ON THE BASE SIZE OF A TRANSITIVE GROUP WITH SOLVABLE POINT STABILIZER[1]

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We prove that the base size of a transitive group G with solvable point stabilizer is not greater than k provided the same statement holds for every group of G-induced automorphisms of each nonabelian composition factor of G.

Keywords: solvable subgroup, finite simple group, solvable radical.

1 Introduction

The term "group" always means "finite group". We use symbols $A \subseteq G$, $A \leqslant G$, and $A \trianglelefteqslant G$ if A is a subset of G, A is a subgroup of G, and A is a normal subgroup of G, respectively. If Ω is a (finite) set, then by $\mathrm{Sym}(\Omega)$ we denote the group of all permutations of Ω. We also denote $\mathrm{Sym}(\{1, \ldots, n\})$ by Sym_n. Given $H \leqslant G$ we denote by $H_G = \cap_{g \in G} H^g$ the core of H.

Assume that G acts on Ω. An element $x \in \Omega$ is called a *G-regular point* if $|xG| = |G|$, i.e., if the G-orbit of x is regular. Define an action of G on Ω^k by

$$g : (i_1, \ldots, i_k) \mapsto (i_1 g, \ldots, i_k g).$$

If G acts faithfully and transitively on Ω, then the minimal k such that Ω^k possesses a G-regular orbit is called a *base size* of G and is denoted by $\mathrm{Base}(G)$. For every natural m the number of G-regular orbits in Ω^m is denoted by $\mathrm{Reg}(G, m)$ (this number equals 0 if $m < \mathrm{Base}(G)$). If H is a subgroup of G and G acts on the set Ω of right cosets of H by right multiplications, then G/H_G acts faithfully and transitively on Ω. In this case we denote $\mathrm{Base}(G/H_G)$ and $\mathrm{Reg}(G/H_G, m)$ by $\mathrm{Base}_H(G)$ and $\mathrm{Reg}_H(G, m)$ respectively. We also say that $\mathrm{Base}_H(G)$ is the *base size* of G *with respect* to H.

There are a lot of papers dedicated to this subject. We mention only a few the most recent papers, whose subject is very close to this article. In [9] S.Dolfi proved that in every π-solvable group G there exist elements $x, y \in G$ such that the equality $H \cap H^x \cap H^y = O_\pi(G)$ holds, where H is a π-Hall subgroup of G (see also [10]). V.I.Zenkov in [11] constructed an example of a finite group G with a solvable π-Hall subgroup H such that the intersection of five subgroups conjugate with H in G is equal to $O_\pi(G)$, while the intersection of every four conjugates of H is greater than $O_\pi(G)$. In [12] it is proven that if for every finite almost simple group S (possessing a solvable π-Hall subgroup) and for every solvable π-Hall subgroup H of S the inequalities $\mathrm{Base}_H(S) \leqslant 5$ and $\mathrm{Reg}_H(S) \geqslant 5$ hold, then for every finite group G (possessing a solvable π-Hall subgroup) and for every solvable π-Hall subgroup the inequality $\mathrm{Base}_H(G) \leqslant 5$ holds. In the present paper we generalize above mentioned result from [12]. Namely, we prove the following

Theorem 1. *Let G be a finite group and let*

$$\{e\} = G_0 < G_1 < G_2 < \ldots < G_n = G \qquad (1)$$

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be a composition series of G that is a refinement of a chief series. Assume that the following condition **(Orb-solv)** *holds: If G_i/G_{i-1} is nonabelian, then for every solvable subgroup S of $\mathrm{Aut}_G(G_i, G_{i-1})$ we have*

$$\mathrm{Base}_S(\mathrm{Aut}_G(G_i, G_{i-1})) \leqslant k \textit{ and } \mathrm{Reg}_S(\mathrm{Aut}_G(G_i, G_{i-1}), k) \geqslant 5.$$

Then, for every maximal solvable subgroup S of G, we have $\mathrm{Base}_S(G) \leqslant k$.

The example constructed by V.I.Zenkov shows that k in this theorem is at least 5. The author of the paper insert to the "Kourovka notebook" [13] the following problem 17.41.

Problem. Let S be a solvable subgroup of a finite group G with $S(G) = \{e\}$.

(a) (L.Babai, A.J.Goodman, L.Pyber) Does there exists 7 conjugates of S such that their intersection is trivial?

(b) Does there exists 5 conjugates of S such that their intersection is trivial?

Theorem 1 reduces both parts of the Problem to investigation of finite almost simple groups.

2 Notation and preliminary results

By $|G|$ and $|g|$ we denote the cardinality of G and the order of $g \in G$, respectively. By $A : B$, $A^{\cdot}B$, and $A\,.\,B$ we denote a split, a nonsplit, and an arbitrary extension of a group A by a group B. For a group G and a subgroup S of Sym_n by $G \wr S$ we always denote the permutation wreath product. We identify $G \wr M$ with the natural split extension $(G_1 \times \ldots \times G_n) : M$, where $G_1 \simeq \ldots \simeq G_m \simeq G$ and M permutes $G_1, \ldots, G_n$. Given group G, we denote by $\Phi(G)$, $F(G)$, $F^*(G)$, $E(G)$, and $S(G)$ the Frattini subgroup of G, the Fitting subgroup of G, the generalized Fitting subgroup of G, the socle of G, and the maximal normal solvable subgroup of G, respectively. We denote by e the identity element of G.

Let A, B, H be subgroups of G such that $B \trianglelefteqslant A$. Then $N_H(A/B) := N_H(A) \cap N_H(B)$ is the *normalizer* of A/B in H. If $x \in N_H(A/B)$, then x induces an automorphism of A/B by $Ba \mapsto Bx^{-1}ax$. Thus there exists a homomorphism $N_H(A/B) \to \mathrm{Aut}(A/B)$. The image of $N_H(A/B)$ under this homomorphism is denoted by $\mathrm{Aut}_H(A/B)$ and is called a *group of induced automorphisms* of A/B, while the kernel of this homomorphism is denoted by $C_H(A/B)$ and is called the *centralizer* of A/B in H. By definition, $\mathrm{Aut}_H(A) := \mathrm{Aut}_H(A/\{e\})$.

The following statement is evident.

Lemma 2. *If S is a maximal solvable subgroup of G, then $N_G(S) = S$.*

Lemma 3. [7, Lemma 1.2] *Let H be a normal subgroup of a finite group G, $S = (A/H)/(B/H)$ be a composition factor of G/H and L be a subgroup of G.*

Then $\mathrm{Aut}_L(A/B) \simeq \mathrm{Aut}_{LH/H}((A/H)/(B/H))$.

Lemma 4. *Let S be a maximal solvable subgroup of G and let N be a normal subgroup of G containing $S(G)$. Then $N_N(N \cap S) = N \cap S$.*

Proof. Assume that the claim is false and G is a counter example of minimal order. Assume that $S(G) \neq \{e\}$ and consider the natural homomorphism

$$\overline{} : G \to G/S(G).$$

Clearly $\overline{S}$ is a maximal solvable subgroup of $\overline{G}$ and $S(\overline{G}) = \overline{S(G)} = \{e\}$. Moreover, $|\overline{G}| < |G|$. Since G is a counter example of minimal order it follows that $N_{\overline{N}}(\overline{N} \cap \overline{S}) = \overline{N} \cap \overline{S}$. Now $S(G)$ lies in both N

and S, hence $N_N(N \cap S)$ is a complete preimage of $N_{\overline{N}}(\overline{N} \cap \overline{S}) = \overline{N} \cap \overline{S}$, and so $N_N(N \cap S) = N \cap S$. Thus $S(G) = \{e\}$.

Set $M = N_G(N \cap S)$, $L = N_N(N \cap S) = N \cap M$. In view of [2, Proposition 3], $N \cap S \neq \{e\}$, so $S(M) \geqslant S \cap M \neq \{e\}$ and M is a proper subgroup of G. Clearly $S(M) \leqslant S \leqslant M$ and L is normal in M. So $LS(M)$ is normal in M. Since $|M| < |G|$, we obtain

$$N_{LS(M)}(S \cap LS(M))) = S \cap LS(M) = (S \cap L)S(M) \leqslant S.$$

Now suppose that $x \in L$. By construction, $L \cap S = N \cap S$ and $L = N_N(L \cap S)$, so $L \cap S \trianglelefteq L$. Moreover $L \leqslant M$, hence x normalizes $S(M)$, and so x normalizes $(S \cap L)S(M) = N_{LS(M)}(S \cap LS(M)))$, in particular, $x \in S$. Thus $L = S \cap N$. A contradiction with G being counter example. □

Let L be a nonabelian finite simple group and let G be such that there exists a normal subgroup $T = L_1 \times \ldots \times L_n$ of G satisfying the following conditions:

(1) $L_1 \simeq \ldots \simeq L_k \simeq L$;

(2) subgroups $L_1, \ldots, L_k$ are conjugate in G;

(3) $C_G(T) = \{e\}$.

Condition (2) implies that $N_G(L_1), \ldots, N_G(L_k)$ are conjugate in G. We have that G acts on the right cosets of $N_G(L_1)$ by right multiplication, let $\rho : G \to \mathrm{Sym}_n$ be the corresponding permutation representation. Since the action by right multiplication of G on the right cosets of $N_G(L_1)$ coincide with the action by conjugation of G on the set $\{L_1, \ldots, L_n\}$ we obtain that $G\rho$ is a transitive subgroup of Sym_n. By [3, Hauptsatz 1.4, p. 413] there exists a monomorphism

$$\varphi : G \to (N_G(L_1) \times \ldots \times N_G(L_n)) : (G\rho) = N_G(L_1) \wr (G\rho) = M.$$

Since $C_G(L_i)$ is a normal subgroup of $N_G(L_i)$, it follows that $C_G(L_1) \times \ldots \times C_G(L_n)$ is a normal subgroup of M. Consider the natural homomorphism

$$\psi : M \to M/(C_G(L_1) \times \ldots \times C_G(L_n)).$$

Denoting $\mathrm{Aut}_G(L_i) = N_G(L_i)/C_G(L_i)$ by A_i we obtain a homomorphism

$$\varphi \circ \psi : G \to (A_1 \times \ldots \times A_n) : (G\rho) \simeq A_1 \wr (G\rho) =: \overline{G}.$$

The kernel of the homomorphism is equal to $C_G(L_1, \ldots, L_n) = \{e\}$, i. e., $\varphi \circ \psi$ is a monomorphism and we identify G with the subgroup $G(\varphi \circ \psi)$ of $\overline{G}$.

Lemma 5. *Let $T = L_1 \times \ldots \times L_k$ be a normal subgroup of G, and* (1), (2), (3) *are fulfilled. Assume also that G/T is solvable and S is a maximal solvable subgroup of G such that $G = ST$. We identify G, S, and T with their images under $\varphi \circ \psi$. Then $\overline{G}$, defined above, possesses a solvable subgroup $\overline{S}$ with $\overline{S} \geqslant S$ and $\overline{G} = \overline{S}T$.*

Proof. By construction, $A_i = \mathrm{Aut}_{\overline{G}}(L_i) = \mathrm{Aut}_G(L_i) \simeq \mathrm{Aut}_G(L_1)$ for all i. Since $[L_i, L_j] = 1$ for $i \neq j$ and $G = ST$, we obtain that

$$A_i = \mathrm{Aut}_G(L_i) = N_G(L_i)/C_G(L_i) = N_S(L_i)T/C_G(L_i),$$

and so $A_i/L_i \simeq N_S(L_i)/(N_S(L_i) \cap L_iC_G(L_i))$ is solvable. Therefore $\overline{G}/(L_1 \times \ldots \times L_n) \simeq (A_1/L_1) \wr (G\rho)$ is solvable. Denote $S \cap T$ by H, then H is solvable and, by Lemma 4, $N_T(H) = H$. Moreover, if

$H_i = H \cap L_i$, then $N_{L_i}(H_i) = H_i$ (otherwise we would obtain $N_T(H) \neq H$). It follows that A_i is equal to $N_{A_i}(H_i)L_i$ and $N_{A_i}(H_i)$ is solvable. Hence,

$$A_1 \times \ldots \times A_n = (N_{A_1}(H_1) \times \ldots \times N_{A_n}(H_n))T = N_{A_1\times\ldots\times A_n}(H)T$$

and $N_{A_1\times\ldots\times A_n}(H)$ is solvable. Since $\overline{G} = (A_1 \times \ldots \times A_n)S$, and since S normalizes H, it follows $\overline{G} = N_{\overline{G}}(H)T$. Moreover $N_{\overline{G}}(H)$ is solvable and S lies in $N_{\overline{G}}(H)$. □

Lemma 6. *Let G be a transitive subgroup of* Sym_n. *Denote* $\Omega = \{1, \ldots, n\}$. *Let H be the stabilizer of* 1 *in G.*

(a) $(1, i_2, \ldots, i_k)$ *and* $(1, j_2, \ldots, j_k)$ *are in the same G-orbit if and only if* $(i_2, \ldots, i_k)$ *and* $(j_2, \ldots, j_k)$ *are in the same H-orbit;*

(b) *every G-orbit of Ω^k contains an element* $(1, i_2, \ldots, i_k)$*;*

(c) $(1, i_2, \ldots, i_k)$ *is a G-regular point if and only if* $(i_2, \ldots, i_k)$ *is an H-regular point;*

(d) *the number of G-orbits in Ω^k is equal to the number of H-orbits in* $(\Omega \setminus \{1\})^{k-1}$*;*

Proof. (a) Evident.

(b) Follows from the fact that G is transitive.

(c) If $(1, i_2, \ldots, i_k)$ is a G-regular point, then $(1, i_2, \ldots, i_k)g = (1, i_2, \ldots, i_k)$ implies $g = e$. Assume that $h \in H$ is chosen so that $(i_2, \ldots, i_k)h = (i_2, \ldots, i_k)$. Since H is the stabilizer of 1, it follows that $(1, i_2, \ldots, i_k)h = (1, i_2, \ldots, i_k)$, hence $h = e$ and $(i_2, \ldots, i_k)$ is an H-regular point. Conversely, if $(i_2, \ldots, i_k)$ is an H-regular point and $(1, i_2, \ldots, i_k)g = (1, i_2, \ldots, i_k)$, we obtain $g \in H$, and $(i_2, \ldots, i_k)g = (i_2, \ldots, i_k)$, hence $g = e$ and $(1, i_2, \ldots, i_k)$ is a G-regular point.

(d) Clear from (a), (b) and (c). □

Recall that G is called almost simple if there exists a nonabelian simple group L such that $L \simeq \mathrm{Inn}(L) \leqslant G \leqslant \mathrm{Aut}(L)$.

Let G be a subgroup of Sym_n. A partition $\{P_1, P_2, \ldots, P_m\}$ of $\{1, \ldots, n\}$ is called an *asymmetric partition* for G, if only the identity element of G fixes the partition, i. e., the equality $P_j x = P_j$ for all $j = 1, \ldots, m$ implies $x = e$. Clearly for every G the partition $P_1 = \{1\}, P_2 = \{2\}, \ldots, P_n = \{n\}$ is always asymmetric.

Lemma 7. [6, Theorem 1.2] *Let G be a solvable group of permutations of* $\{1, 2, \ldots, n\}$. *Then there exists an asymmetric partition* $\{P_1, P_2, \ldots, P_m\}$ *of this set with* $m \leqslant 5$.

Lemma 8. *Let G be a finite group and let M be a solvable subgroup of* Sym_n. *Assume that for every maximal solvable subgroup S of G the inequalities*

$$\mathrm{Base}_S(G) \leqslant k \textit{ and } \mathrm{Reg}_S(G, k) = s \geqslant 5$$

hold. Then, for every maximal solvable subgroup L of $G \wr M$ we have $\mathrm{Base}_L(G \wr M) \leqslant k$. *Moreover*

$$\mathrm{Reg}_L(G \wr M, k) \geqslant s.$$

Proof. We have $G \wr M = (G_1 \times \ldots \times G_n) : M$. Moreover $S(G \wr M) = S(G_1) \times \ldots \times S(G_n)$, since $C_M(G_1 \times \ldots \times G_n) = \{e\}$. Assume by contradiction that $G \wr M$ is a counter example to the lemma with $|G \wr M|$ minimal. Then clearly $S(G \wr M) = \{e\}$, i.e., $S(G) = \{e\}$, otherwise we substitute G by $G/S(G)$ and proceed by induction.

Since $G \wr M$ is a counter example to the lemma, there exists a maximal solvable subgroup S of $G \wr M$ such that for every $x_1, \ldots, x_k \in G \wr M$ we have $S^{x_1} \cap \ldots \cap S^{x_k} \neq \{e\}$. It is clear that $(G_1 \times \ldots \times G_n)S = G \wr M$, otherwise consider the image $\overline{S}$ of S under the natural homomorphism $G \wr M \to G \wr M/(G_1 \times \ldots \times G_n)$. We obtain that $(G_1 \times \ldots \times G_n)S \simeq G \wr \overline{S}$, so we substitute $G \wr M$ by $G \wr \overline{S}$ and proceed by induction. The fact that $G \wr M$ is a minimal counter example implies also that M is transitive, otherwise we would obtain that $G \wr M \leqslant (G \wr M_1) \times (G \wr M_2)$ and proceed by induction. Indeed denote the projections of $G \wr M$ onto $G \wr M_1$ and $G \wr M_2$ by π_1 and π_2 respectively. Up to renumbering we may suppose that there exists m such that $G \wr M_1 = (G_1 \times \ldots \times G_m) : M_1$ and $G \wr M_2 = (G_{m+1} \times \ldots \times G_n) : M_2$. Denote $G_1 \times \ldots \times G_m$ by E_1 and $G_{m+1} \times \ldots \times G_n$ by E_2. Since $G \wr M = (G_1 \times \ldots \times G_n)S$, $E_1 \leqslant \mathrm{Ker}(\pi_2)$ and $E_2 \leqslant \mathrm{Ker}(\pi_1)$, it follows that $(G \wr M)\pi_i = E_i(S\pi_i)$ (we identify $E_i\pi_i$ with E_i, since $E_i\pi_i \simeq E_i$). Then, by induction for each $i \in \{1, 2\}$ there exist elements $x_{1,i}, \ldots, x_{k,i}$ of $E_i(S\pi_i)$ such that

$$(S\pi_i)^{x_{1,i}} \cap \ldots \cap (S\pi_i)^{x_{k,i}} = \{e\}. \quad (2)$$

Since $G\pi_i = E_i(S\pi_i)$, we may assume that $x_{1,i}, \ldots, x_{k,i}$ are in E_i. Consider $x_1 = x_{1,1}x_{1,2}, \ldots, x_k = x_{k,1}x_{k,2}$. Since (2) is true for every i, for elements $x_1, \ldots, x_k$ we have

$$S^{x_1} \cap \ldots \cap S^{x_k} = \{e\},$$

and G is not a counter example.

Consider $L = S \cap G_1 \times \ldots \times G_n$ and denote by π_i the natural projection $G_1 \times \ldots \times G_n \to G_i$. Put $L_i = L^{\pi_i}$. Clearly $L \leqslant L_1 \times \ldots \times L_n$. If $x \in S$ and $G_i^x = G_j$, then $L_i^x = L_j$, since L is normal in S. Hence S normalizes $L_1 \times \ldots \times L_n$ and so $L = L_1 \times \ldots \times L_n$, by the maximality of S.

Clearly $N_{G_1 \times \ldots \times G_n}(L_1 \times \ldots \times L_n) = N_{G_1}(L_1) \times \ldots \times N_{G_n}(L_n)$. By Lemma 4 we obtain that $N_{G_1 \times \ldots \times G_n}(L_1 \times \ldots \times L_n) = L_1 \times \ldots \times L_n$, hence $N_{G_i}(L_i) = L_i$ for $i = 1, \ldots, n$. Denote by Ω_i the set $\{L_i^x \mid x \in G_i\}$, then G_i acts on Ω_i by conjugation. Since $N_{G_i}(L_i) = L_i$, it follows that L_i is the point stabilizer under this action. Set $\Omega = \Omega_1 \times \ldots \times \Omega_n$. For every $x \in G \wr M$ and for every i we have $L_i^x \leqslant G_j$ for some j. We show that $L_i^x \in L_j^{G_j}$, i.e., there exists $y \in G_j$ such that $L_j^y = L_i^x$. Since $(G_1 \times \ldots \times G_n) : M = (G_1 \times \ldots \times G_n)S$, it follows that there exists $s \in S$ with $G_i^s = G_j$. We also have $L_i^s = L_j$, since L is normal in S. Thus $L_i^x = L_j^{s^{-1}x}$. Now $s^{-1}x = g_1 \cdot \ldots \cdot g_n \cdot h$, where $g_i \in G_i$ for $i = 1, \ldots, n$ and $h \in M$. Since M permutes the G_i-s, it follows that for every $i = 1, \ldots, n$, either $G_i^h \cap G_i = \{e\}$, or h centralizes G_i. Thus we obtain that $L_j^{s^{-1}x} = L_j^{g_j}$. So $G \wr M$ acts by conjugation on Ω and S is the stabilizer of the point $(L_1, \ldots, L_n)$. Therefore we need to show that Ω^k possesses at least s $(G \wr M)$-regular orbits.

Now there exist G_1-regular points $\omega_1, \ldots, \omega_s \in \Omega_1^k$ lying in distinct G_1-orbits. If we choose $h_1 = e, h_2, \ldots, h_n \in M$ so that $G_1^{h_i} = G_i$, then $\omega_1^{h_i}, \ldots, \omega_s^{h_i} \in \Omega_i^k$ are G_i-regular points, and, as we noted above, they are in distinct G_i-orbits. We set $\omega_{i,j} = \omega_i^{h_j}$. By Lemma 7 there exists an asymmetric partition $P_1 \sqcup P_2 \sqcup P_3 \sqcup P_4 \sqcup P_5 = \{1, \ldots, n\}$ for M. If we choose $\omega = (\omega_{i_1,1}, \ldots, \omega_{i_n,n})$ so that $i_i = i_j$ if and only if i, j lie in the same P_l, then ω is a $(G \wr M)$-regular point in Ω^k. Clearly we can choose such ω, simce $s \geqslant 5$. Indeed, consider $g = (g_1 \ldots g_n)h$, where $g_i \in G_i$ for $i = 1, \ldots, n$ and $h \in M$, and assume that $\omega g = \omega$. It follows that $\omega h^{-1} = \omega(g_1 \ldots g_n)$. We obtain that $\omega_{i,j}h^{-1} = \omega_{i,jh^{-1}} = \omega_{m,jh^{i-1}}g_{jh^{i-1}}$ for some m. Since for every $t = 1, \ldots, n$ points $\omega, \omega' \in \Omega_t$ are in the same $G \wr M$-orbit if and only if ω, ω' are in the same G_t-orbit, it follows that $i = m$. By construction, j and jh^{-1} lie in the same P_l, hence h^{-1} stabilizes the asymmetric partition $P_1 \sqcup P_2 \sqcup P_3 \sqcup P_4 \sqcup P_5$, and so $h^{-1} = h = e$ and $g \in G_1 \times \ldots \times G_n$. By construction, ω is a $G_1 \times \ldots \times G_n$-regular point, i.e., $g = e$. Moreover, distinct point constructed in this way are in distinct $(G \wr M)$-regular orbits. Clearly we can construct at least s points in this way (at least one of P_1, P_2, P_3, P_4, P_5 is nonempty) and the lemma follows. □

3 Proof of the main theorem

Assume that the claim is false and G is a counter example of minimal order.

Assume that $S(G) \neq 1$. Then there exists a minimal elementary abelian normal subgroup K of G. Since elements from distinct minimal normal subgroups commute, we may suppose that $G_1 \leqslant K$ and there exists l such that $G_l = K$, i.e., composition series (1) is a refinement of a chief series starting with K. In this case, if

$$\overline{} : G \to G/K = \overline{G}$$

is a natural homomorphism, then

$$\{\overline{e}\} = \overline{G}_l < \overline{G}_{l+1} < \ldots < \overline{G}_n = \overline{G}$$

is a composition series of $\overline{G}$ that is a refinement of a chief series of $\overline{G}$. Moreover, for every non-abelian $\overline{G}_i/\overline{G}_{i-1}$, Lemma 3 implies $\mathrm{Aut}_{\overline{G}}(\overline{G}_i/\overline{G}_{i-1}) \simeq \mathrm{Aut}_G(G_i/G_{i-1})$, and so $\mathrm{Aut}_{\overline{G}}(\overline{G}_i/\overline{G}_{i-1})$ satisfies **(Orb-solv)**. Thus $\overline{G}$ satisfies conditions of the theorem. In view of the minimality of G, there exist $x_1, \ldots, x_k \in G$ such that

$$\overline{S}^{\bar{x}_1} \cap \ldots \cap \overline{S}^{\bar{x}_k} = S(\overline{G}).$$

Now $K \leqslant S(G)$, hence $\overline{S(G)} = S(\overline{G})$. Therefore $S^{x_1} \cap \ldots \cap S^{x_k} = S(G)$ and the claim holds, i.e., G is not a counter example.

Thus we may assume that $S(G) = \{e\}$. Consider the generalized Fitting subgroup $F^*(G)$ of G. Since $S(G) = \{e\}$, we obtain that $F^*(G) = E(G) = L_1 \times \ldots \times L_n$ is a product of nonabelian simple groups and, by [1, Theorem 9.8], $C_G(F^*(G)) = Z(F^*(G)) = \{e\}$. In particular, $S(E(G)S) = \{e\}$. If $E(G)S \lneq G$, then, in view of the minimality of G, there exist $x_1, \ldots, x_k \in E(G)S$ such that $S^{x_1} \cap \ldots \cap S^{x_k} = S(E(G)S) = \{e\}$. Thus the claim holds in this case, i.e., G is not a counter example. It follows that $G = E(G)S$. Moreover, since $L_1, \ldots, L_n$ are nonabelian simple, [5, Theorem 3.3.10] implies that G, acting by conjugation, interchanges the elements of $\{L_1, \ldots, L_n\}$.

Set $E_1 := \langle L_1^S \rangle$. Since $E(G) = L_1 \times \ldots \times L_k$, we obtain that $E(G) = E_1 \times E_2$, where E_1 and E_2 are S-invariant subgroups. By Remak theorem [4, Theorem 4.3.9] there exists a homomorphism $G \to G/C_G(E_1) \times G/C_G(E_2)$, such that the image of G is a subdirect product of $G/C_G(E_1)$ and $G/C_G(E_2)$, while the kernel is equal to $C_G(E_1) \cap C_G(E_2) = C_G(E(G)) = \{e\}$. Denote the projections of G onto $G/C_G(E_1)$ and $G/C_G(E_2)$ by π_1 and π_2 respectively. Since $G = E(G)S$, $E_1 \leqslant \mathrm{Ker}(\pi_2)$ and $E_2 \leqslant \mathrm{Ker}(\pi_1)$, it follows that $G\pi_1 = E_1(S\pi_1)$ and $G\pi_2 = E_2(S\pi_2)$ (we identify $E_i\pi_i$ and E_i since $E_i\pi_i \simeq E_i$).

Suppose that $E_1 \neq E(G)$. Then, by induction for each $i \in \{1, 2\}$ there exist elements $x_{1,i}, \ldots, x_{k,i}$ of $E_i(S\pi_i)$ such that

$$(S\pi_i)^{x_{1,i}} \cap \ldots \cap (S\pi_i)^{x_{k,i}} = \{e\}. \tag{3}$$

Since $G\pi_i = E_i(S\pi_i)$, we may assume that $x_{1,i}, \ldots, x_{k,i}$ are in E_i. Consider $x_1 = x_{1,1}x_{1,2}, \ldots, x_k = x_{k,1}x_{k,2}$. Since (3) is true for every i, for elements $x_1, \ldots, x_k$ we have

$$S^{x_1} \cap \ldots \cap S^{x_k} = \{e\},$$

and G is not a counter example.

Therefore $E_1 = E(G)$ and S acts transitively on $\{L_1, \ldots, L_n\}$. Since $\mathrm{Aut}_G(L_1)$ satisfies **(Orb-solv)**, we may assume that $m > 1$. By Lemma 5, we may assume that $G = (A_1 \times \ldots \times A_k) : K = A_1 \wr K$, where $A_i = \mathrm{Aut}_G(L_i)$ and $K = G\rho = S\rho \leqslant \mathrm{Sym}_n$ (in particular, K is solvable). Lemma 8 implies that $\mathrm{Base}_S(G) \leqslant k$ for every maximal solvable subgroup S of G. This final contradiction completes the proof.

4 Final notes

In this section we discuss the meaning of $\mathrm{Reg}_S(G,k)$ and lower bounds for $\mathrm{Reg}_S(G,k)$, where S is a solvable subgroup of G and $k \geqslant \mathrm{Base}_S(G)$. If we have a group G and a solvable subgroup S of G, then Theorem 1 gives us an idea, how to find $\mathrm{Base}_S(G)$, or, at least, how to find an upper bound for $\mathrm{Base}_S(G)$. However, for computation purposes one need to find also the base of G with respect to S, i.e., elements $x_1,\dots,x_k$ such that $S^{x_1}\cap\ldots\cap S^{x_k} = S_G$. In general there is no way to find these elements and we can suggest just a probabilistic approach in this direction. Denote by Ω the set of right cosets of S in G. If one knows that $\mathrm{Reg}_S(G,k)\geqslant s$ and $|G:S| = |\Omega| = n$, then $|\Omega^k| = n^k$, while Ω^k possesses at least $s|G/S_G|$ regular points. So the probability that k randomly chosen elements from Ω form a base of G with respect to S is not less than

$$\varepsilon = \frac{s\cdot|G/S_G|}{n^k} \geqslant \frac{s}{n^{k-1}}.$$

In Theorem 1 the condition **(Orb-solv)** demands that

$$\mathrm{Base}_S(\mathrm{Aut}_G(G_i,G_{i-1})) \leqslant k \text{ and } \mathrm{Reg}_S(\mathrm{Aut}_G(G_i,G_{i-1}),k) \geqslant 5.$$

We show that if $k\geqslant 6$, then we can guarantee that $\mathrm{Reg}_S(\mathrm{Aut}_G(G_i,G_{i-1}),k)\geqslant 5$. More precisely, the following lemma holds.

Lemma 9. *Let G be a transitive permutation group acting on $\Omega=\{1,\dots,n\}$ and let the stabilizer S of 1 be solvable. Assume that $k=\max\{\mathrm{Base}(G),6\}$. Then $\mathrm{Reg}(G,k)\geqslant 5$.*

Proof. In view of Lemma 6, we have that S acts on $\Theta=\Omega\setminus\{1\}$ and the number of G-regular orbits on Ω^k is equal to the number of S-regular orbits on Θ^{k-1}. Thus we need to prove that $\mathrm{Reg}(S,k-1)\geqslant 5$, where S acts on Θ. By the conditions of the lemma there exists $\theta_1,\dots,\theta_{k-1}\in\Theta$ such that $(\theta_1,\dots,\theta_{k-1})$ is an S-regular point in Θ^{k-1}.

Consider $\Delta=\{\theta_1,\dots,\theta_{k-1}\}$, let T be the stabilizer of Δ in S, i.e., $T=\{x\in S \mid \Delta x=\Delta\}$. It is clear that $(\theta_{1\sigma},\dots,\theta_{(k-1)\sigma})$ is an S-regular point for every $\sigma\in\mathrm{Sym}_{k-1}$. Moreover if $\sigma,\tau\in\mathrm{Sym}_{k-1}$, then $(\theta_{1\sigma},\dots,\theta_{(k-1)\sigma})$ and $(\theta_{1\tau},\dots,\theta_{(k-1)\tau})$ are in the same S-orbit if and only if there exists $x\in T$ such that $(\theta_{1\sigma},\dots,\theta_{(k-1)\sigma})^x=(\theta_{1\tau},\dots,\theta_{(k-1)\tau})$. Consider the restriction homomorphism $\varphi: T\to\mathrm{Sym}(\Delta)$. Since $(\theta_1,\dots,\theta_{k-1})$ is an S-regular point (and so a T-regular point), it follows that $Ker(\varphi)=\{e\}$, i.e., φ is injective.

Assume that $k\geqslant 9$ first. Consider the asymmetric partition $P_1\sqcup P_2\sqcup P_3\sqcup P_4\sqcup P_5=\{\theta_1,\theta_2,\dots,\theta_{k-1}\}$ for T^φ. Without loss of generality we may assume that $|P_1|\geqslant|P_2|\geqslant|P_3|\geqslant|P_4|\geqslant|P_5|$. Since $k\geqslant 9$ it follows that either $|P_1|\geqslant 3$, or $|P_1|=|P_2|=|P_3|=2$.

If $|P_1|\geqslant 3$, then, up to renumbering, we may assume that $\theta_1,\theta_2,\theta_3\in P_1$. In this case for every distinct $\sigma,\tau\in\mathrm{Sym}_3$ we have that $(\theta_{1\sigma},\theta_{2\sigma},\theta_{3\sigma},\theta_4\dots,\theta_{k-1})$ and $(\theta_{1\tau},\theta_{2\tau},\theta_{3\tau},\theta_4,\dots,\theta_{k-1})$ are in distinct T^φ-orbits, thus these points are in distinct T-orbits, and so in distinct S-orbits. So $\mathrm{Reg}(S,k-1)\geqslant |\,\mathrm{Sym}_3\,|=6$ in this case.

If $|P_1|=|P_2|=|P_3|=2$, then, up to renumbering, we may assume that $\theta_1,\theta_2\in P_1$, $\theta_3,\theta_4\in P_2$, and $\theta_5,\theta_6\in P_3$. In this case for every distinct $\sigma,\tau\in\mathrm{Sym}(\{1,2\})\times\mathrm{Sym}(\{3,4\})\times\mathrm{Sym}(\{5,6\})$ we have that

$$(\theta_{1\sigma},\theta_{2\sigma},\theta_{3\sigma},\theta_{4\sigma},\theta_{5\sigma},\theta_{6\sigma},\theta_7\dots,\theta_{k-1}) \text{ and } (\theta_{1\tau},\theta_{2\tau},\theta_{3\tau},\theta_{4\tau},\theta_{5\tau},\theta_{6\tau},\theta_7\dots,\theta_{k-1})$$

are in distinct T^φ-orbits, thus these points are in distinct T-orbits, and so in distinct S-orbits. So $\mathrm{Reg}(S,k-1)\geqslant|\,\mathrm{Sym}(\{1,2\})\times\mathrm{Sym}(\{3,4\})\times\mathrm{Sym}(\{5,6\})|=8$ in this case.

Now assume that $6 \leqslant k \leqslant 8$. Denote by Ξ the subset $\{(\theta_{1\sigma}, \ldots, \theta_{(k-1)\sigma}) \mid \sigma \in \mathrm{Sym}_{k-1}\}$ of Δ^{k-1}. Then T^φ acts on Ξ and every point of Ξ is T^φ-regular. Moreover $|\Xi| = |\,\mathrm{Sym}_{k-1}| = (k-1)!$. We also have that T^φ is a solvable subgroup of Sym_{k-1}. It is immediate (from [8], for example), that $|T^\varphi| \leqslant 12$ for $k = 6$, $|T^\varphi| \leqslant 36$ for $k = 7$, and $|T^\varphi| \leqslant 72$ for $k = 8$. Now the number of T^φ-orbits on Ξ is equal $\frac{(k-1)!}{|T^\varphi|}$ and direct computations show that this number is at least 10. □

References

[1] *I.M. Isaacs*, Finite group theory. AMS Providence, RI: Amer. Math. Soc., 2008. 351 p.

[2] *R.M. Guralnick, J.S. Wilson*, The probability of generating a finite soluble group, Proc. London Math. Soc. (3), v. 81 (2000), 405–427.

[3] *B. Huppert*, Endliche Gruppen I. Berlin: Springer, 1967. 808 p.

[4] *M.I. Kargapolov, Yu.I. Merzlyakov,* Foundations of the theory of groups. M.: Nauka. 1982. 288 p (In Russian).

[5] *D.J.S. Robinson*, A course in the theory of groups. (Springer, 1996).

[6] *A. Seress*, The minimal base size of primitive solvable permutation groups, J. London Math. Soc., v. 53 (1996), 243–255.

[7] *E.P. Vdovin*, Carter subgroups of finite almost simple groups, Algebra and Logic, v. 46 (2007), 90–119.

[8] *J.H.Conway, R.T.Curtis, S.P.Norton, R.A.Parker, R.A.Wilson, Atlas of Finite Groups,* Clarendon Press, Oxford, 1985.

[9] *S. Dolfi*, Large orbits in coprime actions of solvable groups, Trans. AMS, 2008. v. 360 (2008), 135–152.

[10] *E.P. Vdovin*, Regular orbits of solvable linear p'-groups, Sib. Electr. Math. Reports, v. 4 (2007), 345–360.

[11] *V.I. Zenkov*, On the intersections of solvable Hall subgroups in finite nonsolvable groups, Tr. In-ta Matematiki i meckhaniki UrO RAN, v. 13 (2007), 86–89.

[12] *E.P.Vdovin, V.I.Zenkov*, On the intersection of solvable Hall subgroups in finite groups, Proc. Stekl. Inst. Math. Suppl. 3, 2009, 234–243.

[13] *The Kourovka notebook. Unsolved problems in group theory*. Edited by V. D. Mazurov and E. I. Khukhro. 17-th. ed., Russian Academy of Sciences Siberian Division, Institute of Mathematics, Novosibirsk, 2010.