Maximum path information and Fokker-Planck Equation

W. Li[†‡*], Q.A. Wang[‡], A. Le Méhauté[‡]

[†]Complexity Science Center and Institute of Particle Physics,
Hua-Zhong Normal University, Wuhan 430079, P.R. China

[‡]Institut Supérieur des Matériaux du Mans,
44, Avenue F.A. Bartholdi, 72000 Le Mans, France

Abstract

We present in this paper a rigorous method to derive the nonlinear Fokker-Planck (FP) equation of anomalous diffusion directly from a generalization of the principle of least action of Maupertuis proposed by Wang [1] for smooth or quasi-smooth irregular dynamics evolving in Markovian process. The FP equation obtained may take two different but equivalent forms. It was also found that the diffusion constant may depend on both q (the index of Tsallis entropy [2]) and the time t.

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The Fokker-Planck equation is a differential equation describing the time evolution of probability distribution of state during stochastic processes. The FP equation and its generalizations play very crucial roles in statistical physics. The FP equation is not only applicable to the systems near the thermal equilibrium, but to the systems far from the thermal equilibrium as well. This latter application has special meaning in dealing with a large class of self-organized, complex dynamical systems. In this sense, the FP equation not only describes stationary properties but also the dynamics of evolving systems.

*Corresponding author. Electronic address: liw@phy.ccnu.edu.cn. Tel: 86-27-6786-7046.

The FP equation was first derived by Fokker [3] and [4] as one to describe Brownian motion. Later on, many books and review articles were published [5, 6, 7, 8, 9]. The usual way of deriving the FP equation starts from calculating the transition probability $P(x, t+\tau|x', t)$ for small τ, where the particle travels from x to x'. There are various ways to derive the expression of the transition probability [10]. In this paper, we employ the concept of maximum path information, related to non-extensive Tsallis entropy, to derive the expression of the transition probability for the motion of particle under the influence of external forces. On the basis of the transition probability, an nonlinear FP equation can be obtained.

A path information based on Shannon entropy [11] has been defined as [1]

$$H_s(a,b) = -\sum_{i=1}^{w} p_{ab}(i) \ln p_{ab}(i), \tag{1}$$

where $p_{ab}(i)$ is called the transition probability that a system moving from point a to point b will choose path i among all possibly existed w paths. In the case of non-extensive statistics, the corresponding path information naturally adopts the formula of Tsallis entropy [2]

$$H_t(a,b) = -k\sum_{i=1}^{w} \frac{p_{ab}(i) - p_{ab}^q(i)}{1-q}, \tag{2}$$

where q, the entropy index, specifies a particular statistics. In general, the larger the path information, the less we know about paths states of the system.

Of course, the transition probability $p_{ab}(i)$ satisfies the following normalization condition

$$\sum_{i=1}^{w} p_{ab}(i) = 1. \tag{3}$$

For classical dynamical systems we also suppose each possible path is characterized by its action $A_{ab}(i)$

$$A_{ab}(i) = \int_{t_{ab}} L_i(t)dt, \tag{4}$$

where $L_i(t)$ is the Lagrangian of the system at time t via the path i. The average action is represented by

$$\langle A_{ab}\rangle = \sum_{i=1}^{w} A_{ab}(i)p_{ab}(i). \tag{5}$$

In order to obtain the form of path probability, we seek to optimize the path information $H_t(a,b)$ under the constraints of Eqs. (3) and (5). That is,

$$\delta(H_t(a,b) + \alpha\sum_{i=1}^{w} p_{ab}(i) + \eta\sum_{i=1}^{w} A_{ab}(i)p_{ab}(i)) = 0 \tag{6}$$

Through a simple algebra, the optimization yields the following expression of path probability

$$p_{ab}(i) = \frac{1}{Z_q}[1-(1-q)\eta A_{ab}(i)]^{\frac{1}{1-q}}, \tag{7}$$

where $Z_q = \sum_{i=1}^{w}[1-(1-q)\eta A_{ab}(i)]^{\frac{1}{1-q}}$.

In order to obtain a general derivation of FP equation at the existence of any form of external forces (drifts), we adopt here the Euler's method to calculate the action. The detailed method is as follows. The path through which the particle travels from point a to point b is cut into N segments with each having a spatial length $\Delta x_k = (x_k - x_{k-1})$ $(k = 1, 2, ..., N)$. $t = t_k - t_{k-1}$ is the time interval spent by the system on every segment. According to the theorem of large numbers, the fluctuation of calculation will go to 0 as N approaches infinity. The action A_k on the segment k is simply

$$A_k = \frac{m(\Delta x_k)^2}{2t} + \frac{\Delta x_k}{2}F_k - U(x_{k-1})t, \tag{8}$$

where $F_k = -(\frac{\partial U}{\partial x})_k$ and $U(x_{k-1})$ is the potential energy at the point x_{k-1}. Here in this paper F_k and $U(x_{k-1})$ will be considered as constant. From now on, we will write $U(x_{k-1})$ as U for simplicity.

By using Eq. (7) the transition probability $p_{k/k-1}$ from point $k-1$ to point k via the path i can be written as

$$\begin{aligned} p_{k/k-1}(i) &= \frac{1}{Z_q(k,k-1)}\{1-(1-q)\eta[\frac{m(\Delta x_k)^2}{2t} \\ &\quad +\frac{F_k t}{2}\Delta x_k - Ut]\}^{\frac{1}{1-q}}, \end{aligned} \tag{9}$$

where $Z_q(k,k-1)$ can be calculated from the normalization condition $\int_{-\infty}^{+\infty} p_{k/k-1}(i)dx_k = 1$

$$\begin{aligned} Z_q(k,k-1) &= \int_{-\infty}^{+\infty} dx_k\{1-(1-q)\eta[\frac{m(\Delta x_k)^2}{2t} \\ &\quad +\frac{F_k t}{2}\Delta x_k - Ut]\}^{\frac{1}{1-q}}. \end{aligned} \tag{10}$$

Introducing the methods by Tsallis and Prato [12, 13], after a tedious calculation, we obtain the exact form of $Z_q(k,k-1)$

$$Z_q(k,k-1) = A(q)\sqrt{\frac{2\pi t}{\eta m}}[1-(q-1)\eta(\frac{F_k^2 t^3}{8m}+Ut)]^{\frac{q-3}{2q-2}}, \qquad (11)$$

where $A(q)$ can be written as

$$A(q) = \begin{cases} \Gamma^{-1}(\frac{1}{q-1})\Gamma(\frac{q-3}{2q-2})\sqrt{\frac{1}{q-1}} & , \quad q>1; \\ \Gamma(\frac{2-q}{1-q})\Gamma^{-1}(\frac{3-q}{2-2q})\sqrt{\frac{1}{1-q}} & , \quad 0<q<1. \end{cases}$$

It is not difficult to prove that $Z_q(k,k-1)$ restores to $Z_1(k,k-1)$ at the $q \to 1$ limit, which is

$$Z_1(k,k-1) = exp[\eta(\frac{F_k^2 t^3}{8m}+Ut)]\sqrt{\frac{2\pi t}{\eta m}}. \qquad (12)$$

Hence the transition probability $p_{k/k-1}(i)$ has the form

$$\begin{aligned} p_{k/k-1}(i) &= B(q)t^{-1/2}[1-(q-1)\eta(\frac{F_k^2 t^3}{8m}+Ut)]^{\frac{3-q}{2q-2}}\{1- \\ &\quad (1-q)\eta[\frac{m(\Delta x_k)^2}{2t}+\frac{F_k t}{2}\Delta x_k - Ut]\}^{\frac{1}{1-q}}, \end{aligned} \qquad (13)$$

where $B(q) = A^{-1}(q)\sqrt{\frac{m\eta}{2\pi}}$.

Now we are ready to derive the FP equation for the system travelling through the k-th segment of path i connecting points a and b. It is readily that

$$\begin{aligned} \frac{\partial p_{k/k-1}(i)}{\partial t} &= p_{k/k-1}(i)\{-\frac{1}{2}t^{-1} + A_1^{-1}\frac{(q-3)\eta}{2}(\frac{3F_k^2 t^2}{8m} + \\ &\quad U) + A_2^{-1}\eta[\frac{m}{2t^2}(\Delta x_k)^2 - \frac{F_k \Delta x_k}{2} + U]\}, \end{aligned} \qquad (14)$$

where $A_1 = 1-(q-1)\eta(\frac{F_k^2 t^3}{8m}+Ut)$, and $A_2 = 1-(1-q)\eta[\frac{m}{2t}(\Delta x_k)^2 + \frac{F_k \Delta x_k}{2}t - Ut]$. We also have

$$\frac{\partial p_{k/k-1}(i)}{\partial x_k} = -p_{k/k-1}(i)A_2^{-1}\eta(mt^{-1}\Delta x_k + \frac{F_k t}{2}), \qquad (15)$$

and

$$\frac{\partial^2[p_{k/k-1}(i)]^\gamma}{\partial x_k^2} = -\gamma\eta A_2^{-1}[p_{k/k-1}(i)]^\gamma[mt^{-1}-(\gamma-1+q) \times\eta A_2^{-1}(mt^{-1}\Delta x_k+\frac{F_k t}{2})^2], \tag{16}$$

where γ is a constant that might depend on q.

Combining the equations (14) and (15), one obtains the following expression

$$(\frac{\partial}{\partial t}+F_k\frac{\partial}{\partial x_k})p_{k/k-1}(i) = -\frac{p_{k/k-1}(i)}{2m}[u_1(x_k,t,q)+v(x_k,t,q)], \tag{17}$$

where

$$u_1(x_k,t,q) = mt^{-1}-\eta A_2^{-1}(mt^{-1}\Delta x_k+\frac{F_k t}{2})^2 \tag{18}$$

and

$$v(x_k,t,q) = (\frac{3F_k^2t^2}{8m}+U)(3-q)\eta m A_1^{-1}+2\eta m A_2^{-1}(\frac{F_k^2 t}{2} +\frac{F_k^2t^2}{8m}+\frac{mF_k\Delta x_k}{t}+F_k\Delta x_k-U). \tag{19}$$

Writing (16) in another form one gets

$$\frac{\partial^2[p_{k/k-1}(i)]^\gamma}{\partial x_k^2} = -\gamma\eta A_2^{-1}[p_{k/k-1}(i)]^\gamma u_2(x,t,q,\gamma), \tag{20}$$

where

$$u_2(x_k,t,q,\gamma) = mt^{-1}-(\gamma-1+q)\eta A_2^{-1}(mt^{-1}\Delta x_k+\frac{F_k t}{2})^2. \tag{21}$$

It is obvious that $u_2(x_k,t,q,2-q) = u_1(x_k,t,q)$.

Relating Eqs.(17) and (20), together with $u_2(x_k,t,q,2-q) = u_1(x_k,t,q)$, one obtains the following equation

$$(\frac{\partial}{\partial t}+F_k\frac{\partial}{\partial x_k})p_{k/k-1}(i) = D(q,t)\frac{\partial^2[p_{k/k-1}(i)]^{2-q}}{\partial x_k^2}, \tag{22}$$

where

$$D(q,t) = \frac{1}{2\eta m}B^{q-1}A_1^{(3-q)/2}t^{(1-q)/2}[1-\frac{v(x_k,t,q)}{u_1(x_k,t,q)(2-q)}]. \tag{23}$$

One can check that $D(1,t) = 1/2\eta m$, which is consistent with the results in [1]. Apparently, Eq. (22) is the exact FP equation for the system in an infinitesimally interval in the existence of external forces.

Besides Eq. (22), the FP equation can also take another form,

$$\begin{aligned}\frac{\partial}{\partial t}[p_{k/k-1}(i)]^{2-q} &= -\frac{\partial}{\partial x_k}\{F_k[p_{k/k-1}(i)]^{2-q})\} + D'(q,t) \\ &\quad \times\frac{\partial^2}{\partial x_k^2}[p_{k/k-1}(i)]^{2-q}, \end{aligned} \tag{24}$$

where

$$D'(q,t) = \frac{A_2}{2\eta m}[1 - \frac{v(x_k,t,q)}{u_1(x_k,t,q)(2-q)}]. \tag{25}$$

We note that $D(q,t)$ in Eq. (22) and $D'(q,t)$ in Eq. (24) are both q and t dependent. The dependence on q is a direct consequence of the non-extensive statistics where q is the identity of the system described. It has been shown above that when $q = 1$, the normal diffusion constant can be restored. The dependence on t is also quite natural because we are now dealing with evolutionary processes where the phase space through which the diffusion occurs is changing with time. As $t \to \infty$, one readily obtains the diffusion constant for the stationary state.

The nonlinear FP equation derived above, Eq. (22) and Eq. (24), is well applied to describing the evolutionary processes and stochastic processes of a large class of self-organized systems that are far from thermal equilibrium, as well as chemical equilibrium, such as transportation and diffusion occurred in fractal or curved space. For example, it can be employed to describe the broad range of markets and exchanges characterized by the anomalous (super) diffusion and power-law distributions [15]. Another hope is that this equation can be applied to the complex biological systems where evolution and anomalous diffusion are taking place from time to time. Compared to the normal FP equation and some of its other nonlinear forms [16], our FP equation is more general because it can describe both regular dynamics and irregular dynamics that occurred in a large category of non-equilibrium and chaotic systems [17, 18]. Another important feature of our FP equation is that the diffusion coefficient is both q and t dependent.

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