal{D}(Q)$. Our assumption $\mathbb{W}(a\otimes 1) = (1\otimes b)\mathbb{W}$ means

$$(x\otimes y \mid \mathbb{W} \mid az\otimes u) = (x\otimes b^*y \mid \mathbb{W} \mid z\otimes u)$$

for all $x,z\in\mathcal{H}$, $y\in\mathcal{D}(Q)$ and $u\in\mathcal{D}(Q^{-1})$. The modularity condition for $\mathbb{W}$ yields

$$\begin{aligned}\big(\overline{az}\otimes Qy \mathrel{\big|} \widetilde{\mathbb{W}} \mathrel{\big|} \overline{x}\otimes Q^{-1}u\big) &= \big(x\otimes y \mathrel{\big|} \mathbb{W} \mathrel{\big|} az\otimes u\big)\\ &= \big(x\otimes b^*y \mathrel{\big|} \mathbb{W} \mathrel{\big|} z\otimes u\big) = \big(\overline{z}\otimes Qb^*y \mathrel{\big|} \widetilde{\mathbb{W}} \mathrel{\big|} \overline{x}\otimes Q^{-1}u\big).\end{aligned}$$

In this formula, $\widetilde{\mathbb{W}}(\overline{x}\otimes Q^{-1}u)$ runs through a dense subset of $\overline{\mathcal{H}}\otimes\mathcal{H}$. Therefore, we may replace $\widetilde{\mathbb{W}}(\overline{x}\otimes Q^{-1}u)$ by $\overline{x}\otimes Q^{-1}u$ and get

$$(\overline{az}\otimes Qy \mid \overline{x}\otimes Q^{-1}u) = (\overline{z}\otimes Qb^*y \mid \overline{x}\otimes Q^{-1}u),$$

that is,

$$(x\mid az)\cdot(Qy\mid Q^{-1}u) = (x\mid z)\cdot(Qb^*y\mid Q^{-1}u).$$

Since this holds for all $x,z\in\mathcal{H}$, $y\in\mathcal{D}(Q)$ and $u\in\mathcal{D}(Q^{-1})$, we get $a=\lambda\cdot 1_{\mathcal{H}}$ for some $\lambda\in\mathbb{C}$ and $b=\lambda\cdot 1_{\mathcal{H}}$ for the same λ.

To prove the statement in full generality, we first regularise a and b. For $a\in\mathbb{B}(\mathcal{H})$ and $n\in\mathbb{N}$, we define

$$\widehat{R}_n(a) := \int_{-\infty}^{+\infty}\widehat{Q}^{-\mathrm{i}t}a\widehat{Q}^{\mathrm{i}t}\delta_n(t)\,\mathrm{d}t \quad\text{and}\quad R_n(a) := \int_{-\infty}^{+\infty}Q^{-\mathrm{i}t}aQ^{\mathrm{i}t}\delta_n(t)\,\mathrm{d}t,$$

where

$$\delta_n(t) := \sqrt{\frac{n}{2\pi}}\exp\left(-\frac{nt^2}{2}\right)$$

is a δ-like sequence of Gaussian functions. Since

$$\mathbb{W}^*(\widehat{Q}\otimes Q)\mathbb{W} = \widehat{Q}\otimes Q,$$

our condition $\mathbb{W}(a\otimes 1) = (1\otimes b)\mathbb{W}$ implies

$$\mathbb{W}(\widehat{R}_n(a)\otimes 1) = (1\otimes R_n(b))\mathbb{W}.$$

We will show below that

$$R_n(b)^*\mathcal{D}(Q)\subseteq\mathcal{D}(Q). \tag{2.7}$$

The first part of the proof now yields $\widehat{R}_n(a) = R_n(b) = \lambda_n 1$ for all $n \in \mathbb{N}$. If $n \to \infty$, then $\widehat{R}_n(a)$ and $R_n(b)$ converge weakly towards a and b, respectively. Hence we get $a = b = \lambda 1$ for some $\lambda \in \mathbb{C}$ in full generality.

It only remains to establish (2.7). Let $x, y \in \mathcal{D}(Q)$. Then the function

$$f_{x,y}(z) := (Q^{\mathrm{i}(\overline{z}-\mathrm{i})}x \mid b^* \mid Q^{\mathrm{i}z}y)$$

is well-defined, bounded, and continuous in the strip $\Sigma := \{z \in \mathbb{C} \colon -1 \leq \operatorname{Im} z \leq 0\}$ and holomorphic in the interior of Σ. In particular, for $t \in \mathbb{R}$:

$$f_{x,y}(t) = (Qx \mid Q^{-\mathrm{i}t}b^*Q^{\mathrm{i}t} \mid y), \qquad f_{x,y}(t-\mathrm{i}) = (x \mid Q^{-\mathrm{i}t}b^*Q^{\mathrm{i}t} \mid Qy). \tag{2.8}$$

By Cauchy's Theorem, the integrals of $f_{x,y}(z)\delta_n(z)$ along the lines $\mathbb{R} + \mathrm{i}s$ for $0 \leq 1 \leq s$ do not depend on s. For $s = 0$ and $s = 1$, (2.8) shows that the integrals are $(Qx \mid R_n(b)^* \mid y)$ and

$$\left(x \,\middle|\, \int_{-\infty}^{+\infty} Q^{-\mathrm{i}t}b^*Q^{\mathrm{i}t}\delta_n(t-\mathrm{i})\,\mathrm{d}t \,\middle|\, Qy\right),$$

respectively. Their equality shows that $(Qx \mid R_n(b)^*y)$ depends continuously on x. This yields $R_n(b)^*y \in \mathcal{D}(Q^*) = \mathcal{D}(Q)$, that is, (2.7).

Finally, if $a, b \in \mathcal{M}(\mathbb{K}(\mathcal{H}) \otimes D)$ satisfy $\mathbb{W}(a \otimes 1) = (1 \otimes b)\mathbb{W}$ as elements of $\mathcal{M}(\mathbb{K}(\mathcal{H} \otimes \mathcal{H}) \otimes D)$, then the first part of the theorem applies to the slices $(\mathrm{id} \otimes \mu)(a)$ and $(\mathrm{id} \otimes \mu)(b)$ for all $\mu \in D'$. Thus $(\mathrm{id} \otimes \mu)(a) = (\mathrm{id} \otimes \mu)(b) = \lambda_\mu \cdot 1$ for all $\mu \in D'$. This implies that $a = b \in \mathbb{C} \cdot 1 \otimes \mathcal{M}(D)$. ☐

Corollary 2.9. *Let (C, Δ_C) be a quantum group constructed from a manageable (or, more generally, from a modular) multiplicative unitary $\mathbb{W} \in \mathbb{B}(\mathcal{H} \otimes \mathcal{H})$. If $c \in \mathcal{M}(C)$, then $\Delta_C(c) \in \mathcal{M}(C \otimes 1)$ or $\Delta_C(c) \in \mathcal{M}(1 \otimes C)$ if and only if $c \in \mathbb{C} \cdot 1$.*

More generally, if D is a C^*-algebra and $c \in \mathcal{M}(C \otimes D)$, then $(\Delta_C \otimes \mathrm{id}_D)(c) \in \mathcal{M}(C \otimes 1 \otimes D)$ or $(\Delta_C \otimes \mathrm{id}_D)(c) \in \mathcal{M}(1 \otimes C \otimes D)$ if and only if $c \in \mathbb{C} \cdot 1 \otimes \mathcal{M}(D)$.

Proof. Using (2.2), we rewrite the equation $\Delta_C(c) = 1 \otimes c'$ for $c, c' \in \mathcal{M}(C \otimes D)$ as $\mathbb{W}^C(c \otimes 1) = (1 \otimes c')\mathbb{W}^C$. Now Theorem 2.6 yields $c \in \mathbb{C} \cdot 1 \otimes \mathcal{M}(D)$. If $\Delta_C(c) = c' \otimes 1$ instead, then we apply the unitary antipodes. With $a := (R_C \otimes \mathrm{id}_D)(c)$ and $a' := (R_C \otimes \mathrm{id}_D)(c')$, we get $\Delta_C(a) = 1 \otimes a'$. The argument above shows $a \in \mathbb{C} \cdot 1 \otimes \mathcal{M}(D)$ and hence $c \in \mathbb{C} \cdot 1 \otimes \mathcal{M}(D)$. ☐

3. Linking unitaries

Definition 3.1. A unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ is called a *linking unitary* from C to A if its image $\mathbb{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A)$ satisfies the following two conditions:

$$\mathbb{V}_{23}\mathbb{W}^C_{12} = \mathbb{W}^C_{12}\mathbb{V}_{13}\mathbb{V}_{23} \qquad \text{in } \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_C \otimes \mathcal{H}_A), \tag{3.2}$$

$$\mathbb{W}^A_{23}\mathbb{V}_{12} = \mathbb{V}_{12}\mathbb{V}_{13}\mathbb{W}^A_{23} \qquad \text{in } \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A \otimes \mathcal{H}_A). \tag{3.3}$$

Example 3.4. Take V to be $\mathrm{W}^C \in \mathcal{UM}(\hat{C} \otimes C)$. Then (3.2) and (3.3) are the pentagon equation (2.1). Thus $\mathrm{W}^C \in \mathcal{UM}(\hat{C} \otimes C)$ is a linking unitary. We will see that it is the *identity* on C.

Lemma 3.5. *A unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ is a linking unitary if and only if it is a bicharacter, that is,*

$$(\Delta_{\hat{C}} \otimes \mathrm{id}_A)V = V_{23}V_{13} \qquad \textit{in } \mathcal{UM}(\hat{C} \otimes \hat{C} \otimes A), \tag{3.6}$$

$$(\mathrm{id}_{\hat{C}} \otimes \Delta_A)V = V_{12}V_{13} \qquad \textit{in } \mathcal{UM}(\hat{C} \otimes A \otimes A). \tag{3.7}$$

Proof. The representation of $\hat{C} \otimes \hat{C} \otimes A$ on $\mathcal{H}_C \otimes \mathcal{H}_C \otimes \mathcal{H}_A$ is faithful, so that (3.6) is equivalent to an equation of unitary operators on $\mathcal{H}_C \otimes \mathcal{H}_C \otimes \mathcal{H}_A$. Using (2.3), we rewrite (3.6) as

$$\Sigma_{12}(\mathbb{W}^C_{12})^*\mathbb{V}_{23}\mathbb{W}^C_{12}\Sigma_{12} = \mathbb{V}_{23}\mathbb{V}_{13}.$$

This is equivalent to (3.2). A similar argument shows that (3.7) is equivalent to (3.3). ☐

Example 3.8. Any strong quantum group homomorphism $f\colon C \to A$ yields a linking unitary as $V_f := (\mathrm{id}_{\hat{C}} \otimes f)\mathrm{W}^C$. This follows from Lemma 3.5 and (2.4) and (2.5).

Remark 3.9. Definition 3.1 has the merit of being formulated entirely in the language of multiplicative unitaries and pentagon equations. But the same quantum group may be generated by different multiplicative unitaries. Since W^C only depends on (C, Δ_C) by [5], the characterisation of linking unitaries in Lemma 3.5 shows that they only depend on (A, Δ_A) and (C, Δ_C).

The following result generalises [5, Lemma 40] and is proved by the same idea.

Proposition 3.10. *Let $V \in \mathcal{UM}(\hat{C} \otimes A)$ be a linking unitary. Let R and τ denote the unitary antipodes and scaling groups of quantum groups. Then*

$$(R_{\hat{C}} \otimes R_A)(V) = V, \tag{3.11}$$

$$(\tau_t^{\hat{C}} \otimes \tau_t^A)(V) = V \qquad \text{for all } t \in \mathbb{R}. \tag{3.12}$$

Proof. Let $\varphi \in \hat{C}_*$ and $\psi \in A_*$ be entire analytic for $(\tau_t^{\hat{C}})$ and (τ_t^A), respectively. Let $\varphi_t := \varphi \circ \tau_t^{\hat{C}}$ and $\psi_t := \psi \circ \tau_t^A$ for all $t \in \mathbb{R}$. Analytic continuation yields

$$\varphi_{z+z'} = \varphi_z \circ \tau_{z'}^{\hat{C}}, \qquad \text{and} \qquad \varphi_{z+z'} = \psi_z \circ \tau_{z'}^A \qquad \text{for all } z, z' \in \mathbb{C}.$$

Polar decomposition of the antipodes $\kappa_{\hat{C}}$ and κ_A ([8, Theorem 1.5]) shows that

$$\varphi_z \circ \kappa_{\hat{C}} = \varphi_{z+\mathrm{i}/2} \circ R_{\hat{C}}, \qquad \text{and} \qquad \psi_z \circ \kappa_A = \psi_{z+\mathrm{i}/2} \circ R_A.$$

Let $\bar{\kappa}_A$ be the closure of κ_A with respect to the strict topology on $\mathcal{M}(A)$. Then [8, Theorem 1.6(4)] yields

$$\bar{\kappa}_A(\omega \otimes \mathrm{id})V = (\omega \otimes \mathrm{id})(V^*)$$

for all $\omega \in \hat{C}_*$. Applying ψ_z to both sides and using that ω is arbitrary, we get

$$\bigl(\mathrm{id} \otimes \psi_{z+\mathrm{i}/2} \circ R_A\bigr)V = (\mathrm{id} \otimes \psi_z)(V^*).$$

Interchanging the roles of A and $\hat{C}$ and replacing V by $\Sigma V^*\Sigma$ and ψ by φ, we get

$$\bigl(\varphi_{z+\mathrm{i}/2} \circ R_{\hat{C}} \otimes \mathrm{id}\bigr)(V^*) = (\varphi_z \otimes \mathrm{id})V.$$

Both formulas together yield

$$\begin{aligned}(\varphi_{z+\mathrm{i}/2} \otimes \psi_{z+\mathrm{i}/2}) \circ (R_{\hat{C}} \otimes R_A)(V) &= (\varphi_{z+\mathrm{i}/2} \circ R_{\hat{C}} \otimes \psi_{z+\mathrm{i}/2} \circ R_A)(V) \\ = (\varphi_{z+\mathrm{i}/2} \circ R_{\hat{C}} \otimes \psi_z)(V^*) &= \psi_z(\varphi_{z+\mathrm{i}/2} \circ R_{\hat{C}} \otimes \mathrm{id})(V^*) = (\varphi_z \otimes \psi_z)(V).\end{aligned} \tag{3.13}$$

Inserting $\varphi \circ \kappa_{\hat{C}}$ and $\psi \circ \kappa_A$ into (3.13) instead of φ and ψ yields

$$(\varphi_{z+\mathrm{i}} \otimes \psi_{z+\mathrm{i}})(V) = (\varphi_{z+\mathrm{i}/2} \otimes \psi_{z+\mathrm{i}/2}) \circ (R_{\hat{C}} \otimes R_A)(V) = (\varphi_z \otimes \psi_z)(V).$$

This shows that $(\varphi_z \otimes \psi_z)(V)$ is a periodic function of period i. Being bounded as well, Liouville's Theorem shows that it is constant, that is,

$$(\varphi_z \otimes \psi_z)(V) = (\varphi \otimes \psi)(V) \tag{3.14}$$

for all $z \in \mathbb{C}$. Putting $z = -\mathrm{i}/2$ in (3.13) and using (3.14) yields

$$(\varphi \otimes \psi) \circ (R_{\hat{C}} \otimes R_A)(V) = (\varphi \otimes \psi)(V).$$

This proves $(R_{\hat{C}} \otimes R_A)(V) = V$. Finally, (3.14) also yields $(\tau_t^{\hat{C}} \otimes \tau_t^A)(V) = V$ for all $t \in \mathbb{R}$. □

We will interpret linking unitaries as arrows in a category of quantum groups. Before we discuss the category structure, that is, the composition of arrows, we treat some canonical functors on this category. Given any quantum group (C, Δ_C), we form closely related quantum groups $(C^{\text{op}}, \Delta_C^{\text{op}})$, $(C^{\text{cop}}, \Delta_C^{\text{cop}})$, and $(\hat{C}, \Delta_{\hat{C}})$ by taking the opposite multiplication, the opposite comultiplication, and the dual, respectively. These constructions should be functors in our new category, that is, they should act on linking unitaries. To make this explicit, recall that

$$\widehat{C^{\text{op}}} \cong \hat{C}^{\text{cop}}, \qquad \widehat{C^{\text{cop}}} \cong \hat{C}^{\text{op}}, \quad \text{and} \quad \hat{\hat{A}} \cong A$$

as quantum groups. As vector spaces, $C = C^{\text{op}} = C^{\text{cop}}$, and this extends to the multiplier algebras.

Thus we may interpret $V \in \mathcal{UM}(\hat{C} \otimes A)$ as a unitary multiplier of $\widehat{C^{\text{op}}} \otimes A^{\text{op}}$ or $\widehat{C^{\text{cop}}} \otimes A^{\text{cop}}$, and $\sigma(V^*)$ for the flip map $\sigma\colon \hat{C} \otimes A \to A \otimes \hat{C}$ as a unitary multiplier of $\hat{\hat{A}} \otimes \hat{C}$.

Proposition 3.15. *A linking unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ yields linking unitaries*

$$V^{\text{op}} := V \qquad \text{in } \mathcal{UM}(\widehat{C^{\text{op}}} \otimes A^{\text{op}}), \tag{3.16}$$

$$V^{\text{cop}} := V \qquad \text{in } \mathcal{UM}(\widehat{C^{\text{cop}}} \otimes A^{\text{cop}}), \tag{3.17}$$

$$\hat{V} := \sigma(V^*) \qquad \text{in } \mathcal{UM}(\hat{\hat{A}} \otimes \hat{C}). \tag{3.18}$$

Proof. Since $\widehat{C^{\text{op}}} \cong \hat{C}^{\text{cop}}$, the first two legs on the left hand side of (3.6) are exchanged for the opposites, and this has the same effect as changing the order of multiplication on the right hand side of (3.6). There is no change in (3.7) when we pass to opposite quantum groups. Hence V^{op} in (3.16) is a linking unitary.

Recall that $A^{\text{op}} \cong A^{\text{cop}}$ via the unitary antipode R_A. The corresponding isomorphism $C^{\text{op}} \cong C^{\text{cop}}$ induces isomorphism $\hat{C}^{\text{cop}} \cong \widehat{C^{\text{op}}}$ and $\hat{C}^{\text{op}} \cong \widehat{C^{\text{cop}}}$. These induced isomorphisms are given by the unitary antipode $R_{\hat{C}}$ because $(R_{\hat{C}} \otimes \text{id}_C)(\mathrm{W}^C) = (\text{id}_{\hat{C}} \otimes R_C)(\mathrm{W}^C)$. When we apply these isomorphisms for $\hat{C}$ and A to our linking unitary V^{op}, we get a linking unitary $V^{\text{cop}} := (R_{\hat{C}} \otimes R_A)(V^{\text{op}})$ for the coopposite quantum groups. Finally, Proposition 3.10 yields $V^{\text{cop}} = V$.

The following computation shows that $\hat{V} := \sigma(V^*)$ satisfies (3.6):

$$\begin{aligned}(\Delta_A \otimes \text{id}_{\hat{C}})(\sigma(V^*)) &= \sigma_{23}\sigma_{12}((\text{id}_{\hat{C}} \otimes \Delta_A)V^*) = \sigma_{23}\sigma_{12}(V_{13}^* V_{12}^*)\\ &= \sigma_{23}\sigma_{12}(V_{13}^*)\sigma_{13}\sigma_{23}(V_{12}^*) = \sigma_{23}(V_{23}^*) \cdot \sigma_{13}(V_{13}^*) = \hat{V}_{23}\hat{V}_{13}\end{aligned}$$

if V satisfies (3.7); here σ_{12} denotes the automorphism exchanging the first and second tensor factors. Similarly, $\hat{V}$ satisfies (3.7) if V satisfies (3.6). Thus $\hat{V}$ is a linking unitary from $\hat{A}$ to $\hat{C}$. □

Remark 3.19. The opposite, coopposite, and dual quantum groups are indeed quantum groups in our sense, that is, they may be obtained from a modular multiplicative unitary. This is well-known for the dual quantum group, which is obtained from the multiplicative unitary $\Sigma\mathbb{W}^*\Sigma \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ if $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ gives rise to (C, Δ_C) and $\Sigma \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ is the flip of the tensor factors.

The multiplicative unitary for the opposite quantum group acts on $\overline{\mathcal{H}} \otimes \overline{\mathcal{H}}$ for the complex-conjugate Hilbert $\overline{\mathcal{H}}$. An operator w on $\mathcal{H}$ induces a transpose operator w^{T} on $\overline{\mathcal{H}}$ by $w(\overline{\xi}) := \overline{w^*\xi}$ for all $\xi \in \mathcal{H}$. The operator $(\mathbb{W}^*)^{\mathsf{T}\otimes\mathsf{T}}$ is a multiplicative unitary on $\overline{\mathcal{H}} \otimes \overline{\mathcal{H}}$ that gives rise to the opposite quantum group.

Finally, the coopposite quantum group is isomorphic to the opposite quantum group, so that it is generated by the same modular multiplicative unitary.

Proposition 3.15 shows that the linking unitaries V^{op} and V^{cop} constructed out of a given linking unitary V are the same when viewed as elements of $\mathcal{UM}(\hat{C}\otimes A)$. However, since our identifications do not preserve the quantum group structure, these operators become different when we realise them concretely as unitary operators on a tensor product of two Hilbert spaces.

Now we define the composition for linking unitaries. Let (B,Δ_B) be another quantum group.

Definition 3.20. A unitary $\mathrm{V}^{C\to B}\in\mathcal{UM}(\hat{C}\otimes B)$ is called a *composition* of two linking unitaries $\mathrm{V}^{C\to A}\in\mathcal{UM}(\hat{C}\otimes A)$ and $\mathrm{V}^{A\to B}\in\mathcal{UM}(\hat{A}\otimes B)$ if its image $\mathbb{V}^{C\to B}$ in $\mathcal{U}(\mathcal{H}_C\otimes\mathcal{H}_B)$ satisfies

$$\mathbb{V}^{A\to B}_{23}\mathbb{V}^{C\to A}_{12}=\mathbb{V}^{C\to A}_{12}\mathbb{V}^{C\to B}_{13}\mathbb{V}^{A\to B}_{23}\qquad\text{in }\mathcal{U}(\mathcal{H}_C\otimes\mathcal{H}_A\otimes\mathcal{H}_B).$$

We also briefly write $\mathrm{V}^{C\to B}=\mathrm{V}^{A\to B}*\mathrm{V}^{C\to A}$.

Since we may rewrite the condition in Definition 3.20 as

$$\mathbb{V}^{C\to B}_{13}=(\mathbb{V}^{C\to A}_{12})^*\mathbb{V}^{A\to B}_{23}\mathbb{V}^{C\to A}_{12}(\mathbb{V}^{A\to B}_{23})^*,\tag{3.21}$$

it is clear that there is at most one composition for two given linking unitaries. Existence, however, is less clear. This is established in the following lemma:

Lemma 3.22. *For any two linking unitaries $\mathrm{V}^{C\to A}$ and $\mathrm{V}^{A\to B}$, there is a unique composition $\mathrm{V}^{C\to B}$, and it is a linking unitary from C to B.*

Proof. Define $\tilde{V}:=(\mathbb{V}^{C\to A}_{12})^*\mathbb{V}^{A\to B}_{23}\mathbb{V}^{C\to A}_{12}(\mathbb{V}^{A\to B}_{23})^*\in\mathcal{UM}(\hat{C}\otimes\mathbb{K}(\mathcal{H}_B)\otimes B)$. We are going to use Theorem 2.6 to show that $\tilde{V}\in\mathcal{UM}(\hat{C}\otimes 1\otimes B)$.

$$\begin{aligned}
&\mathbb{W}^A_{23}\tilde{V}_{124}(\mathbb{W}^A_{23})^*\\
&=\mathbb{W}^A_{23}(\mathbb{V}^{C\to A}_{12})^*\mathbb{V}^{A\to B}_{24}\mathbb{V}^{C\to A}_{12}(\mathbb{V}^{A\to B}_{24})^*(\mathbb{W}^A_{23})^*\\
&=(\mathbb{V}^{C\to A}_{13})^*(\mathbb{V}^{C\to A}_{12})^*\mathbb{W}^A_{23}\mathbb{V}^{A\to B}_{24}\mathbb{V}^{C\to A}_{12}(\mathbb{V}^{A\to B}_{24})^*(\mathbb{W}^A_{23})^*\\
&=(\mathbb{V}^{C\to A}_{13})^*(\mathbb{V}^{C\to A}_{12})^*\mathbb{V}^{A\to B}_{34}\mathbb{W}^A_{23}(\mathbb{V}^{A\to B}_{34})^*\mathbb{V}^{C\to A}_{12}\mathbb{V}^{A\to B}_{34}(\mathbb{W}^A_{23})^*(\mathbb{V}^{A\to B}_{34})^*\\
&=(\mathbb{V}^{C\to A}_{13})^*(\mathbb{V}^{C\to A}_{12})^*\mathbb{V}^{A\to B}_{34}\mathbb{W}^A_{23}\mathbb{V}^{C\to A}_{12}(\mathbb{W}^A_{23})^*(\mathbb{V}^{A\to B}_{34})^*\\
&=(\mathbb{V}^{C\to A}_{13})^*(\mathbb{V}^{C\to A}_{12})^*\mathbb{V}^{A\to B}_{34}\mathbb{V}^{C\to A}_{12}\mathbb{V}^{C\to A}_{13}(\mathbb{V}^{A\to B}_{34})^*\\
&=(\mathbb{V}^{C\to A}_{13})^*\mathbb{V}^{A\to B}_{34}\mathbb{V}^{C\to A}_{13}(\mathbb{V}^{A\to B}_{34})^*=\tilde{V}_{134};
\end{aligned}$$

the first step uses (3.3); the second step uses (3.2) for $\mathrm{V}^{A\to B}$; the third step uses that $\mathbb{V}^{A\to B}_{34}$ and $\mathbb{V}^{C\to A}_{12}$ commute; the fourth step again uses (3.3); and the last step uses again that $\mathbb{V}^{A\to B}_{34}$ and $\mathbb{V}^{C\to A}_{12}$ commute. Now Theorem 2.6 shows that $V\in\mathcal{UM}(\hat{C}\otimes 1\otimes B)$, so that (3.21) has a solution.

The following computation yields (3.2) for V:

$$\begin{aligned}
&\Sigma_{12}(\mathbb{W}^C_{12})^*\mathbb{V}^{C\to B}_{24}\mathbb{W}^C_{12}\Sigma_{12}\\
&=\Sigma_{12}(\mathbb{W}^C_{12})^*(\mathbb{V}^{C\to A}_{23})^*\mathbb{V}^{A\to B}_{34}\mathbb{V}^{C\to A}_{23}(\mathbb{V}^{A\to B}_{34})^*\mathbb{W}^C_{12}\Sigma_{12}\\
&=\Sigma_{12}(\mathbb{V}^{C\to A}_{23})^*(\mathbb{V}^{C\to A}_{13})^*(\mathbb{W}^C_{12})^*\mathbb{V}^{A\to B}_{34}\mathbb{V}^{C\to A}_{23}(\mathbb{V}^{A\to B}_{34})^*\mathbb{W}^C_{12}\Sigma_{12}\\
&=(\mathbb{V}^{C\to A}_{13})^*(\mathbb{V}^{C\to A}_{23})^*\mathbb{V}^{A\to B}_{34}\Sigma_{12}(\mathbb{W}^C_{12})^*\mathbb{V}^{C\to A}_{23}\mathbb{W}^C_{12}\Sigma_{12}(\mathbb{V}^{A\to B}_{34})^*\\
&=(\mathbb{V}^{C\to A}_{13})^*(\mathbb{V}^{C\to A}_{23})^*\mathbb{V}^{A\to B}_{34}\mathbb{V}^{C\to A}_{23}\mathbb{V}^{C\to A}_{13}(\mathbb{V}^{A\to B}_{34})^*\\
&=(\mathbb{V}^{C\to A}_{13})^*\mathbb{V}^{C\to B}_{24}\mathbb{V}^{A\to B}_{34}\mathbb{V}^{C\to A}_{13}(\mathbb{V}^{A\to B}_{34})^*\\
&=\mathbb{V}^{C\to B}_{24}(\mathbb{V}^{C\to A}_{13})^*\mathbb{V}^{A\to B}_{34}\mathbb{V}^{C\to A}_{13}(\mathbb{V}^{A\to B}_{34})^*\\
&=\mathbb{V}^{C\to B}_{24}\mathbb{V}^{C\to B}_{14}.
\end{aligned}$$

The first step uses (3.2); the second step uses properties of Σ and that $\mathbb{W}^C_{12}$ and $\mathbb{V}^{A\to B}_{34}$ commute; the third step again uses (3.2); the fourth step uses (3.21); the fifth step uses that $\mathbb{V}^{C\to A}_{13}$ and $\mathbb{V}^{C\to A}_{24}$ commute; and the last step uses (3.21) again.

Similarly, one shows (3.3). Hence $\mathrm{V}^{C\to B}$ is indeed a linking unitary. □

Proposition 3.23. *The composition of linking unitaries is associative, and the multiplicative unitary of C is an identity on C. Thus the linking unitaries with the above composition form the arrows of a category, with locally compact quantum groups as objects.*

Proof. The defining properties of a linking unitary amount to

$$\mathrm{W}^A * \mathrm{V}^{C\to A} = \mathrm{V}^{C\to A}, \qquad \mathrm{V}^{C\to A} * \mathrm{W}^C = \mathrm{V}^{C\to A},$$

that is, W^C is a unit object on C for the composition of linking unitaries.

Associativity means

$$\mathrm{V}^{B\to D} * (\mathrm{V}^{A\to B} * \mathrm{V}^{C\to A}) = (\mathrm{V}^{B\to D} * \mathrm{V}^{A\to B}) * \mathrm{V}^{C\to A}.$$

Define

$$\begin{aligned} \mathrm{V}^{C\to B} &:= \mathrm{V}^{A\to B} * \mathrm{V}^{C\to A}, & V^{C\to D} &:= \mathrm{V}^{B\to D} * \mathrm{V}^{C\to B}, \\ \mathrm{V}^{A\to D} &:= \mathrm{V}^{B\to D} * \mathrm{V}^{A\to B}, \quad \text{and} & \tilde{V}^{C\to D} &:= \mathrm{V}^{A\to D} * \mathrm{V}^{C\to A}. \end{aligned}$$

Equation (3.21) yields

$$\begin{aligned} \mathbb{V}^{C\to D}_{14} &= (\mathbb{V}^{C\to B}_{13})^* \mathbb{V}^{B\to D}_{34} \mathbb{V}^{C\to B}_{13} (\mathbb{V}^{B\to D}_{34})^* \\ &= (\mathbb{V}^{C\to B}_{13})^* (\mathbb{V}^{C\to A}_{12})^* \mathbb{V}^{B\to D}_{34} \mathbb{V}^{A\to B}_{23} \mathbb{V}^{C\to A}_{12} (\mathbb{V}^{A\to B}_{23})^* (\mathbb{V}^{B\to D}_{34})^* \\ &= \mathbb{V}^{A\to B}_{23} (\mathbb{V}^{C\to A}_{12})^* (\mathbb{V}^{A\to B}_{23})^* \mathbb{V}^{B\to D}_{34} \mathbb{V}^{A\to B}_{23} \mathbb{V}^{C\to A}_{12} (\mathbb{V}^{A\to B}_{23})^* (\mathbb{V}^{B\to D}_{34})^*. \end{aligned}$$

A similar computation yields

$$\begin{aligned} \tilde{\mathbb{V}}^{C\to D}_{14} &= (\mathbb{V}^{C\to A}_{12})^* \mathbb{V}^{A\to D}_{24} \mathbb{V}^{C\to A}_{12} (\mathbb{V}^{A\to D}_{24})^* \\ &= (\mathbb{V}^{C\to A}_{12})^* (\mathbb{V}^{A\to B}_{23})^* \mathbb{V}^{B\to D}_{34} \mathbb{V}^{A\to B}_{23} \mathbb{V}^{C\to A}_{12} (\mathbb{V}^{A\to B}_{23})^* (\mathbb{V}^{B\to D}_{34})^* \mathbb{V}^{A\to B}_{23} \end{aligned}$$

We may rewrite these two computations as

$$\begin{aligned} (\mathbb{V}^{A\to B}_{23})^* \mathbb{V}^{C\to D}_{14} &= (\mathbb{V}^{C\to A}_{12})^* (\mathbb{V}^{A\to B}_{23})^* \mathbb{V}^{B\to D}_{34} \mathbb{V}^{A\to B}_{23} \mathbb{V}^{C\to A}_{12} (\mathbb{V}^{A\to B}_{23})^* (\mathbb{V}^{B\to D}_{34})^* \\ &= \tilde{\mathbb{V}}^{C\to D}_{14} (\mathbb{V}^{A\to B}_{23})^*. \end{aligned}$$

Since $\tilde{\mathbb{V}}^{C\to D}_{14}$ and $\mathbb{V}^{A\to B}_{23}$ commute, we get $\mathbb{V}^{C\to D}_{14} = \tilde{\mathbb{V}}^{C\to D}_{14}$, that is, the composition of linking unitaries is associative. □

It is routine to check that the passage to opposite and coopposite quantum groups is a covariant functor, that is, the constructions in Proposition 3.15 are compatible with composition. Similarly, taking duals is a contravariant functor. This follows from the following computation:

$$\widehat{\mathrm{V}^{C\to B}_{13}} = \Sigma_{13} \mathrm{V}^{A\to B}_{23} (\mathrm{V}^{C\to A}_{12})^* (\mathrm{V}^{A\to B}_{23})^* \mathrm{V}^{C\to A}_{12} \Sigma_{13} = \widehat{\mathrm{V}^{A\to B}_{12}}^* \widehat{\mathrm{V}^{C\to A}_{23}} \widehat{\mathrm{V}^{A\to B}_{12}} \widehat{\mathrm{V}^{C\to A}_{23}}^*.$$

Of course, the composition of quantum group homomorphisms generates the more obvious composition in the case of strong quantum group homomorphisms. We can generalise this as follows.

Example 3.24. Let $\mathrm{V}^{C\to A} \in \mathcal{UM}(\hat{C} \otimes A)$ and $\mathrm{V}^{A\to B} \in \mathcal{UM}(\hat{A} \otimes B)$ be linking unitaries.

If $\mathrm{V}^{A\to B}$ comes from a strong quantum group homomorphism $f\colon A \to B$, that is, $\mathrm{V}^{A\to B} = (\mathrm{id} \otimes f)(\mathrm{W}^A)$, then the composition $\mathrm{V}^{C\to B}$ is $(\mathrm{id} \otimes f)(\mathrm{V}^{C\to A})$.

Dually, assume that $\mathrm{V}^{C\to A}$ is constructed from a strong quantum group homomorphism $f\colon \hat{A} \to \hat{C}$, that is, $\mathrm{V}^{C\to A} = (f\otimes\mathrm{id})(\mathrm{W}^A)$. Then the composition $\mathrm{V}^{C\to B}$ is $(f\otimes\mathrm{id})(\mathrm{V}^{A\to B})$.

It is more convenient to prove these statements at the end of the next section (Examples 4.22 and 4.23), using an equivalent description of linking unitaries involving right coactions.

4. Right and left coactions

Now we develop an alternative definition of quantum group homomorphisms using left or right coactions instead of linking unitaries.

Definition 4.1. A *right quantum group homomorphism* from (C,Δ_C) to (A,Δ_A) is a homomorphism $\Delta_R\colon C\to C\otimes A$ such that the following two diagrams commute:

$$\begin{CD} C @>{\Delta_R}>> C\otimes A \\ @V{\Delta_C}VV @VV{\Delta_C\otimes\mathrm{id}_A}V \\ C\otimes C @>>{\mathrm{id}_C\otimes\Delta_R}> C\otimes C\otimes A, \end{CD} \tag{4.2}$$

$$\begin{CD} C @>{\Delta_R}>> C\otimes A \\ @V{\Delta_R}VV @VV{\mathrm{id}_C\otimes\Delta_A}V \\ C\otimes A @>>{\Delta_R\otimes\mathrm{id}_A}> C\otimes A\otimes A. \end{CD} \tag{4.3}$$

The second condition (4.3) means that Δ_R is an A-comodule structure on C.

Example 4.4. A strong quantum group homomorphism $\varphi\colon C\to\mathcal{M}(A)$ yields a right quantum group homomorphism by $\Delta_R := (\mathrm{id}_C\otimes\varphi)\Delta_C$. The right quantum group homomorphism induced by the identity on C is Δ_C.

Recall that $\mathrm{W}\in\mathcal{UM}(\hat{C}\otimes C)$ denotes the reduced bicharacter. The following proposition shows that right quantum group homomorphisms are equivalent to linking unitaries.

Theorem 4.5. *For any right quantum group homomorphism $\Delta_R\colon C\to C\otimes A$, there is a unique unitary $V\in\mathcal{UM}(\hat{C}\otimes A)$ with*

$$(\mathrm{id}_{\hat{C}}\otimes\Delta_R)(\mathrm{W}) = \mathrm{W}_{12}V_{13}. \tag{4.6}$$

This unitary is a linking unitary.

Conversely, let V be a linking unitary from C to A, and let $\mathbb{V}\in\mathcal{U}(\mathcal{H}_C\otimes\mathcal{H}_A)$ be the corresponding unitary operator on Hilbert space. Then

$$\Delta_R(x) := \mathbb{V}(x\otimes 1)\mathbb{V}^* \qquad \textit{for all } x\in C \tag{4.7}$$

defines a right quantum group homomorphism from C to A.

These two maps between linking unitaries and right quantum group homomorphisms are bijective and inverse to each other.

Proof. First we check that $\tilde{V} := \mathrm{W}_{12}^*\cdot(\mathrm{id}_{\hat{C}}\otimes\Delta_R)(\mathrm{W})$ belongs to $\mathcal{UM}(\hat{C}\otimes 1\otimes A)$, that is, $\tilde{V} = V_{13}$ for some $V\in\mathcal{UM}(\hat{C}\otimes A)$. This is the unique V that verifies (4.6).

We compute

$$\begin{aligned}\mathbb{W}^C_{23}\tilde{V}_{124}(\mathbb{W}^C_{23})^* &= \mathbb{W}^C_{23}(\mathbb{W}^C_{12})^*(\mathbb{W}^C_{23})^* \cdot \mathbb{W}^C_{23}(\mathrm{id}_{\hat{C}} \otimes \Delta_R)(\mathrm{W})_{124}(\mathbb{W}^C_{23})^* \\ &= (\mathbb{W}^C_{13})^*(\mathbb{W}^C_{12})^* \cdot (\mathrm{id}_{\hat{C}} \otimes \Delta_C \otimes \mathrm{id}_A)(\mathrm{id}_{\hat{C}} \otimes \Delta_R)(\mathrm{W}) \\ &= (\mathbb{W}^C_{13})^*(\mathbb{W}^C_{12})^* \cdot (\mathrm{id}_{\hat{C}} \otimes (\mathrm{id}_C \otimes \Delta_R)\Delta_C)\mathrm{W} \\ &= (\mathbb{W}^C_{13})^*(\mathbb{W}^C_{12})^* \cdot (\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_C \otimes \Delta_R)(\mathrm{W}_{12}\mathrm{W}_{13}) \\ &= (\mathbb{W}^C_{13})^*\big((\mathrm{id}_{\hat{C}} \otimes \Delta_R)\mathrm{W}\big)_{134};\end{aligned}$$

the first equality is the definition of $\tilde{V}$, the second one uses (2.1) and (2.2), the third one (4.2), the fourth one uses (2.4), and the last one is trivial. Now Theorem 2.6 yields $\tilde{V} \in \mathcal{UM}(\hat{C} \otimes 1 \otimes A)$. Hence there is a unique $V \in \mathcal{UM}(\hat{C} \otimes A)$ that verifies (4.6).

Next we verify that V is a linking unitary. We check (3.6):

$$\begin{aligned}\big((\Delta_{\hat{C}} \otimes \mathrm{id}_A)V\big)_{124} &= (\Delta_{\hat{C}} \otimes \mathrm{id}_C \otimes \mathrm{id}_A)\big(\mathrm{W}^*_{12} \cdot (\mathrm{id}_{\hat{C}} \otimes \Delta_R)(\mathrm{W})\big) \\ &= ((\Delta_{\hat{C}} \otimes \mathrm{id}_C)\mathrm{W}^*)_{123} \cdot (\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_{\hat{C}} \otimes \Delta_R)(\Delta_{\hat{C}} \otimes \mathrm{id}_C)(\mathrm{W}) \\ &= (\mathrm{W}_{23}\mathrm{W}_{13})^*(\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_{\hat{C}} \otimes \Delta_R)(\mathrm{W}_{23}\mathrm{W}_{13}) \\ &= \mathrm{W}^*_{13}\mathrm{W}^*_{23}\mathrm{W}_{23}V_{24}\mathrm{W}_{13}V_{14} = V_{24}V_{14}.\end{aligned}$$

The first two equalities use (4.6) and that $\Delta_{\hat{C}}$ is a *-homomorphism; the third equality uses (2.5); the fourth one uses (4.6) again; and the final step uses that W_{13} and V_{24} commute.

The following computation yields (3.7):

$$\begin{aligned}\big((\mathrm{id}_{\hat{C}} \otimes \Delta_A)V\big)_{134} &= \mathrm{W}^*_{12}(\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_C \otimes \Delta_A)(\mathrm{id}_{\hat{C}} \otimes \Delta_R)\mathrm{W} \\ &= \mathrm{W}^*_{12}(\mathrm{id}_{\hat{C}} \otimes \Delta_R \otimes \mathrm{id}_A)(\mathrm{id}_{\hat{C}} \otimes \Delta_R)\mathrm{W} \\ &= \mathrm{W}^*_{12}(\mathrm{id}_{\hat{C}} \otimes \Delta_R \otimes \mathrm{id}_A)(\mathrm{W}_{12}V_{14}) = V_{13}V_{14}.\end{aligned}$$

The first equality follows from (4.6); the second one from (4.3); the third and fourth equalities from (4.6). Thus we have constructed a linking unitary V from a right quantum group homomorphism.

Conversely, let $V \in \mathcal{UM}(\hat{C} \otimes A)$ be a linking unitary. We claim that (4.7) defines a morphism from C to $C \otimes A$. Recall that slices of $\mathbb{W}$ by linear functionals $\omega \in \mathbb{B}(\mathcal{H})_*$ generate a dense subspace of C. On $x := (\omega \otimes \mathrm{id}_{\mathcal{H}})(\mathbb{W})$, we compute

$$\Delta_R(x) = (\omega \otimes \mathrm{id}_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{H}})(\mathbb{V}_{23}\mathbb{W}_{12}\mathbb{V}^*_{23}) = (\omega \otimes \mathrm{id}_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{H}})(\mathbb{W}_{12}\mathbb{V}_{13}),$$

and this belongs to $\mathcal{M}(C \otimes A)$. Thus $\Delta_R(C) \subseteq \mathcal{M}(C \otimes A)$. It is clear from the definition that Δ_R is non-degenerate.

We may also rewrite the above computation as $(\omega \otimes \mathrm{id}_{C\otimes A}) \circ (\mathrm{id}_{\hat{C}} \otimes \Delta_R)(\mathrm{W}) = (\omega \otimes \mathrm{id}_{C\otimes A})(\mathrm{W}_{12}V_{13})$ for all $\omega \in \mathbb{B}(\mathcal{H})_*$. Since ω is arbitrary, (4.6) holds for Δ_R and our original linking unitary V.

Now we use (4.6) to check that Δ_R is a right quantum group homomorphism. Diagram (4.2) amounts to

$$(\mathrm{id}_{\hat{C}} \otimes \Delta_C \otimes \mathrm{id}_A)(\mathrm{id}_{\hat{C}} \otimes \Delta_R)(\mathrm{W}) = (\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_C \otimes \Delta_R)(\mathrm{id}_{\hat{C}} \otimes \Delta_C)(\mathrm{W})$$

because slices of W generate C. This follows from (4.6) and (2.4): both sides are equal to $\mathrm{W}_{12}\mathrm{W}_{13}V_{14}$. Similarly, (4.3) amounts to

$$(\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_C \otimes \Delta_A)(\mathrm{id}_{\hat{C}} \otimes \Delta_R)(\mathrm{W}) = (\mathrm{id}_{\hat{C}} \otimes \Delta_R \otimes \mathrm{id}_A)(\mathrm{id}_{\hat{C}} \otimes \Delta_R)(\mathrm{W}),$$

which follows from (4.6) and (3.7) because both sides are equal to $\mathrm{W}_{12}V_{13}V_{14}$.

Thus we have associated a right quantum group homomorphism Δ_R to a linking unitary V. Since these are related by (4.6), we get back the original linking unitary from this right quantum group homomorphism. It only remains to check that, if we start with a right quantum group homomorphism Δ_R, define a linking unitary by (4.6) and then a right quantum group homomorphism by (4.7), we get back the original Δ_R. We may rewrite (3.2) as

$$\mathbb{V}_{23}\mathbb{W}_{12}\mathbb{V}_{23}^{*} = \mathbb{W}_{12}\mathbb{V}_{13} = (\mathrm{id}_{\hat{C}} \otimes \Delta_R)(\mathbb{W}),$$

using (4.6). This implies that the original Δ_R satisfies (4.7) because the slices of W by linear functionals on $\hat{C}$ span a dense subspace of C. □

Definition 4.8. A *left quantum group homomorphism* from (C, Δ_C) to (A, Δ_A) is a morphism $\Delta_L\colon C \to A \otimes C$ such that the following two diagrams commute:

$$\text{(4.9)} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta_L} & A \otimes C \\ {\scriptstyle\Delta_C}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \Delta_C} \\ C \otimes C & \xrightarrow[\Delta_L \otimes \mathrm{id}_C]{} & A \otimes C \otimes C, \end{array}$$

$$\text{(4.10)} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta_L} & A \otimes C \\ {\scriptstyle\Delta_L}\downarrow & & \downarrow{\scriptstyle \Delta_A \otimes \mathrm{id}_C} \\ A \otimes C & \xrightarrow[\mathrm{id}_A \otimes \Delta_L]{} & A \otimes A \otimes C. \end{array}$$

Let $\sigma\colon C \otimes A \to A \otimes C$ and $\sigma\colon A \otimes C \to C \otimes A$ denote the flip maps. If Δ_R is a right quantum group homomorphism from C to A, then $\sigma \circ \Delta_R$ is a left quantum group homomorphism from C^{cop} to A^{cop}. And if Δ_L is a left quantum group homomorphism from C to A, then $\sigma \circ \Delta_L$ is a right quantum group homomorphism from C^{cop} to A^{cop}. We have seen in Proposition 3.15 that linking unitaries from C to A and from C^{cop} to A^{cop} are essentially the same thing. Since right quantum group homomorphisms correspond bijectively to linking unitaries by Theorem 4.5, we conclude that left quantum group homomorphisms also correspond bijectively to linking unitaries from C to A and hence to right quantum group homomorphisms. We now make these bijections more explicit:

Theorem 4.11. *For any left quantum group homomorphism $\Delta_L\colon C \to C \otimes A$, there is a unique unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ with*

$$\text{(4.12)} \qquad (\mathrm{id}_{\hat{C}} \otimes \Delta_L)(\mathrm{W}) = V_{12}\mathrm{W}_{13}.$$

This unitary is a linking unitary.

Conversely, let V be a linking unitary from C to A, and let $\mathbb{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A)$ be the corresponding unitary operator on Hilbert space. Then

$$\text{(4.13)} \qquad \Delta_L(x) := \sigma((\mathbb{V}^{\mathrm{cop}})(x \otimes 1)(\mathbb{V}^{\mathrm{cop}})^{*}) \qquad \text{for all } x \in C$$

defines a left quantum group homomorphism from C to A.

These two maps between linking unitaries and left quantum group homomorphisms are bijective and inverse to each other.

Despite Proposition 3.15, $\mathbb{V} \neq \mathbb{V}^{\mathrm{cop}}$ because the quantum groups C and C^{cop} are represented differently on $\mathcal{H}_C$.

Lemma 4.14. *Let $\Delta_L\colon C \to A \otimes C$ and $\Delta_R\colon C \to C \otimes B$ be a left and a right quantum group homomorphism. Then the following diagram commutes:*

$$\begin{array}{ccc} C & \xrightarrow{\Delta_L} & A \otimes C \\ {\scriptstyle\Delta_R}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \Delta_R} \\ C \otimes B & \xrightarrow[\Delta_L \otimes \mathrm{id}_B]{} & A \otimes C \otimes B. \end{array}$$

Proof. Since slices of W^C span a dense subspace of C, (4.14) commutes if and only if

$$(\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_A \otimes \Delta_R)(\mathrm{id}_{\hat{C}} \otimes \Delta_L)(\mathrm{W}) = (\mathrm{id}_{\hat{C}} \otimes \Delta_L \otimes \mathrm{id}_B)(\mathrm{id}_{\hat{C}} \otimes \Delta_R)(\mathrm{W}). \tag{4.15}$$

Let V and $\tilde{V}$ be the linking unitaries associated to Δ_L and Δ_R, respectively. Equations (4.6) and (4.12) imply that both sides of (4.15) are equal to $V_{12}\mathrm{W}_{13}\tilde{V}_{14}$. □

We may also characterise when a left and a right quantum group homomorphism correspond to the same linking unitary:

Lemma 4.16. *Let $\Delta_L\colon C \to A \otimes C$ and $\Delta_R\colon C \to C \otimes B$ be a left and a right quantum group homomorphism. Then they are associated to the same linking unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ if and only if the following diagram commutes:*

$$\begin{array}{ccc} C & \xrightarrow{\Delta_C} & C \otimes C \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \Delta_L} \\ C \otimes C & \xrightarrow[\Delta_R \otimes \mathrm{id}_C]{} & C \otimes A \otimes C. \end{array}$$

Proof. The above diagram commutes if and only if

$$(\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_C \otimes \Delta_L)(\mathrm{id}_{\hat{C}} \otimes \Delta_C)(\mathrm{W}) = (\mathrm{id}_{\hat{C}} \otimes \Delta_R \otimes \mathrm{id}_C)(\mathrm{id}_{\hat{C}} \otimes \Delta_C)(\mathrm{W}) \tag{4.17}$$

because slices of W span a dense subspace of C. Let Δ_L and Δ_R be associated to the linking unitaries $\tilde{V} \in \mathcal{UM}(\hat{C} \otimes A)$ and $V \in \mathcal{UM}(\hat{C} \otimes A)$, respectively.

Using (2.4), (4.12) and (4.6), we may rewrite (4.17) as

$$\mathrm{W}_{12}\tilde{V}_{13}\mathrm{W}_{14} = \mathrm{W}_{12}V_{13}\mathrm{W}_{14}.$$

Thus (4.17) is equivalent to $V = \tilde{V}$. □

Lemma 4.18. *Right or left quantum group homomorphisms are continuous as coactions.*

Proof. It suffices to prove the assertion for right quantum group homomorphisms, the left case follows by passing to coopposites. Let $\Delta_R\colon C \to \mathcal{M}(C \otimes A)$ be a right quantum group homomorphism with associated linking unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$. We must show that the linear span of $\Delta_R(C)(1 \otimes A)$ is dense in $C \otimes A$. We may replace C by the dense subspace of slices $(\hat{c}\mu \otimes \mathrm{id}_C)\mathrm{W}^C$ for $\mu \in \hat{C}$ and $\hat{c} \in \hat{C}$, where $\hat{c}\mu \in \hat{C}'$ is defined by $\hat{c}\mu(x) := \mu(x\hat{c})$ for $\hat{c} \in \hat{C}$, $\mu \in \hat{C}'$, and $x \in \hat{C}$. We have

$$((\hat{c}\mu \otimes \mathrm{id}_C \otimes \mathrm{id}_A)(\mathrm{id}_{\hat{C}} \otimes \Delta_R)\mathrm{W}^C)(1 \otimes a) = (\mu \otimes \mathrm{id}_C \otimes \mathrm{id}_A)(\mathrm{W}^C_{12}V_{13}(\hat{c} \otimes 1 \otimes a)).$$

Here $V_{13}(\hat{c} \otimes 1 \otimes a)$ ranges over a linearly dense subset of $\hat{C} \otimes 1 \otimes A$. Hence we do not change the closed linear span if we replace this expression by $\hat{c} \otimes 1 \otimes a$. This leads to

$$(\mu \otimes \mathrm{id}_C \otimes \mathrm{id}_A)(\mathrm{W}^C_{12} \cdot (\hat{c} \otimes 1 \otimes a)) = \big((\hat{c}\mu \otimes \mathrm{id}_C)\mathrm{W}^C\big) \otimes a,$$

and these elements span a dense subspace of $C \otimes A$ as asserted. □

By passing to the corresponding linking unitaries, we also get a notion of composition for right and left quantum group homomorphisms. We only make this explicit for right quantum group homomorphisms. Let $\alpha\colon C \to C \otimes A$ and $\beta\colon A \to A \otimes B$ be two right quantum group homomorphisms associated to the linking unitaries $\mathrm{V}^{C \to A}$ and $\mathrm{V}^{A \to B}$. Let $\beta * \alpha$ be the right quantum group homomorphism associated to the linking unitary $\mathrm{V}^{A \to B} * \mathrm{V}^{C \to A}$. We may describe $\beta * \alpha$ directly in terms of right and left quantum group homomorphisms:

Proposition 4.19. *There is a unique right quantum group homomorphism $\gamma\colon C \to C \otimes B$ that makes the following diagram commute:*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & C \otimes A \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \beta} \\ C \otimes B & \xrightarrow[\alpha\otimes \mathrm{id}_B]{} & C \otimes A \otimes B. \end{array} \tag{4.20}$$

This is exactly the composition $\beta * \alpha$.

Proof. Since slices of W by continuous linear functionals on $\hat{C}$ generate a dense subspace of C, the diagram (4.20) commutes if and only if

$$(\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_C \otimes \beta)(\mathrm{id}_{\hat{C}} \otimes \alpha)(\mathrm{W}^C) = (\mathrm{id}_{\hat{C}} \otimes \alpha \otimes \mathrm{id}_B)(\mathrm{id}_{\hat{C}} \otimes \gamma)(\mathrm{W}^C).$$

Equation (4.6) implies $\mathrm{id}_{\hat{C}} \otimes \alpha(\mathrm{W}^C) = \mathrm{W}^C_{12}\mathrm{V}^{C\to A}_{13}$, and $\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_A \otimes \beta$ maps this to the element represented by the unitary operator

$$\mathbb{W}^C_{12}\mathbb{V}^{A\to B}_{34}\mathbb{V}^{C\to A}_{13}(\mathbb{V}^{A\to B}_{34})^* = \mathbb{W}^C_{12}\mathbb{V}^{C\to A}_{13}\mathbb{V}^{C\to B}_{14}$$

by (4.7) and (3.21). Thus

$$(\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_C \otimes \beta)(\mathrm{id}_{\hat{C}} \otimes \alpha)(\mathrm{W}^C) = \mathrm{W}^C_{12}\mathrm{V}^{C\to A}_{13}\mathrm{V}^{C\to B}_{14},$$

where $\mathrm{V}^{C\to B} := \mathrm{V}^{A\to B} * \mathrm{V}^{C\to A}$. Let $\tilde{V}$ be the linking unitary associated to γ. Equation (4.6) implies

$$(\mathrm{id}_{\hat{C}} \otimes \alpha \otimes \mathrm{id}_B)(\mathrm{id}_{\hat{C}} \otimes \gamma)(\mathrm{W}^C) = (\mathrm{id}_{\hat{C}} \otimes \alpha \otimes \mathrm{id}_B)(\mathrm{W}^C_{12}\tilde{V}_{13}) = \mathrm{W}^C_{12}\mathrm{V}^{C\to A}_{13}\tilde{V}_{14}.$$

Hence (4.20) commutes if and only if $\tilde{V} = \mathrm{V}^{C\to B}$. □

Remark 4.21. The composition of linking unitaries at first sight depends on the choice of concrete representations of the quantum groups involved, which depend on the choice of generating modular multiplicative unitaries. Proposition 4.19 is phrased purely in terms of comultiplications. This shows that the composition of linking unitaries does not depend on the choice of a generating multiplicative unitary.

Example 4.22. Let $\mathrm{V}^{C\to A} \in \mathcal{UM}(\hat{C} \otimes A)$ and $\mathrm{V}^{A\to B} \in \mathcal{UM}(\hat{A} \otimes B)$ be linking unitaries.

Assume first that $\mathrm{V}^{A\to B}$ comes from a strong quantum group homomorphism $f\colon A \to B$, that is, $\mathrm{V}^{A\to B} = (\mathrm{id} \otimes f)(\mathrm{W}^A)$. Let α be the right quantum group homomorphism from C to A associated to $\mathrm{V}^{C\to A}$. The right quantum group homomorphism from A to B associated to $\mathrm{V}^{A\to B}$ is $\beta := (\mathrm{id}_A \otimes f)\Delta_A$. The following computation shows that $\gamma = (\mathrm{id}_C \otimes f)\alpha$ satisfies (4.20):

$$\begin{aligned}(\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_C \otimes \beta)(\mathrm{id}_{\hat{C}} \otimes \alpha)\mathrm{W}^C &= (\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_C \otimes \mathrm{id}_A \otimes f)(\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_C \otimes \Delta_A)\mathrm{W}^C_{12}\mathrm{V}^{C\to A}_{13} \\ &= (\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_C \otimes \mathrm{id}_A \otimes f)\mathrm{W}^C_{12}\mathrm{V}^{C\to A}_{13}\mathrm{V}^{C\to A}_{14} \\ &= (\mathrm{id}_{\hat{C}} \otimes \mathrm{id}_C \otimes \mathrm{id}_A \otimes f)(\mathrm{id}_{\hat{C}} \otimes \alpha \otimes \mathrm{id}_B)\mathrm{W}^C_{12}\mathrm{V}^{C\to A}_{13} \\ &= (\mathrm{id}_{\hat{C}} \otimes \alpha \otimes \mathrm{id}_B)(\mathrm{id}_{\hat{C}} \otimes (\mathrm{id}_C \otimes f)\alpha)\mathrm{W}^C;\end{aligned}$$

the first step uses (4.6); the second step uses (3.7); the third and the last step use (4.6). Proposition 4.19 yields $\beta * \alpha = (\mathrm{id}_C \otimes f)\alpha$. Hence the composition $\mathrm{V}^{A\to B} * \mathrm{V}^{C\to A}$ is $(\mathrm{id}_C \otimes f)\mathrm{V}^{C\to A}$.

Example 4.23. Now assume that $\mathrm{V}^{C\to A}$ is constructed from a strong quantum group homomorphism $f\colon \hat{A} \to \hat{C}$, that is, $\mathrm{V}^{C\to A} = (f \otimes \mathrm{id}_A)(\mathrm{W}^A)$. Then the composition $\mathrm{V}^{C\to B}$ is $(f \otimes \mathrm{id})(\mathrm{V}^{A\to B})$. This follows easily from Example 4.22 because $C \mapsto \hat{C}$ is a contravariant functor on linking unitaries.

5. Passage to universal quantum groups

In this section we show that our quantum group homomorphisms are equivalent to strong quantum group homomorphisms between the associated universal quantum groups, which were previously suggested as a suitable notion of quantum group homomorphism.

Let (C, Δ_C) be a quantum group in the sense of [5]. The associated universal quantum group $(C^{\mathrm{u}}, \Delta_{C^{\mathrm{u}}})$, also introduced in [5], is a C*-bialgebra, that is, a C*-algebra equipped with a coassociative comultiplication. While it carries much the same additional structure that locally compact quantum groups carry, it is usually not a quantum group in the sense of [5], that is, it is not generated by a modular multiplicative unitary. Thus the theory developed above does not apply to it.

A left *corepresentation* of $(\hat{C}, \Delta_{\hat{C}})$ on a C*-algebra D is a unitary multiplier $V \in \mathcal{UM}(\hat{C} \otimes D)$ that satisfies $(\Delta_{\hat{C}} \otimes \mathrm{id}_A)(V) = V_{23}V_{13}$. That is, V is a character with respect to the first variable. The universal dual carries a left corepresentation $\mathcal{V} \in \mathcal{UM}(\hat{C} \otimes C^{\mathrm{u}})$ of $\hat{C}$ that is universal in the following sense: for any left corepresentation $U \in \mathcal{UM}(\hat{C} \otimes D)$ there is a unique morphism $\varphi\colon C^{\mathrm{u}} \to D$ with $U = (\mathrm{id}_{\hat{C}} \otimes \varphi)(\mathcal{V})$. This universal property characterises the pair $(C^{\mathrm{u}}, \mathcal{V})$ uniquely up to isomorphism.

The comultiplication on C^{u} is defined so that $\mathrm{id}_{\hat{C}} \otimes \Delta_{C^{\mathrm{u}}}$ maps $\mathcal{V}$ to the left corepresentation $\mathcal{V}_{12}\mathcal{V}_{13}$. Thus $\mathcal{V}$ is a bicharacter and we may interpret it as a quantum group homomorphism from C to C^{u}. This is, however, not literally true because C^{u} is not a quantum group in our sense.

Proposition 5.1. *Let (A, Δ_A) be a C*-bialgebra. Bicharacters in $\mathcal{UM}(\hat{C} \otimes A)$ correspond bijectively to strong quantum group homomorphisms from $(C^{\mathrm{u}}, \Delta_{C^{\mathrm{u}}})$ to (A, Δ_A).*

In particular, if (A, Δ_A) is also a locally compact quantum group, then strong quantum group homomorphisms from $(C^{\mathrm{u}}, \Delta_{C^{\mathrm{u}}})$ to (A, Δ_A) correspond to quantum group homomorphisms from (C, Δ_C) to (A, Δ_A).

Proof. A strong quantum group homomorphism $\varphi\colon C^{\mathrm{u}} \to A$ is also a morphism from C^{u} to A and thus corresponds to a left corepresentation $V \in \mathcal{UM}(\hat{C} \otimes A)$, which is determined by the condition $(\mathrm{id}_{\hat{C}} \otimes \varphi)(\mathcal{V}) = V$. The strong quantum group homomorphisms $\Delta_A \circ \varphi\colon C^{\mathrm{u}} \to A \otimes A$ and $(\varphi \otimes \varphi) \circ \Delta_{C^{\mathrm{u}}}\colon C^{\mathrm{u}} \to A \otimes A$ correspond to the left corepresentations $(\mathrm{id}_{\hat{C}} \otimes \Delta_A)(V)$ and $V_{12}V_{13}$, that is, $\mathrm{id}_{\hat{C}} \otimes (\Delta_A \circ \varphi)(\mathcal{V}) = (\mathrm{id}_{\hat{C}} \otimes \Delta_A)(V)$ and $(\mathrm{id}_{\hat{C}} \otimes (\varphi \otimes \varphi) \circ \Delta_{C^{\mathrm{u}}})(\mathcal{V}) = V_{12}V_{13}$ because $\mathcal{V}$ is a bicharacter. Thus a morphism $\varphi\colon C^{\mathrm{u}} \to A$ is a strong quantum group homomorphism if and only if the corepresentation V also satisfies $(\mathrm{id}_{\hat{C}} \otimes \Delta_A)(V) = V_{12}V_{13}$. That is, V is a bicharacter. □

Corollary 5.2. *Any strong quantum group homomorphism $\varphi\colon C^{\mathrm{u}} \to A$ induces a dual strong quantum group homomorphism $\hat{\varphi}\colon \hat{A}^{\mathrm{u}} \to \hat{C}$.*

Proof. By Proposition 5.1, a strong quantum group homomorphism $\varphi\colon C^{\mathrm{u}} \to A$ corresponds to a bicharacter in $\mathcal{UM}(\hat{C} \otimes A)$. Proposition 3.15 identifies these with bicharacters in $\mathcal{UM}(A \otimes \hat{C})$, which correspond to strong quantum group homomorphisms $\varphi\colon \hat{A}^{\mathrm{u}} \to \hat{C}$ by another application of Proposition 5.1. □

We are going to show that strong quantum group homomorphisms from $(C^{\mathrm{u}}, \Delta_{C^{\mathrm{u}}})$ to (A, Δ_A) lift uniquely to strong quantum group homomorphisms from $(C^{\mathrm{u}}, \Delta_{C^{\mathrm{u}}})$ to $(A^{\mathrm{u}}, \Delta_{A^{\mathrm{u}}})$. This together with Proposition 5.1 will establish a bijection between homomorphisms of quantum groups in our sense and strong strong quantum group homomorphisms between the associated universal quantum groups.

This requires the universal bicharacter $\mathcal{U} \in \mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes C^{\mathrm{u}})$. In the setting of quantum groups with Haar measure, it is constructed in [2, Proposition 6.4]. First we carry this construction over to the setting of [5].

The bicharacter W of C is also a left corepresentation. Hence the universal property yields a reducing *-homomorphism $\Lambda\colon C^{\mathrm{u}} \to C$ with

$$(\mathrm{id}_{\hat{C}} \otimes \Lambda)(\mathcal{V}) = \mathrm{W}. \tag{5.3}$$

The constructions above applied to the dual of C yield a maximal left corepresentation $\tilde{\mathcal{V}} \in \mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes C)$ of C and a reducing *-homomorphism $\hat{\Lambda}\colon \hat{C}^{\mathrm{u}} \to \hat{C}$, such that

$$(\Lambda \otimes \mathrm{id}_C)(\tilde{\mathcal{V}}) = \mathrm{W}. \tag{5.4}$$

We want to find $\mathcal{U} \in \mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes C^{\mathrm{u}})$ with $(\hat{\Lambda} \otimes \mathrm{id}_{C^{\mathrm{u}}})(\mathcal{U}) = \mathcal{V}$ and $(\mathrm{id}_{\hat{C}^{\mathrm{u}}} \otimes \Lambda)(\mathcal{U}) = \tilde{\mathcal{V}}$.

Using (2.2), we may rewrite the fact that $\tilde{\mathcal{V}}$ is a character in the second variable as a pentagon equation

$$\mathrm{W}_{23}\tilde{\mathcal{V}}_{12} = \tilde{\mathcal{V}}_{12}\tilde{\mathcal{V}}_{13}\mathrm{W}_{23} \qquad \text{in } \mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes \mathbb{K}(\mathcal{H}_C) \otimes C). \tag{5.5}$$

Similarly, using (2.3) and that $\mathcal{V}$ is a character in the first variable, we get the pentagon equation

$$\mathcal{V}_{23}\mathrm{W}_{12} = \mathrm{W}_{12}\mathcal{V}_{13}\mathcal{V}_{23} \qquad \text{in } \mathcal{UM}(\hat{C} \otimes \mathbb{K}(\mathcal{H}_C) \otimes C^{\mathrm{u}}). \tag{5.6}$$

In both cases, we should represent the second tensor factors C and $\hat{C}$ (faithfully) on $\mathcal{H}_C$ to make sense of the pentagon equation. We may now characterise $\mathcal{U}$ by a variant of the pentagon equation as in [2, Proposition 6.4].

Proposition 5.7. *There is a unique* $\mathcal{U} \in \mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes C^{\mathrm{u}})$ *such that*

$$\mathcal{V}_{23}\tilde{\mathcal{V}}_{12} = \tilde{\mathcal{V}}_{12}\mathcal{U}_{13}\mathcal{V}_{23}.$$

Moreover, this $\mathcal{U}$ *is a bicharacter, and it satisfies*

$$(\mathrm{id}_{\hat{C}^{\mathrm{u}}} \otimes \Lambda)\mathcal{U} = \tilde{\mathcal{V}}, \tag{5.8}$$

$$(\hat{\Lambda} \otimes \mathrm{id}_{C^{\mathrm{u}}})\mathcal{U} = \mathcal{V}, \tag{5.9}$$

$$(\hat{\Lambda} \otimes \Lambda)\mathcal{U} = \mathrm{W}. \tag{5.10}$$

Proof. Let $\mathcal{U}' := \tilde{\mathcal{V}}_{12}^*\mathcal{V}_{23}\tilde{\mathcal{V}}_{12}\mathcal{V}_{23}^*$. First, we will show that $\mathcal{U}' \in \mathcal{M}(\hat{C}^{\mathrm{u}} \otimes 1 \otimes C^{\mathrm{u}})$, that is, $\mathcal{U}' = \mathcal{U}_{13}$ for some $\mathcal{U} \in \mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes C^{\mathrm{u}})$. Obviously, this unitary is the unique solution of our problem. Then we will establish that $\mathcal{U}$ is a bicharacter.

The first step follows once again from Theorem 2.6. We compute

$$\begin{aligned}
\mathbb{W}_{23}\mathcal{U}'_{124}\mathbb{W}_{23}^* &= \mathrm{W}_{23}\tilde{\mathcal{V}}_{12}^*\mathcal{V}_{24}\tilde{\mathcal{V}}_{12}\mathcal{V}_{24}^*\mathrm{W}_{23}^* \\
&= \tilde{\mathcal{V}}_{13}^*\tilde{\mathcal{V}}_{12}^*\mathrm{W}_{23}\mathcal{V}_{24}\tilde{\mathcal{V}}_{12}\mathcal{V}_{24}^*\mathrm{W}_{23}^* \\
&= \tilde{\mathcal{V}}_{13}^*\tilde{\mathcal{V}}_{12}^*\mathcal{V}_{34}\mathrm{W}_{23}\mathcal{V}_{34}^*\tilde{\mathcal{V}}_{12}\mathcal{V}_{34}\mathrm{W}_{23}^*\mathcal{V}_{34}^* \\
&= \tilde{\mathcal{V}}_{13}^*\tilde{\mathcal{V}}_{12}^*\mathcal{V}_{34}\mathrm{W}_{23}\tilde{\mathcal{V}}_{12}\mathrm{W}_{23}^*\mathcal{V}_{34}^* \\
&= \tilde{\mathcal{V}}_{13}^*\tilde{\mathcal{V}}_{12}^*\mathcal{V}_{34}\tilde{\mathcal{V}}_{12}\tilde{\mathcal{V}}_{13}\mathcal{V}_{34}^* \\
&= \tilde{\mathcal{V}}_{13}^*\mathcal{V}_{34}\tilde{\mathcal{V}}_{13}\mathcal{V}_{34}^* = \mathcal{U}'_{134};
\end{aligned}$$

the first step is the definition of $\mathcal{U}'$; the second step uses (5.5); the third step uses (5.6) twice; the fourth step uses that $\mathcal{V}_{34}^*$ and W_{12} commute; the fifth step again uses (5.5); and the sixth step follows because $\mathcal{V}_{34}$ and $\tilde{\mathcal{V}}_{12}$ commute. Now Theorem 2.6 yields $\mathcal{U}' \in \mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes 1 \otimes C^{\mathrm{u}})$, so that $\mathcal{U}$ exists.

The following computation shows that $\mathcal{U}$ is a character in the second variable:

$$\begin{aligned}(\mathrm{id}_{\hat{C}^{\mathrm{u}}}\otimes\mathrm{id}_C\otimes\Delta_{C^{\mathrm{u}}})\tilde{\mathcal{V}}_{12}^*\mathcal{V}_{23}\tilde{\mathcal{V}}_{12}\mathcal{V}_{23}^* &= \tilde{\mathcal{V}}_{12}^*\mathcal{V}_{23}\mathcal{V}_{24}\tilde{\mathcal{V}}_{12}\mathcal{V}_{24}^*\mathcal{V}_{23}^* \\ &= \mathcal{U}_{13}\mathcal{V}_{23}\tilde{\mathcal{V}}_{12}^*\mathcal{V}_{24}\tilde{\mathcal{V}}_{12}\mathcal{V}_{24}^*\mathcal{V}_{23}^* = \mathcal{U}_{13}\mathcal{V}_{23}\mathcal{U}_{14}\mathcal{V}_{23}^* = \mathcal{U}_{13}\mathcal{U}_{14}.\end{aligned}$$

A similar computation works in the first variable. Thus $\mathcal{U}$ is a bicharacter.

The following computation yields (5.8):

$$\begin{aligned}(\mathrm{id}_{\hat{C}^{\mathrm{u}}}\otimes\mathrm{id}_C\otimes\Lambda)\mathcal{U}_{13} &= (\mathrm{id}_{\hat{C}^{\mathrm{u}}}\otimes\mathrm{id}_C\otimes\Lambda)\tilde{\mathcal{V}}_{12}^*\mathcal{V}_{23}\tilde{\mathcal{V}}_{12}\mathcal{V}_{23}^* \\ &= \tilde{\mathcal{V}}_{12}^*\mathrm{W}_{23}\tilde{\mathcal{V}}_{12}\mathrm{W}_{23}^* = \tilde{\mathcal{V}}_{12}^*\tilde{\mathcal{V}}_{12}\tilde{\mathcal{V}}_{13} = \tilde{\mathcal{V}}_{13}.\end{aligned}$$

A similar computation yields (5.9). Then (5.10) follows from (5.3) or from (5.4). ☐

Definition 5.11. The unitary multiplier $\mathcal{U}$ in Proposition 5.7 is called the *universal bicharacter* of (C,Δ_C).

Lemma 5.12. *Let $X,Y\in\mathcal{UM}(C\otimes A^{\mathrm{u}})$ be characters in the second variable. Let $\pi\colon A^{\mathrm{u}}\to A$ be the reducing *-homomorphism. If $(\mathrm{id}_C\otimes\pi)X=(\mathrm{id}_C\otimes\pi)Y$, then $X=Y$.*

Proof. Copy the proof of [2, Result 6.1]. ☐

Proposition 5.13. *Every strong quantum group homomorphism $C^{\mathrm{u}}\to A$ admits a unique lifting $C^{\mathrm{u}}\to A^{\mathrm{u}}$.*

Proof. Let $\mathcal{U}^A\in\mathcal{UM}(\hat{A}^{\mathrm{u}}\otimes A^{\mathrm{u}})$ be the universal bicharacter of (A,Δ_A) and let $\tilde{\mathcal{V}}corep^A\in\mathcal{UM}(\hat{A}^{\mathrm{u}}\otimes A)$ and $\mathcal{V}^C\in\mathcal{UM}(\hat{C}\otimes C^{\mathrm{u}})$ be the maximal corepresentations of A and $\hat{C}$. Let α be a strong quantum group homomorphism from C^{u} to A. It corresponds to a unique linking unitary

$$\mathrm{V}^{C\to A}=(\mathrm{id}_{\hat{C}}\otimes\alpha)\mathcal{V}^C \qquad \text{in } \mathcal{UM}(\hat{C}\otimes A). \tag{5.14}$$

By Corollary 5.2, there is a strong quantum group homomorphism $\hat{\varphi}\colon\hat{A}^{\mathrm{u}}\to\hat{C}$ such that $\mathrm{V}^{C\to A}=(\hat{\varphi}\otimes\mathrm{id}_A)(\tilde{\mathcal{V}}^A)$. Let

$$V=(\hat{\varphi}\otimes\mathrm{id}_{A^{\mathrm{u}}})\mathcal{U}^A \qquad \text{in } \mathcal{UM}(\hat{C}\otimes A^{\mathrm{u}}).$$

This is a bicharacter because $\mathcal{U}^A$ is and $\hat{\varphi}$ is a strong quantum group homomorphism. Proposition 5.1 now yields a unique strong quantum group homomorphism $\tilde{\varphi}\colon C^{\mathrm{u}}\to A^{\mathrm{u}}$ such that

$$V=(\mathrm{id}_{\hat{C}}\otimes\tilde{\varphi})\mathcal{V}^C \qquad \text{in } \mathcal{UM}(\hat{C}\otimes A^{\mathrm{u}}).$$

Clearly, $\varphi=\Lambda^A\circ\tilde{\varphi}$ is a strong quantum group homomorphism from C^{u} to A. The associated linking unitary is

$$\begin{aligned}(\mathrm{id}_{\hat{C}}\otimes\varphi)\mathcal{V}^C &= (\mathrm{id}_{\hat{C}}\otimes\Lambda^A\circ\tilde{\varphi})\mathcal{V}^C = (\mathrm{id}_{\hat{C}}\otimes\Lambda^A)V = (\mathrm{id}_{\hat{C}}\otimes\Lambda^A)(\hat{\varphi}\otimes\mathrm{id}_{A^{\mathrm{u}}})\mathcal{U}^A \\ &= (\hat{\varphi}\otimes\mathrm{id}_A)(\mathrm{id}_{\hat{A}^{\mathrm{u}}}\otimes\Lambda^A)\mathcal{U}^A = (\hat{\varphi}\otimes\mathrm{id}_A)\tilde{\mathcal{V}}^A = \mathrm{V}^{C\to A}.\end{aligned}$$

This shows that φ and α are two strong quantum group morphisms associated to the same linking unitary $\mathrm{V}^{C\to A}\in\mathcal{UM}(\hat{C}\otimes A)$. The uniqueness part of Proposition 5.1 shows that $\tilde{\varphi}$ lifts α.

Now let $\tilde{\varphi}'\colon C^{\mathrm{u}}\to A^{\mathrm{u}}$ be another lift of α. Then the associated corepresentations $(\mathrm{id}_{\hat{C}}\otimes\tilde{\varphi}')\mathcal{V}^C$ and $(\mathrm{id}_{\hat{C}}\otimes\tilde{\varphi})\mathcal{V}^C$ become equal after applying $\mathrm{id}_{\hat{C}}\otimes\Lambda^A$. Hence Lemma 5.12 yields $(\mathrm{id}_{\hat{C}}\otimes\tilde{\varphi}')\mathcal{V}^C=(\mathrm{id}_{\hat{C}}\otimes\tilde{\varphi})\mathcal{V}^C$. Hence the strong quantum group homomorphisms associated to these bicharacters are equal as well, that is, $\tilde{\varphi}=\tilde{\varphi}'$. ☐

Recall that linking unitaries form a category and that duality is a functor on this category. Strong quantum group homomorphisms $A^{\mathrm{u}}\to C^{\mathrm{u}}$ also form the arrows of a category.

Theorem 5.15. *There is an isomorphism of categories between the category of locally compact quantum groups with linking unitaries and with strong quantum group homomorphisms $C^{\mathrm{u}} \to A^{\mathrm{u}}$ as morphisms, respectively. If $\varphi\colon C^{\mathrm{u}} \to A^{\mathrm{u}}$ is a strong quantum group homomorphism, then the associated linking unitary is $(\Lambda_{\hat{C}} \otimes \Lambda_A\varphi)(\mathcal{U}^C) \in \mathcal{UM}(\hat{C} \otimes A)$. Furthermore, the duality on the level of linking unitaries corresponds to the duality $\varphi \mapsto \hat{\varphi}$ on strong quantum group homomorphisms, where $\hat{\varphi}\colon \hat{A}^{\mathrm{u}} \to \hat{C}^{\mathrm{u}}$ is the unique strong quantum group homomorphism with $(\hat{\varphi} \otimes \mathrm{id}_{A^{\mathrm{u}}})(\mathcal{U}^A) = (\mathrm{id}_{\hat{C}^{\mathrm{u}}} \otimes \varphi)(\mathcal{U}^C)$.*

Proof. Propositions 5.1 and 5.13 yield that the map described above is a bijection from strong quantum group homomorphisms $\hat{C} \to \hat{A}$ to linking unitaries from C to A. It remains to check that this bijection preserves the compositions and the duality on both sides. We first turn to the duality because we need this to establish the compatibility with compositions.

Let $\varphi\colon C^{\mathrm{u}} \to A^{\mathrm{u}}$ be a strong quantum group homomorphism. Let $V := (\Lambda_{\hat{C}} \otimes \Lambda_A\varphi)(\mathcal{U}^C) \in \mathcal{UM}(\hat{C} \otimes A)$ be the associated linking unitary. The duality on the level of linking unitaries yields the linking unitary $\sigma(V^*) \in \mathcal{UM}(A \otimes \hat{C})$ from $\hat{A}$ to $\hat{C}$. By transport of structure, this corresponds to a unique strong quantum group homomorphism $\hat{\varphi}\colon \hat{A}^{\mathrm{u}} \to \hat{C}^{\mathrm{u}}$ with $(\Lambda_A \otimes \Lambda_{\hat{C}}\hat{\varphi})(\mathcal{U}^{\hat{A}}) = \sigma(V)^*$. Now we use $\mathcal{U}^{\hat{A}} = \sigma(\mathcal{U}^A)^*$ to rewrite this as

$$(\Lambda_{\hat{C}} \otimes \Lambda_A\varphi)(\mathcal{U}^C) = (\Lambda_{\hat{C}}\hat{\varphi} \otimes \Lambda_A)(\mathcal{U}^A).$$

Both $(\mathrm{id} \otimes \varphi)(\mathcal{U}^C)$ and $(\hat{\varphi} \otimes \mathrm{id})(\mathcal{U}^A)$ are bicharacters. Applying Lemma 5.12 to both tensor factors, we get first $(\mathrm{id}_{\hat{C}^{\mathrm{u}}} \otimes \Lambda_A\varphi)(\mathcal{U}^C) = (\hat{\varphi} \otimes \Lambda_A)(\mathcal{U}^A)$ and then $(\mathrm{id}_{\hat{C}^{\mathrm{u}}} \otimes \varphi)(\mathcal{U}^C) = (\hat{\varphi} \otimes \mathrm{id}_{A^{\mathrm{u}}})(\mathcal{U}^A)$. This yields the asserted description of duality.

Now let $\varphi\colon C^{\mathrm{u}} \to A^{\mathrm{u}}$ and $\psi\colon A^{\mathrm{u}} \to B^{\mathrm{u}}$ be strong quantum group homomorphisms and let $\mathrm{V}^{C\to A} \in \mathcal{UM}(\hat{C}\otimes A)$ and $\mathrm{V}^{A\to B} \in \mathcal{UM}(\hat{A}\otimes B)$ be the corresponding linking unitaries. Then

$$\begin{aligned}
\mathrm{V}^{A\to B} &= (\Lambda_{\hat{A}} \otimes \Lambda_B\psi)\mathcal{U}^A = (\mathrm{id}_{\hat{A}^{\mathrm{u}}} \otimes \Lambda_B\psi)\mathcal{V}^A,\\
\mathrm{V}^{C\to A} &= (\Lambda_{\hat{C}}\hat{\varphi} \otimes \Lambda_A)\mathcal{U}^A. = (\Lambda_{\hat{C}}\hat{\varphi} \otimes \mathrm{id}_{A^{\mathrm{u}}})\mathcal{V}^A.
\end{aligned}$$

where we use the dual quantum group homomorphism $\hat{\varphi}\colon \hat{A}^{\mathrm{u}} \to \hat{C}^{\mathrm{u}}$. Now we compute

$$\begin{aligned}
(\mathrm{V}^{A\to B} * \mathrm{V}^{C\to A})_{13} &= (\mathrm{V}^{C\to A}_{12})^*\mathrm{V}^{A\to B}_{23}\mathrm{V}^{C\to A}_{12}(\mathrm{V}^{A\to B}_{23})^*\\
&= (\Lambda_A\hat{\varphi} \otimes \mathrm{id}_{\mathbb{B}(\mathcal{H}_A)} \otimes \Lambda_B\psi)((\tilde{\mathcal{V}}^A_{12})^*\mathcal{V}^A_{23}\tilde{\mathcal{V}}^A_{12}(\mathcal{V}^A_{23})^*)\\
&= (\Lambda_A\hat{\varphi} \otimes \mathrm{id}_{\mathbb{B}(\mathcal{H}_A)} \otimes \Lambda_B\psi)(\mathcal{U}^A_{13})
\end{aligned}$$

by Proposition 5.7. Thus

$$\begin{aligned}
\mathrm{V}^{A\to B} * \mathrm{V}^{C\to A} &= (\Lambda_A\hat{\varphi} \otimes \Lambda_B\psi)(\mathcal{U}^A) = (\Lambda_A \otimes \Lambda_B\psi) \circ (\hat{\varphi} \otimes \mathrm{id}_{A^{\mathrm{u}}})(\mathcal{U}^A)\\
&= (\Lambda_A \otimes \Lambda_B\psi) \circ (\mathrm{id}_{\hat{C}^{\mathrm{u}}} \otimes \varphi)(\mathcal{U}^C) = (\Lambda_A \otimes \Lambda_B(\psi \circ \phi))(\mathcal{U}^C).
\end{aligned}$$

Hence $\mathrm{V}^{A\to B} * \mathrm{V}^{C\to A}$ is the linking unitary associated to $\psi \circ \phi$. Thus our bijection is compatible with compositions. ☐

6. Comparison with group homomorphism

To illustrate our theory, we consider the special case of groups and their duals.

Let G be a locally compact group with left Haar measure λ. We define a unitary operator $\mathbb{W}_G$ on $L^2(G,\lambda)\otimes L^2(G,\lambda) \cong L^2(G\times G, \lambda\times\lambda)$ by $(\mathbb{W}_G\xi)(x,y) := \xi(x, x^{-1}y)$ for all $\xi \in L^2(G\times G, \lambda\times\lambda)$ and $x, y \in G$. This is a multiplicative unitary that gives rise to $\mathrm{C}^*_{\mathrm{r}}(G)$ (see [6] for details). The corresponding reduced bicharacter $\mathrm{W}_G \in \mathcal{UM}(\mathrm{C}_0(G)\otimes \mathrm{C}^*_{\mathrm{r}}(G))$ is described by the strictly continuous function $G \ni g \mapsto \lambda_g \in$

$\mathcal{UM}(\mathrm{C}^*_\mathrm{r}(G))$. The dual quantum group is $\mathrm{C}_0(G)$ with its usual comultiplication. The universal quantum group attached to $\mathrm{C}_0(G)$ is again $\mathrm{C}_0(G)$.

Let H be another locally compact group. It is easy to see that strong quantum group homomorphisms from $\mathrm{C}_0(H)$ to $\mathrm{C}_0(G)$ are equivalent to continuous group homomorphisms $G \to H$. By Theorem 5.15, it follows that linking unitaries must also correspond to classical group homomorphisms. And the same holds for right and left quantum group homomorphisms. We are now going to establish this directly.

Let $\varphi\colon G \to H$ be a continuous group homomorphism. Then $V_\varphi(g) := \lambda_{\varphi(g)}$ defines a bicharacter in $\mathcal{UM}(\mathrm{C}_0(G) \otimes \mathrm{C}^*_\mathrm{r}(H))$, that is, a quantum group homomorphism from $\mathrm{C}^*_\mathrm{r}(G)$ to $\mathrm{C}^*_\mathrm{r}(H)$.

Lemma 6.1. *Let G and H be locally compact groups. Then every linking unitary from $\mathrm{C}^*_\mathrm{r}(G)$ to $\mathrm{C}^*_\mathrm{r}(H)$ is induced by a unique continuous group homomorphism $\varphi\colon G \to H$ as above.*

Proof. It is clear that the linking unitary V_φ determines φ. Thus it remains to observe that every bicharacter V in $\mathcal{UM}(\mathrm{C}_0(G) \otimes \mathrm{C}^*_\mathrm{r}(H))$ is of this form for a continuous group homomorphism $\varphi\colon G \to H$. We may view V as a strictly continuous function from G to $\mathcal{UM}(\mathrm{C}^*_\mathrm{r}(H))$. Equation (3.7) means that its values are group-like elements of $\mathcal{UM}(\mathrm{C}^*_\mathrm{r}(H))$ for each $g \in G$, that is, $\Delta_{\mathrm{C}^*_\mathrm{r}(H)}(V(g)) = V(g) \otimes V(g)$. This implies $V(g) = \lambda_{\varphi(g)}$ for some $\varphi(g) \in H$. The map $\varphi\colon G \to H$ must be continuous in order for $g \mapsto \lambda_{\varphi(g)}$ to be strictly continuous. Finally, (3.6) means that the map φ is a group homomorphism. □

Example 6.2. Let $C = \mathrm{C}_0(G)$ and $A = \mathrm{C}_0(G_0)$ for locally compact groups G and G_0. A right quantum group homomorphism from C to $C \otimes A$ corresponds to a continuous map $\alpha\colon G \times G_0 \to G$, which we denote as $\alpha(g,h) := g \cdot h$. The commutativity of (4.2) means that $(g_1 \cdot g_2)\cdot h = g_1 \cdot (g_2 \cdot h)$ for all $g_1, g_2 \in G$, $h \in G_0$, so that $g \cdot h = g \cdot \varphi(h)$ for all $g \in G$, $h \in G_0$ for a continuous map $\varphi\colon G_0 \to G$ defined by $\varphi(h) := 1\cdot h$. The commutativity of (4.3) is equivalent to $\varphi(h_1\cdot h_2) = \varphi(h_1)\cdot\varphi(h_2)$ for all $h_1, h_2 \in G_0$. Thus right quantum group homomorphisms $\mathrm{C}_0(G) \to \mathrm{C}_0(G_0)$ correspond to continuous group homomorphisms $G_0 \to G$.

Example 6.3. Let $C = \mathrm{C}^*_\mathrm{r}(G)$ and $A = \mathrm{C}^*_\mathrm{r}(G_0)$ for second countable locally compact groups G and G_0. We claim that right quantum group homomorphisms from $\mathrm{C}^*_\mathrm{r}(G)$ to $\mathrm{C}^*_\mathrm{r}(G_0)$ correspond bijectively to continuous group homomorphisms $G \to G_0$.

Since $C \otimes A = \mathrm{C}^*_\mathrm{r}(G \times G_0)$, a morphism from C to $C \otimes A$ must come from a continuous representation $g \mapsto u_g$ of G by unitary multipliers of $\mathrm{C}^*_\mathrm{r}(G \times G_0)$. To be a right quantum group homomorphism, it suffices to check that the diagrams (4.2) and (4.3) commute on the unitary multipliers δ_g for $g \in G$. The commutativity of (4.2) becomes $(\Delta_C \otimes \mathrm{id}_A)(u_g) = \delta_g \otimes u_g$, forcing $u_g = \delta_g \cdot u'_g$ for unitary multipliers u'_g of A. The commutativity of (4.3) becomes $\Delta_A(u'_g) = u'_g \otimes u'_g$, forcing $u'_g = \delta_{\varphi(g)}$ for some $\varphi(g) \in G_0$. The map $g \mapsto \varphi(g)$ is a measurable group homomorphism. In the separable case, measurability implies continuity, so that φ is a continuous group homomorphism $G \to G_0$.

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