November 22, 2010

Broué's abelian defect group conjecture holds for the sporadic simple Conway group Co_3

Shigeo Koshitani[a,*], **Jürgen Müller**[b], **Felix Noeske**[c]

[a] *Department of Mathematics, Graduate School of Science, Chiba University, Chiba, 263-8522, Japan*

[b, c] *Lehrstuhl D für Mathematik, RWTH Aachen University, 52062, Aachen, Germany*

Abstract

In the representation theory of finite groups, there is a well-known and important conjecture due to M. Broué. He conjectures that, for any prime p, if a p-block A of a finite group G has an abelian defect group P, then A and its Brauer corresponding block A_N of the normaliser $N_G(P)$ of P in G are derived equivalent (Rickard equivalent). This conjecture is called *Strong Version of Broué's Abelian Defect Group Conjecture*. In this paper, we prove that the strong version of Broué's abelian defect group conjecture is true for the non-principal 2-block A with an elementary abelian defect group P of order 8 of the sporadic simple Conway group Co_3. This result completes the verification of the strong version of Broué's abelian defect group conjecture for all primes p and for all p-blocks of Co_3.

Keywords: Broué's conjecture; abelian defect group; sporadic simple Conway group

1. Introduction and notation

In the representation theory of finite groups, one of the most important and interesting problems is to give an affirmative answer to a conjecture which was introduced by Broué around 1988 [5], and is nowadays called *Broué's Abelian Defect Group Conjecture*. He actually conjectures the following:

Conjecture 1.1 (Strong version of Broué's Abelian Defect Group Conjecture [5], [17])**.** *Let p be a prime, and let $(\mathcal{K}, \mathcal{O}, k)$ be a splitting p-modular system for all subgroups of a finite group G. Assume that A is a block algebra of $\mathcal{O}G$ with a defect group P and that A_N is a block algebra of $\mathcal{O}N_G(P)$ such that A_N is the Brauer correspondent of A, where $N_G(P)$ is the normaliser of P in G. Then A and A_N should be derived equivalent (Rickard equivalent) provided P is abelian.*

In fact, a stronger conclusion than **1.1** is expected, namely that A and A_N are *splendidly Rickard equivalent* in the sense of Linckelmann ([32], [33]), which he calls *splendidly derived equivalent*, see **1.12**. Note that for principal block algebras, this notion coincides with the splendid equivalence given by Rickard in [46].

Conjecture 1.2 (Rickard [46], [47])**.** *Keeping the notation, we suppose that P is abelian as in* **1.1**. *Then there should be a splendid Rickard equivalence between the block algebras A of $\mathcal{O}G$ and A_N of $\mathcal{O}N_G(P)$.*

There are several cases where the conjectures **1.1** and **1.2** of Broué and Rickard, respectively, have been verified. For example, in [19, (0.2)Theorem] it is shown that **1.1** and **1.2** are true for the principal block algebra A of an arbitrary finite group G when the defect group P of A is elementary abelian of order 9 (and hence $p = 3$). As extensions of this, there are results for non-principal 3-blocks with the same defect group $C_3 \times C_3$, see [20], [21], [39], [22] and [24].

On the other hand, let us look at the case where a block A has an elementary abelian defect group P of order 8, namely, $P = C_2 \times C_2 \times C_2$. The numbers of irreducible ordinary characters $k(A)$ and of irreducible Brauer characters $\ell(A)$, respectively, are important in block

* Corresponding author.

E-mail addresses: koshitan@math.s.chiba-u.ac.jp (S.Koshitani), juergen.mueller@math.rwth-aachen.de (J.Müller), Felix.Noeske@math.rwth-aachen.de (F.Noeske).

theory. For the principal 2-blocks they have been known for some time, see [18] and [27], for instance. However, only recently, the numbers of irreducible ordinary characters $k(A)$ and of irreducible Brauer characters $\ell(A)$ for non-principal 2-blocks have been determined in general, see [15]. In [15] it is proved with the help of the classification of finite simple groups, that Alperin's weight conjecture and also the *weak* version (*character theoretic* version) of Broué's abelian defect group conjecture for arbitrary 2-blocks with defect group $C_2 \times C_2 \times C_2$ are both true. The *strong* version of Broué's abelian defect group conjecture, namely, the existence of Rickard splendid equivalences between blocks corresponding via the Brauer correspondence for arbitrary 2-blocks with defect group $C_2 \times C_2 \times C_2$, is still open. There are four cases for the inertial index e of A with the defect group $P = C_2 \times C_2 \times C_2$. Namely, $e = 1, 3, 7$ or 21, since $\mathrm{Aut}(P) \cong \mathrm{GL}_3(2)$ has a unique maximal $2'$-subgroup, up to conjugacy, which is isomorphic to the Frobenius group $F_{21} = C_7 \rtimes C_3$ of order 21. For the cases where $e = 1$ everything is known because the blocks are nilpotent, see Broué-Puig [9]. For the case $e = 3$, there are results of Landrock [27] and Watanabe [55].

Our objective in this paper is to investigate a non-principal 2-block with elementary abelian defect group P of order 8, which has inertial index 21. An interesting candidate for this endeavour is the non-principal 2-block of Conway's third group Co_3, for which we investigate whether the *strong* version of Broué's abelian defect group conjecture holds. For previous results on Co_3, its defect groups, and 2-modular characters confer [12, p.193 Table 6], [26, §7 p.1879], [27, Theorems 3.10 and 3.11], and [53], for example.

Our main theorem of this paper is the following:

Theorem 1.3. *Let G be the sporadic simple Conway group Co_3, and let $(\mathcal{K}, \mathcal{O}, k)$ be a splitting 2-modular system for all subgroups of G, see* **1.11**. *Suppose that A is a non-principal block algebra of $\mathcal{O}G$ with a defect group P which is an elementary abelian group of order 8, and that A_N is a block algebra of $\mathcal{O}N_G(P)$ such that A_N is the Brauer correspondent of A. Then A and A_N are splendidly Rickard equivalent, and hence the conjectures* **1.1** *and* **1.2** *of Broué and Rickard both hold.*

Actually, **1.3** is the last tile in the mosaic proving both Broué's abelian defect group conjecture and Rickard's conjecture for Co_3 in arbitrary characteristic. By [30], [45], and [51] the conjecture is proved for blocks of cyclic defect groups. Hence, since $|G| = 2^{10}{\cdot}3^7{\cdot}5^3{\cdot}7{\cdot}11{\cdot}23$, see [10, p.134], it is sufficient to consider the primes $p \in \{2, 3, 5\}$. For odd p the only block with defect at least 2 is the principal block, whose defect groups are not abelian. For $p = 2$ there is precisely a unique block with a non-cyclic abelian defect group. Its defect group is isomorphic to $C_2 \times C_2 \times C_2$ (see [59, Co_3], [26, p.1879] and [53, p.494 §2]). Therefore we may state the following immediate consequence of **1.3**:

Corollary 1.4. *The strong version of Broué's abelian defect group conjecture* **1.1** *and even Rickard's splendid equivalence conjecture* **1.2** *are true for all primes p and for all block algebras of $\mathcal{O}G$ if $G = \mathsf{Co}_3$.*

As a matter of fact, the main result **1.3** is obtained by proving the following:

Theorem 1.5. *We keep the notation and the assumption as in* **1.3**. *Let H be a maximal subgroup of G with $H = R(3) \times \mathfrak{S}_3 \geqslant N_G(P)$, where $R(3) = {}^2G_2(3) \cong \mathrm{SL}_2(8) \rtimes C_3$ is the smallest Ree group, $\mathfrak{S}_3$ is the symmetric group on 3 letters, and C_3 is the cyclic group of order 3. Let B be a block algebra of $\mathcal{O}H$ such that B is the Brauer correspondent of A, see* [40, Chap.5 Theorem 3.8]. *In addition, let $\mathfrak{f}$ denote the Green correspondence with respect to $(G \times G, \Delta P, G \times H)$, and let $M = \mathfrak{f}(A)$. Then M induces a Morita equivalence between A and B, and hence it is a Puig equivalence.*

The following result is used to get **1.7** from our main result **1.5**.

Theorem 1.6 (Landrock-Michler [29] and Okuyama [42])**.** *Let $p = 2$, and let $R(q) = {}^2G_2(q)$ be a Ree group, where $q = 3^{2n+1}$ for some $n = 0, 1, 2, \cdots$. Let $(\mathcal{K}, \mathcal{O}, k)$ be a splitting 2-modular system for all subgroups of $R(q)$, for all q at the same time, see* [56, Theorem 3.6], *and let $B_0(\mathcal{O}R(q))$ be the principal block algebra of the group algebra $\mathcal{O}R(q)$. Then the block algebras*

$B_0(\mathcal{O}R(3))$ and $B_0(\mathcal{O}R(q))$ are Puig equivalent. In particular, Broué's abelian defect group conjecture **1.1** *and Rickard's conjecture* **1.2** *hold for the principal block algebras of $R(q)$ for any q.*

Proof. This follows from [29, Theorem 5.3] and [42, Example 3.3 and Remark 3.4]. □

Corollary 1.7. *We keep the notation and the assumption as in* **1.3**. *Let $R(q) = {}^2G_2(q)$ be a Ree group, where $q = 3^{2n+1}$ for some $n = 0, 1, 2, \cdots$. We may assume that $(\mathcal{K}, \mathcal{O}, k)$ also is a splitting 2-modular system for all subgroups of $R(q)$, for all q at the same time. Let $B_0(\mathcal{O}R(q))$ be the principal block algebra of the group algebra $\mathcal{O}R(q)$. Then A and $B_0(\mathcal{O}R(q))$ are Puig equivalent.*

Strategy 1.8. Our starting point for this work is the observation that the 2-decomposition matrix for the non-principal block A of Co_3 with an elementary abelian defect group of order 8, see [53], is exactly the same as that for the principal 2-block B of $R(3) \cong \mathrm{SL}_2(8) \rtimes C_3$, see [29]. Therefore it is natural to ask whether these two 2-block algebras are Morita equivalent not only over an algebraically closed field k of characteristic 2 but also over a complete discrete valuation ring $\mathcal{O}$ whose residue field is k. Furthermore, one might even expect that they are *Puig equivalent*, see **1.12**. If this is the case, since the two conjectures of Broué and Rickard **1.1** and **1.2** respectively have been shown to hold for the principal 2-block of $R(3)$ in a paper of Okuyama [42], it follows that these conjectures also hold for the non-principal 2-block of Co_3 with the same defect group $P = C_2 \times C_2 \times C_2$.

The verification that A and B are indeed Morita equivalent relies on theorems by Linckelmann, Broué, Rickard and Rouquier. Linckelmann has shown in [31] that a stable equivalence of Morita type between A and B which maps simple modules to simple modules is in fact a Morita equivalence, see **2.1**. To obtain an appropriate stable equivalence, we employ a variant of a "glueing" theorem, which is due to (originally Broué [6, 6.3.Theorem]), Rickard [46, Theorem 4.1], Rouquier [52, Theorems 5.6 and 6.3, Remark 6.4], and Linckelmann, see [32], [34] and **2.3**: A stable equivalence between two blocks A and B may be derived from Morita equivalences between unique blocks of the centralizers of non-trivial subgroups of P in Co_3 and $R(3)$. Once we have obtained a stable equivalence of Morita type between A and B, it remains to show that it preserves simplicity of modules as stated above. Usually this may be a very hard task.

Contents 1.9. The paper is structured as follows: In Section 2, we give the fundamental lemmas which are used to prove our main results. Furthermore, we establish some properties of the stable equivalences we consider, and collect some further results on Morita equivalences and Green correspondence for ease of reference. In Section 3 we investigate non-principal 2-blocks of the symmetric group $\mathfrak{S}_5$ and the Mathieu group M_{12} whose structure will be used later on in order to get our main theorems. In Section 4 the main objective is to construct the stable equivalence of Morita type between the blocks A and B as outlined above. In order to apply glueing theorems of Rouquier and Linckelmann **2.3**, we begin by analysing the 2-local structure of Co_3 to identify the groups. Then, we combine this knowledge and what we get already in Section 3 to give a stable equivalence F as saught. Section 5 prepares the proof that F maps simple A-modules to simple B-modules. In order to prove this fact, we collect information on simple and indecomposable modules in the three blocks A, B, and A_N. In Section 6 we determine the F-images of the simple A-modules, thus showing that they are indeed all simple. Finally, in Section 7 we combine the previous results to give complete proofs of our main theorems **1.3**, **1.4**, **1.5** and **1.7**. At the end of the paper, we have collected several useful properties of the stable equivalences obtained through **2.3**.

Computations 1.10. A few words on computer calculations are in order. To find our results, next to theoretical reasoning we have to rely fairly heavily on computations. Of course, many of the data contained in explicit libraries and databases are of computational nature, and quite a few traces of further computer calculations are still left in the present exposition. But we would like to point out that we have found many of our intermediate results by explicit computations first, which have subsequently been replaced by more theoretical arguments.

As tools, we use the computer algebra system GAP [13], to calculate with permutation groups and tables of marks, as well as with ordinary and Brauer characters. We also make use of the data library [4], in particular allowing for easy access to the data compiled in [10], [14] and [59], and of the interface [58] to the data library [60]. Moreover, we use the computer algebra system MeatAxe [49] to handle matrix representations over finite fields, as well as its extensions to compute submodule lattices [35], radical and socle series [38], homomorphism spaces and endomorphism rings [37], and direct sum decompositions [36]. We give more comments later on where necessary.

Notation 1.11. Throughout this paper, we use the standard notation and terminology as is used in [40], [54] and [10].

Let k be a field and assume that A and B are finite dimensional k-algebras. We denote by mod-A, A-mod and A-mod-B the categories of finitely generated right A-modules, left A-modules and (A, B)-bimodules, respectively. We write M_A, ${}_AM$ and ${}_AM_B$ when M is a right A-module, a left A-module and an (A, B)-bimodule. In this note, a module always refers to a finitely generated right module, unless stated otherwise. We let $M^\vee = \mathrm{Hom}_A(M_A, A_A)$ be the A-dual of M, so that $M^\vee$ becomes a left A-module via $(a\phi)(m) = a{\cdot}\phi(m)$ for $a \in A$, $\phi \in M^\vee$ and $m \in M$, and we let $M^\circledast = \mathrm{Hom}_k(M, k)$ be the k-dual of M, so that $M^\circledast$ becomes a left A-module as well via $(a\phi)(m) = \phi(ma)$ for $a \in A$, $\phi \in M^\circledast$ and $m \in M$. For A-modules M and N we write $[M, N]^A$ for $\dim_k[\mathrm{Hom}_A(M, N)]$. For an A-module M and the projective cover $P(S)$ of a simple A-module S, we write $[P(S) \mid M]^A$ for the multiplicity of direct summands of M which are isomorphic to $P(S)$. If A is self-injective, the stable module category $\underline{\mathrm{mod}}$-A, is the quotient category of mod-A with respect to the projective A-homomorphisms, that is those factoring through a projective module.

In this paper, G is always a finite group and we fix a prime number p. Assume that $(\mathcal{K}, \mathcal{O}, k)$ is a splitting p-modular system for all subgroups of G, that is to say, $\mathcal{O}$ is a complete discrete valuation ring of rank one such that its quotient field is $\mathcal{K}$ which is of characteristic zero, and its residue field $\mathcal{O}/\mathrm{rad}(\mathcal{O})$ is k, which is of characteristic p, and that $\mathcal{K}$ and k are splitting fields for all subgroups of G. By an $\mathcal{O}G$-lattice we mean a finitely generated right $\mathcal{O}G$-module which is a free $\mathcal{O}$-module. We denote by k_G the trivial kG-module, and similarly by $\mathcal{O}_G$ the trivial $\mathcal{O}G$-lattice. If X is a kG-module, then we write $X^* = \mathrm{Hom}_k(X, k)$ for the *contragredient* of X, namely, $X^* = \mathrm{Hom}_k(X, k)$ which is again a right kG-module via $(\varphi g)(x) = \varphi(xg^{-1})$ for $x \in X$, $\varphi \in X^*$ and $g \in G$; if no confusion may arise we also call this the *dual* of X. Let H be a subgroup of G, and let M and N be an $\mathcal{O}G$-lattice and an $\mathcal{O}H$-lattice, respectively. Then let $M{\downarrow}_H^G = M{\downarrow}_H$ be the restriction of M to H, and let $N{\uparrow}_H^G = N{\uparrow}^G = (N \otimes_{\mathcal{O}H} \mathcal{O}G)_{\mathcal{O}G}$ be the induction (induced module) of N to G. A similar definition holds for kG- and kH-modules. For a subgroup Q of G we write $\mathrm{Scott}(G, Q)$ for the (Alperin-)Scott module with respect to Q in G, see [40, Chap.4 p.297].

We denote by $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ the sets of all irreducible ordinary and Brauer characters of G, respectively. We write 1_G for the trivial character of G, and we write χ^* for the complex conjugate of $\chi \in \mathrm{Irr}(G)$. For $\chi, \psi \in \mathrm{Irr}(G)$ we denote by $(\chi, \psi)^G$ the usual inner product. If A is a block algebra (p-block) of $\mathcal{O}G$, then we write $\mathrm{Irr}(A)$ and $\mathrm{IBr}(A)$ for the sets of all characters in $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ which belong to A, respectively. We denote by $B_0(kG)$ the principal block algebra of kG.

Let G' be another finite group, and let V be an $(\mathcal{O}G, \mathcal{O}G')$-bimodule. Then we can regard V as a right $\mathcal{O}[G \times G']$-module. A similar definition holds for (kG, kG')-bimodules. We denote by $\Delta G = \{(g, g) \in G \times G \,|\, g \in G\}$ the diagonal copy of G in $G \times G$. For an $(\mathcal{O}G, \mathcal{O}G')$-bimodule V and a common subgroup Q of G and G', we set $V^{\Delta Q} = \{v \in V \mid qv = vq \text{ for all } q \in Q\}$. If Q is a p-group, the Brauer construction is defined to be the quotient $V(\Delta Q) = V^{\Delta Q}/[\sum_{R \gneqq Q} \mathrm{Tr}{\uparrow}_R^Q(V^{\Delta R}) + \mathrm{rad}\mathcal{O}{\cdot}V^{\Delta Q}]$, where $\mathrm{Tr}{\uparrow}_R^Q$ is the usual trace map. The Brauer homomorphism $\mathrm{Br}_{\Delta Q} : (\mathcal{O}G)^{\Delta Q} \to kC_G(Q)$ is obtained from composing the canonical epimorphism $(\mathcal{O}G)^{\Delta Q} \twoheadrightarrow (\mathcal{O}G)(\Delta Q)$ and the canonical isomorphism $(\mathcal{O}G)(\Delta Q) \overset{\approx}{\to} kC_G(Q)$.

For a positive integer n, $\mathfrak{A}_n$ and $\mathfrak{S}_n$ denote the alternating and the symmetric groups on n letters, M_n denotes the Mathieu group, and C_n and D_n denote the cyclic group and the dihedral group of order n, respectively. We denote by $Z(G)$ the center of G, and by S^g a set $g^{-1}Sg$ for $g \in G$ and a subset S of G.

Equivalences 1.12. Let A and A' be block algebras of $\mathcal{O}G$ and $\mathcal{O}G'$, respectively. Then we say that A and A' are *Puig equivalent* if A and A' have a common defect group P, and if there is a Morita equivalence between A and A' which is induced by an (A, A')-bimodule $\mathfrak{M}$ such that, as a right $\mathcal{O}[G \times G']$-module, $\mathfrak{M}$ is a trivial source module and ΔP-projective. A similar definition holds for blocks of kG and kG'. Due to a result of Puig (and independently of Scott), see [44, Remark 7.5], this is equivalent to a condition that A and A' have source algebras which are isomorphic as interior P-algebras, see [33, Theorem 4.1].

We say that A and A' are *stably equivalent of Morita type* if there exists an (A, A')-bimodule $\mathfrak{M}$ such that ${}_A\mathfrak{M}$ is projective as a left A-module, $\mathfrak{M}_{A'}$ is projective as a right A'-module, ${}_A(\mathfrak{M} \otimes_{A'} \mathfrak{M}^\vee)_A \cong {}_AA_A \oplus (\text{proj } (A, A)\text{-bimod})$ and ${}_{A'}(\mathfrak{M}^\vee \otimes_A \mathfrak{M})_{A'} \cong {}_{A'}A'_{A'} \oplus (\text{proj } (A', A')\text{-bimod})$.

We say that A and A' are *splendidly stably equivalent of Morita type* if A and A' have a common defect group P and the stable equivalence of Morita type is induced by an (A, A')-bimodule $\mathfrak{M}$ which is a trivial source $\mathcal{O}[G \times G']$-module and is ΔP-projective, see [33, Theorem 3.1].

We say that A and A' are *derived equivalent (or Rickard equivalent)* if $\mathrm{D}^b(\text{mod-}A)$ and $\mathrm{D}^b(\text{mod-}A')$ are equivalent as triangulated categories, where $\mathrm{D}^b(\text{mod-}A)$ is the bounded derived category of mod-A. In that case, there even is a *Rickard complex* $M^\bullet \in \mathrm{C}^b(A\text{-mod-}A')$, where the latter is the category of bounded complexes of finitely generated (A, A')-bimodules, all of whose terms are projective both as left A-modules and as right A'-modules, such that $M^\bullet \otimes_{A'} (M^\bullet)^\vee \cong A$ in $K^b(A\text{-mod-}A)$ and $(M^\bullet)^\vee \otimes_A M^\bullet \cong A'$ in $K^b(A'\text{-mod-}A')$, where $K^b(A\text{-mod-}A)$ is the homotopy category associated with $\mathrm{C}^b(A\text{-mod-}A)$. In other words, in that case we even have $K^b(\text{mod-}A) \cong K^b(\text{mod-}A')$.

We say that A and A' are *splendidly Rickard equivalent* if $K^b(\text{mod-}A)$ and $K^b(\text{mod-}A')$ are equivalent via a Rickard complex $M^\bullet \in \mathrm{C}^b(A\text{-mod-}A')$ as above, such that additionally each of its terms is a direct sum of ΔP-projective trivial source modules as an $\mathcal{O}[G \times G']$-module.

2. Preliminaries

In this section we give several theorems crucial to the later sections of this paper. We state these results in a more general context; in particular, G is an arbitrary finite group and $(K, \mathcal{O}, k)$ is a p-modular splitting system for G. As we draw upon these lemmas frequently in the sequel, we state these explicitly for the convenience of the reader and ease of reference.

As stated in the introduction, our approach centers around **2.1** which allows us to verify that a stable equivalence of Morita type is in fact a Morita equivalence. The stable equivalences investigated are obtained with the help of **2.3**, and are realised by tensoring with a bimodule given through Green correspondence. We proceed to study several properties of these stable equivalences, and give some further results needed in the upcoming parts of this paper. We refer the reader also to the appendix for a more detailed discussion of further properties of stable equivalences obtained through **2.3**.

Lemma 2.1 (Linckelmann [31]). *Let A and B be finite-dimensional k-algebras such that A and B are both self-injective and indecomposable as algebras, but not simple. Suppose that there is an (A, B)-bimodule M such that M induces a stable equivalence between the algebras A and B.*

(i) *If M is indecomposable then for any simple A-module S, the B-module $(S \otimes_A M)_B$ is non-projective and indecomposable.*
(ii) *If for all simple A-module S the B-module $S \otimes_A M$ is simple then M induces a Morita equivalence between A and B.*
(iii) *If $(M, M^\vee)$ induces a stable equivalence of Morita type between A and B then there is a unique (up to isomorphism) non-projective indecomposable (A, B)-bimodule M' such that $M' \mid M$, and $(M', M'^\vee)$ induces a stable equivalence of Morita type between the algebras A and B.*

Proof. (i) and (ii) respectively are given in [31, Theorem 2.1(ii) and (iii)]. Part (iii) follows by [31, Theorem 2.1(i) and Remark 2.7]. □

We obtain a suitable stable equivalence to apply **2.1** through a “glueing theorem” as given in **2.3**.

Lemma 2.2 (Koshitani-Linckelmann [23])**.** *Let A be a block algebra of kG with defect group P, and let (P, e) be a maximal A-Brauer pair such that $H = N_G(P, e) = N_G(P)$. Let B be a block algebra of kH such that B is the Brauer correspondent of A. Let $\mathfrak{f}$ be the Green correspondence with respect to $(G \times G, \Delta P, G \times H)$, and set $M = \mathfrak{f}(A)$, in particular M is an indecomposable (A, B)-bimodule with vertex ΔP.*

Take any subgroup Q of $Z(P)$, and set $G_Q = C_G(Q)$ and $H_Q = C_H(Q)$. Let e_Q and f_Q be block idempotents of kG_Q and kH_Q satisfying $(Q, e_Q) \subseteq (P, e)$ and $(Q, f_Q) \subseteq (P, e)$, respectively, see [54, (40.9) Corollary]*. Let $\mathfrak{f}_Q$ be the Green correspondence with respect to $(G_Q \times G_Q, \Delta P, G_Q \times H_Q)$. Then we have*

$$e_Q M(\Delta Q) f_Q = \mathfrak{f}_Q(e_Q kG_Q)$$

and this is a unique (up to isomorphism) indecomposable direct summand of $(e_Q kG_Q){\downarrow}_{G_Q \times H_Q}$ with vertex ΔP.

Proof. We know $M = \mathfrak{f}(A) \mid A{\downarrow}^{G\times G}_{G\times H} \mid {}_{kG}kG_{kH}$. Hence, $M(\Delta Q) \mid (kG)(\Delta Q) = kC_G(Q) = kG_Q$. Thus,

$$e_Q M(\Delta Q) f_Q \,\big|\, e_Q kG_Q f_Q \,\big|\, (e_Q kG_Q){\downarrow}^{G_Q\times G_Q}_{G_Q\times H_Q}.$$

By [23, Theorem], $e_Q M(\Delta Q) f_Q$ is an indecomposable $k[G_Q \times H_Q]$-module with vertex ΔP. Thus Green correspondence yields $e_Q M(\Delta Q) f_Q = \mathfrak{f}_Q(e_Q kG_Q)$. □

Lemma 2.3 (Linckelmann [33], [34])**.** *Let A be a block algebra of $\mathcal{O}G$ with a defect group P, and let (P, e) be a maximal A-Brauer pair in G. Set $H = N_G(P, e)$, Assume that*

(1) *P is abelian,*
(2) *for each Q with $1 \neq Q \leqslant P$, $kC_G(Q)$ has a unique block algebra A_Q with the defect group P,*
(3) *for each Q with $1 \neq Q \leqslant P$, $kC_H(Q)$ has a unique block algebra B_Q with the defect group P.*

Let B a block algebra of $\mathcal{O}H$ which is the Brauer correspondent of A. For each subgroup Q of P, let e_Q and f_Q be the block idempotents of A_Q and B_Q, respectively, and hence $A_Q = kC_G(Q)e_Q$ and $B_Q = kC_H(Q)f_Q$. Note that $e_P = e = f_P$ and $A_P = B_P$. Let $\mathfrak{f}$ be the Green correspondence with respect to $(G \times G, \Delta P, G \times H)$, and set ${}_AM_B = \mathfrak{f}(A)$, see **2.4***. Moreover, let $\mathfrak{f}_Q$ be the Green correspondence with respect to $(C_G(Q) \times C_G(Q), \Delta P, C_G(Q) \times C_H(Q))$. Now, assume further that*

(4) *for each non-trivial proper subgroup Q of P, the (A_Q, B_Q)-bimodule $\mathfrak{f}_Q(A_Q)$ induces a Morita equivalence between A_Q and B_Q.*

Then the (A, B)-bimodule M induces a stable equivalence of Morita type between A and B.

Proof. First, note $H = N_G(P)$. Secondly, it follows from **2.2** that $e_Q{\cdot}M(\Delta Q){\cdot}f_Q = \mathfrak{f}_Q(A_Q)$ for each $Q \leqslant P$ since P is abelian by (1). Then since $A_P = B_P$ and since $A_P = \mathfrak{f}_P(A_P) = e{\cdot}M(\Delta P){\cdot}e$, the (A_P, B_P)-bimodule $e_P{\cdot}M(\Delta P){\cdot}e_P$ induces a Morita equivalence between A_P and B_P.

Now, for each $Q \leqslant P$, it follows from the uniqueness of e_Q and f_Q that

$$(Q, e_Q) \subseteq (P, e) \qquad \text{and} \qquad (Q, f_Q) \subseteq (P, e).$$

Next, we want to claim

$$E_G\Big((Q, e_Q), (R, e_R)\Big) = E_H\Big((Q, f_Q), (R, f_R)\Big) \qquad \text{for } \ Q,\, R \leqslant P,$$

where $E_G\Big((Q, e_Q), (R, e_R)\Big)$ is the set $\{\varphi : Q \to R \mid \text{there is } g \in G \text{ with } \varphi(u) = u^g, \text{ for all } u \in Q, \ \text{ and } (Q, e_Q)^g \subseteq (R, e_R)\}$, see [33, p.821]. This is known by using [2, Proposition 4.21 and Theorem 3.4] and [8, Theorem 1.8(1)] since P is abelian, see [21, The proof of 1.15. Lemma] for details. Therefore we can apply Linckelmann's result [33, Theorem 3.1]. □

We remark that in [33, Theorem 3.1] and [34, Theorem A.1], Linckelmann proves more general theorems than **2.3**. However, we formulate with **2.3** a version which is specifically tailored to our practical purposes, and use this ad hoc version in the sequel.

In the notation of **2.3**, we have that the bimodule M realising a stable equivalence between A and B is a Green correspondent of A. In fact it is a direct summand of $1_A \cdot kG \cdot 1_B$ as the next lemma shows.

Lemma 2.4. *Let A be a block algebra of kG with defect group P. Assume that (P, e) is a maximal A-Brauer pair such that $H = N_G(P, e) = N_G(P)$. Let B be a block algebra of kH such that B is the Brauer correspondent of A. Let $\mathfrak{f}$ be the Green correspondence with respect to $(G \times G, \Delta P, G \times H)$. Then we have $\mathfrak{f}(A) \mid 1_A{\cdot}kG{\cdot}1_B$.*

Proof. It follows from [3, Theorem 5(i)] that $(A{\downarrow}^{G\times G}_{G\times H}){\cdot}1_B = 1_A{\cdot}kG{\cdot}1_B$ has a unique (up to isomorphism) indecomposable direct summand with vertex ΔP. Clearly, $1_A{\cdot}kG{\cdot}1_B \mid (A{\downarrow}^{G\times G}_{G\times H})$, hence by Green correspondence we have $\mathfrak{f}(A) \mid 1_A{\cdot}kG{\cdot}1_B$. □

We remark that a stable equivalence of Morita type induced by the Green correspondent $\mathfrak{f}(A)$ in the context of **2.4** preserves vertices and sources, see **A.3**.

Lemma 2.5. *Let G, H, and L be finite groups, all of which have a common non-trivial p-subgroup P, and assume that $H \leqslant G$. Let A, B, and C be block algebras of kG, kH, and kL, respectively, all of which have P as their defect group. In addition, suppose that a pair $({}_A\mathfrak{M}_B, {}_B\mathfrak{M}'{}_A)$ induces a stable equivalence between A and B such that ${}_A\mathfrak{M}_B \mid k_{\Delta P}{\uparrow}^{G\times H}$, ${}_B\mathfrak{M}'{}_A \mid k_{\Delta P}{\uparrow}^{H\times G}$, and that $\mathfrak{M}$ and $\mathfrak{M}'$ preserve vertices and sources. Similarly, suppose that a pair $({}_B\mathfrak{N}_C, {}_C\mathfrak{N}'{}_B)$ induces a stable equivalence between B and C such that ${}_B\mathfrak{N}_C \mid k_{\Delta P}{\uparrow}^{H\times L}$, ${}_C\mathfrak{N}'{}_B \mid k_{\Delta P}{\uparrow}^{L\times H}$, and that $\mathfrak{N}$ and $\mathfrak{N}'$ preserve vertices and sources. Then we have (A, C)- and (C, A)-bimodules M and M', respectively, which satisfy the following:*

(1) ${}_A(\mathfrak{M} \otimes_B \mathfrak{N})_C = {}_AM_C \oplus$ (proj (A, C)-bimodule) *and*
${}_C(\mathfrak{N}' \otimes_B \mathfrak{M}')_A = {}_CM'_A \oplus$ (proj (C, A)-bimodule).
(2) *${}_AM_C$ and ${}_CM'_A$ are both non-projective indecomposable.*
(3) *The pair (M, M') induces a stable equivalence between A and C.*
(4) *The functors*

$$- \otimes_A M : \text{mod-}A \longrightarrow \text{mod-}C$$

and

$$- \otimes_C M' : \text{mod-}C \longrightarrow \text{mod-}A$$

preserve vertices and sources of indecomposable modules. That is, for non-projective indecomposable A- and C-modules X and Y corresponding via $X \otimes_A M = Y \oplus$ (proj) and $Y \otimes_C M' = X \oplus$ (proj), respectively, there is a non-trivial p-subgroup Q and an indecomposable kQ-module S such that Q is a common vertex of X and Y and that S is a common source of X and Y.
(5) ${}_AM_C \mid k_{\Delta P}{\uparrow}^{G\times L}$ *and* ${}_CM'{}_A \mid k_{\Delta P}{\uparrow}^{L\times G}$.
(6) *In particular, if a pair $(\mathfrak{M}, \mathfrak{M}^\vee)$ induces a stable equivalence of Morita type between A and B, and if a pair $(\mathfrak{N}, \mathfrak{N}^\vee)$ induces a stable equivalence of Morita type between B and C, then we can replace M' above by $M^\vee$ and we have that the pair $(M, M^\vee)$ induces a stable equivalence of Morita type between A and C.*

Proof. Obviously, the pair $({}_A(\mathfrak{M} \otimes_B \mathfrak{N})_C, {}_C(\mathfrak{N}' \otimes_B \mathfrak{M}')_A)$ induces a stable equivalence between A and C. Clearly, ${}_A(\mathfrak{M} \otimes_B \mathfrak{N})$, $(\mathfrak{M} \otimes_B \mathfrak{N})_C$, ${}_C(\mathfrak{N}' \otimes_B \mathfrak{M}')$, and $(\mathfrak{N}' \otimes_B \mathfrak{M}')_A$ are all projective. Since A and C are symmetric algebras, it follows from **2.1**(iii) that there are (A, C)- and (C, A)-bimodules M and M' which satisfy the conditions (1)–(4).

Next we want to show (5). It follows from [40, Chap.5 Lemma 10.9(iii)] that

$$
\begin{aligned}
M \Big| \mathfrak{M} \otimes_B \mathfrak{N} \Big| & (k_{\Delta P}{\uparrow}^{G\times H}) \otimes_{kH} (k_{\Delta P}{\uparrow}^{H\times L}) \\
&\cong (kG \otimes_{kP} kH) \otimes_{kH} (kH \otimes_{kP} kL) \cong kG \otimes_{kP} [(kH){\downarrow}^{H\times H}_{P\times P}] \otimes_{kP} kL \\
&\cong kG \otimes_{kP} \Big(\bigoplus_{h\in[P\backslash H/P]} k[PhP] \Big) \otimes_{kP} kL \cong \bigoplus_{h\in[P\backslash H/P]} k[PhP]{\uparrow}^{G\times L}_{P\times P}.
\end{aligned}
$$

Since ${}_AM_C$ is indecomposable, there is an element $h \in H$ such that $M \mid k[PhP]{\uparrow}^{G\times L}_{P\times P}$. Set $(P\times P)_h = \{(u, h^{-1}uh) \in P \times P \mid u \in P \cap hPh^{-1}\}$. Then

$$(P\times P)_h = \{(huh^{-1}, u) \in P\times P \mid u \in P \cap h^{-1}Ph\} = (h,1)\cdot\Delta[P\cap P^h]\cdot(h^{-1},1).$$

We get by [40, Chap.5 Lemma 10.9(iii)] that $k[PhP] \cong k_{(h,1)\Delta[P\cap P^h](h^{-1},1)}{\uparrow}^{P\times P}$, and hence $M \mid k_{(h,1)\Delta[P\cap P^h](h^{-1},1)}{\uparrow}^{G\times L}$. Now, since $(h^{-1},1) \in H\times L \leqslant G\times L$, we have that

$$M \mid k_{\Delta[P\cap P^h]}{\uparrow}^{G\times L} \cong kG \otimes_{kQ} kL$$

where $Q = P\cap P^h$. Then for any X in mod-A the module $X\otimes_A M$ has a vertex contained in Q. If Q is a proper subgroup of P then, since (M, M') induces a stable equivalence between A and C, any module in mod-C has a vertex properly contained in P, a contradiction since P is a defect group of C. Hence $Q = P$, so that $h \in N_H(P) \subseteq N_G(P)$. Therefore $M \mid k_{\Delta P}{\uparrow}^{G\times L}$. An analogous argument gives the claim for M'.

(6) Follows from (1)–(5) and **2.1**(iii). □

Next, we give some results on Morita equivalences and tensor products, which will be useful in Section 4.

Lemma 2.6. *The following hold:*

(i) *Let A, B, C and D be finite dimensional k-algebras. Assume that an (A,B)-bimodule M realizes a Morita equivalence between A and B, and so does a (C,D)-bimodule N between C and D. Then the $(A\otimes C, B\otimes D)$-bimodule $M\otimes N$ induces a Morita equivalence between $A\otimes C$ and $B\otimes D$.*

(ii) *Keep the notation as in* (i). *Assume that P is a common p-subgroup of finite groups G and H, and that Q is a subgroup of P. Suppose moreover that A and B respectively are block algebras of kG and kH, $C = D = kQ$ and $N = {}_{kQ}kQ_{kQ}$. If a (kG, kH)-bimodule M satisfies that $M \mid k_{\Delta P}{\uparrow}^{G\times H}$, then $(M\otimes N) \mid k_{\Delta[P\times Q]}{\uparrow}^{(G\times Q)\times(H\times Q)}$.*

Proof. The proof of (i) is straightforward. For (ii) observe that $k_{\Delta P}{\uparrow}^{(G\times Q)\times(H\times Q)}$ is isomorphic to $k[G\times Q]\otimes_{k[P\times Q]} k[H\times Q]$, and hence to $(kG\otimes_{kP} kH)\otimes kQ$ as $k[G\times Q]\otimes k[H\times Q]$-bimodules. The latter is isomorphic to $k_{\Delta P}{\uparrow}^{G\times H} \otimes {}_{kQ}kQ_{kQ}$. □

Note that we cannot replace the *Morita* equivalence in **2.6** by a *stable* equivalence in general, see [48, Question 3.8].

Lemma 2.7. *Let G and H be finite groups, let A and B, respectively, be block algebras of kG and kH. Let X be an indecomposable kG-module in A, and let Y be an indecomposable kH-module in B. Then the following hold:*

(i) *If B is of defect zero, then a block algebra $A\otimes B$ of $k[G\times H]$ is Puig equivalent to A.*

(ii) *Set $Z = X\otimes Y$. Then Z is an indecomposable $k[G\times H]$-module in $A\otimes B$. If X and Y are are trivial source modules, then Z is a trivial source module as well.*

(iii) *If Y is projective, and Q is a vertex of X, then $Q\times\langle 1\rangle$ is a vertex of Z, and Z is a trivial source module if and only if X is.*

Proof. (i) By [54, p.341 line −9], k is a source algebra of B. Hence the assertion follows from Lemma **2.6**(i).

(ii)–(iii) These follow from [25, Proposition 1.2]. □

Finally, we collect a few facts about Green correspondence, its compatability with Brauer correspondence, and its transitivity (see [40, Chap.4, §4], for example).

Lemma 2.8. *Let P be a p-subgroup of a finite group G, and let N and H be subgroups of G with $N_G(P) \leqslant N \leqslant H \leqslant G$. Furthermore, assume that f, f_1 and f_2 are the Green correspondences with respect to (G, P, H), (H, P, N) and (G, P, N), respectively. Then from the definition and properties of Green correspondence and the Krull-Schmidt Theorem we get the following:*

(i) *We have $\mathfrak{A}(G, P, N) \subseteq \mathfrak{A}(G, P, H) \cap \mathfrak{A}(H, P, N)$, where $\mathfrak{A}(G, P, N)$ and the others are defined as in* [40, Chap.4, §4].

(ii) *For any indecomposable kG-module X with vertex in $\mathfrak{A}(G, P, N)$, the isomorphism $f_1\Big(f(X)\Big) \cong f_2(X)$ holds.*

(iii) *Let $N = N_G(P)$, and let A, B, and A_N be block algebras of kG, kH, and kN, respectively, such that they are Brauer correspondents with respect to P. Then any indecomposable kG-module X belonging to A such that a vertex of X is in $\mathfrak{A}(G, P, N)$ has its Green correspondent $f(X)$ belong to B.*

Proof. (i) and (ii) are clear.

(iii) It follows from Green's result [40, Chap.5 Corollary 3.11] and Brauer's first main theorem that $f_2(X)$ belongs to A_N. The Green correspondent $f(X)$ has a vertex in $\mathfrak{A}(G, P, N)$, and hence in $\mathfrak{A}(H, P, N)$. By (ii), $f_2 = f_1 \circ f$. Hence $f_2(X) = f_1 \circ f(X)$ lies in the Brauer correspondent of A which is A_N. Therefore, by the above, the block of $f(X)$ corresponds to A_N, namely, it is B. □

3. Non-principal 2-blocks of $\mathfrak{S}_5$ and M_{12}

By the "glueing" theorem given in **2.3**, we want to obtain a stable equivalence of Morita type between the non-principal 2-block of Co_3 with a defect group $P = C_2 \times C_2 \times C_2$ and its Brauer correspondent in the normalizer $N_{\mathsf{Co}_3}(P)$. In order to do it, we need to consider non-trivial subgroups of P and establish Morita equivalences between unique blocks of the associated centralizers in Co_3 and $N_{\mathsf{Co}_3}(P)$. The objective of this section is to show the existence of various Morita equivalences which will be required to apply **2.3**. The relevance of the groups related to $\mathfrak{S}_5$ and M_{12}, respectively, will be revealed in in **4.2** in the next section.

For the remainder of this paper, we let the characteristic p of k be 2.

Lemma 3.1. *Set $G = \mathfrak{S}_5$.*

(i) *There exists a unique block algebra A of kG with defect one. In fact, a defect group T of A is generated by a transposition.*

(ii) *Set $H = N_G(T)$. Then $H = C_G(T) \cong T \times \mathfrak{S}_3 \cong D_{12}$.*

(iii) *A is a nilpotent block algebra, $k(A) = 2$, $\ell(A) = 1$, and we can write $\mathrm{Irr}(A) = \{\chi_4, \chi_4'\}$ and $\mathrm{IBr}(A) = \{4_{kG}\}$, where the number 4 denotes the degree (dimension).*

(iv) *The unique simple kG-module 4_{kG} is a trivial source module.*

(v) *Let B be a block algebra of kH such that B is the Brauer correspondent of A. Then $k(B) = 2$, $\ell(B) = 1$, and we can write $\mathrm{Irr}(B) = \{\theta_2, \theta_2'\}$ and $\mathrm{IBr}(B) = \{2_{kH}\}$, where the number 2 again gives the degree (dimension).*

(vi) *Let $\mathfrak{f}$ be the Green correspondence with respect to $(G\times G, \Delta T, G\times H)$, and set $M = \mathfrak{f}(A)$. Then ${}_AM_B = 1_A{\cdot}kG{\cdot}1_B$ and M induces a Puig equivalence between A and B.*

Proof. (i)–(iii) and (v) are immediate by [10, p.2], and [14, A_5.2 (mod 2)] or [59, A_5.2 (mod 2)].

(iv) It follows from [10, p.2] that $1_H{\uparrow}^G = 1_G + \chi_4 + \chi_5$, where $\chi_i \in \mathrm{Irr}(G)$ and $\chi_i(1) = i$ for $i = 4, 5$. Thus, by (ii), $1_H{\uparrow}^G{\cdot}1_A = \chi_4$, and hence $k_H{\uparrow}^G{\cdot}1_A = 4_{kG}$.

(vi) We first show that $1_A{\cdot}kG{\cdot}1_B$ induces a Morita equivalence between A and B. To this end let $1_{\widehat{A}}{\cdot}\mathcal{O}G{\cdot}1_{\widehat{B}}$ be its lift to $\mathcal{O}$, which is projective both as a left $\mathcal{O}G$-module and as a right $\mathcal{O}H$-module. Moreover, it follows from (iii), (v), and [10, p.2] that

$$\chi_4{\downarrow}_H{\cdot}1_B = \theta_2, \quad \chi_4'{\downarrow}_H{\cdot}1_B = \theta_2'$$

by interchanging θ_2 and θ_2' if necessary. Therefore

$$\chi_4 \otimes_{\mathcal{K}A} (1_{\widehat{A}}{\cdot}\mathcal{K}G{\cdot}1_{\widehat{B}}) = \theta_2, \qquad \chi_4' \otimes_{\mathcal{K}A} (1_{\widehat{A}}{\cdot}\mathcal{K}G{\cdot}1_{\widehat{B}}) = \theta_2'.$$

Hence by [5, 0.2 Théorèm] , we get that $1_{\widehat{A}}\cdot\mathcal{O}G\cdot 1_{\widehat{B}}$ induces a Morita equivalence between $\widehat{A}$ and $\widehat{B}$, and so does $1_A\cdot kG\cdot 1_B$ between A and B. As $1_A\cdot kG\cdot 1_B$ is a trivial source $k[G\times H]$-module with vertex ΔP, we infer that this even is a Puig equivalence.

Finally, let (T, e) be a maximal A-Brauer pair. Then we know $N_G(T, e) = H$ by (ii). Hence **2.4** implies that $M | 1_A\cdot kG\cdot 1_B$. But it follows from Morita's Theorem, see [11, Sect. 3D Theorem (3.54)] that $1_A\cdot kG\cdot 1_B$ already is indecomposable as an (A, B)-bimodule, implying that $M = 1_A\cdot kG\cdot 1_B$. □

Lemma 3.2. *Set $R = C_2$ and consider a group $G = R\times\mathfrak{S}_5$, and let T be as in* **3.1**, *$Q = R\times T$ (and hence $Q \cong C_2 \times C_2$), and set $H = N_G(Q)$. Let A be a unique non-principal block algebra of kG with defect group Q, and let B be a block algebra of kH such that B is the Brauer correspondent of A. Then we get the following:*

(i) *$H = C_G(Q) = Q \times \mathfrak{S}_3$.*
(ii) *Let $\mathfrak{f}$ be the Green correspondence with respect to $(G\times G, \Delta Q, G\times H)$, and set $M = \mathfrak{f}(A)$. Then $M \cong 1_A\cdot kG\cdot 1_B$, and M induces a Puig equivalence between A and B.*

Proof. This follows from **3.1**(vi) and **2.6**. □

Lemma 3.3. *Let $Q = C_2 \times C_2$, and let $G = Q \times \mathfrak{S}_5$ and $P = Q \times T$, where T is as in* **3.1**. *Set $H = N_G(P)$. Let A be a unique non-principal block algebra of kG with defect group P, and let B be a block algebra of kH such that B is the Brauer correspondent of A. Then we get the following:*

(i) *$H = C_G(P) = Q \times (T \times \mathfrak{S}_3) = P \times \mathfrak{S}_3$.*
(ii) *Let f be the Green correspondence with respect to $(G\times G, \Delta P, G\times H)$, and set $M = \mathfrak{f}(A)$. Then $M = 1_A\cdot kG\cdot 1_B$ and M induces a Puig equivalence between A and B.*

Proof. This follows from **3.1**(vi) and **2.6**. □

We next turn to the Mathieu group M_{12}.

Lemma 3.4. *Let $G = \mathsf{M}_{12}$.*

(i) *There exists a unique block algebra A of kG with defect group $Q = C_2 \times C_2$.*
(ii) *We can write $\mathrm{IBr}(A) = \{16, 16^*, 144\}$, where the numbers 16 and 144 denote dimensions (degrees). Moreover, all the simple kG-modules in A are trivial source modules.*
(iii) *Let $H = N_G(Q)$. Then $H \cong \mathfrak{A}_4 \times \mathfrak{S}_3 \cong (Q \rtimes C_3) \times \mathfrak{S}_3$.*
(iv) *Let B be a block algebra of kH such that B is the Brauer correspondent of A. Let $\mathfrak{f}$ be the Green correspondence with respect to $(G \times G, \Delta Q, G \times H)$, and set $M = \mathfrak{f}(A)$. Then M induces a Puig equivalence between A and B.*

Proof. (i)–(iii) except the last part of (ii) are easy by [10, p.33], and [14, M_{12} (mod 2)] or [59, M_{12} (mod 2)]. Actually, using the character table of G, it turns out that the conjugacy class 3B of G is a defect class of A. Hence Q is a Sylow 2-subgroup of the centralizer $C_G(3B) = \mathfrak{A}_4 \times C_3$, while the normalizer $N_G(3B) = \mathfrak{A}_4 \times \mathfrak{S}_3$ is a maximal subgroup of G, containing Q as normal subgroup.

It remains to show the last statement in (ii). By [10, p.33], G has a maximal subgroup $L \cong \mathrm{PSL}_2(11)$. Then again [10, p.33] yields that $1_L{\uparrow}^G\cdot 1_A = \chi_{16} + \chi_{16}^*$, where $\chi_{16}(1) = \chi_{16}^*(1) = 16$. Set $X_{kG} = k_L{\uparrow}^G\cdot 1_A$. Then $X = 16 + 16^*$ as composition factors. Since $\chi_{16} \neq \chi_{16}^*$, we get by [40, Chap.4 Theorem 8.9(i)] that $[X, X]^G = 2$. Therefore $X = 16 \oplus 16^*$. Hence 16 and 16^* are both trivial source kG-modules. Finally, we know that $k_W{\uparrow}^G\cdot 1_A = 144$, where W is a maximal subgroup of G with $W = 2_+^{1+4}.\mathfrak{S}_3$. This shows that 144 is also a trivial source kG-module.

(iv) All elements of $Q - \{1\}$ are conjugate in H, hence the character table of G [10, p.33] shows that they all belong to the conjugacy class 2A of G. Take any element $t \in Q - \{1\}$, and set $R = \langle t\rangle$. Thus we have

$$C_G(R) \cong R \times \mathfrak{S}_5 \qquad \text{and} \qquad C_H(R) \cong Q \times \mathfrak{S}_3 \cong R \times (C_2 \times \mathfrak{S}_3).$$

The algebra $kC_G(R)$ has a unique block algebra A_R with the defect group Q since $k\mathfrak{S}_5$ has a unique block algebra with defect group C_2, and similarly $kC_H(R)$ has a unique block algebra

B_R with the defect group Q since $k\mathfrak{S}_3$ has a unique block algebra of defect zero. Moreover, we know by **3.2** that $\mathfrak{f}_R(A_R)$ induces a Morita equivalence between A_R and B_R, where $\mathfrak{f}_R$ is the Green correspondence with respect to $(C_G(R) \times C_G(R), \Delta Q, C_G(R) \times C_H(R))$. Thus it follows from **2.3** that M induces a stable equivalence of Morita type between A and B.

Now, let f be the Green correspondence with respect to (G, Q, H). Take any simple kG-module S in A. It follows from (ii), [16, 3.7.Corollary], and [41, Lemma 2.2] that $f(S)$ is a simple kH-module. Hence from **A.3**(v) and **2.1**(i) we obtain that $S \otimes_A M$ is a simple kH-module in B. We then finally know that M realizes a Morita equivalence between A and B by **2.1**(ii). □

Lemma 3.5. *Set $R = C_2$, and let $G = R \times \mathsf{M}_{12}$.*

(i) *There exists a unique block algebra A of kG with defect group $P = R \times C_2 \times C_2$.*

(ii) *We can write $\mathrm{IBr}(A) = \{16, 16^*, 144\}$, where the numbers 16 and 144 give the dimensions (degrees). Moreover, all the simple kG-modules 16, 16^*, 144 in A are trivial source modules.*

(iii) *Let $H = N_G(P)$. Then $H = R \times \mathfrak{A}_4 \times \mathfrak{S}_3 \cong (P \rtimes C_3) \times \mathfrak{S}_3$. Note that $P \rtimes C_3 \cong R \times (Q \rtimes C_3)$ and $Q \rtimes C_3 \cong \mathfrak{A}_4$, where $Q = C_2 \times C_2$.*

(iv) *Let B be a block algebra of kH such that B is the Brauer correspondent of A. Let $\mathfrak{f}$ be the Green correspondence with respect to $(G \times G, \Delta P, G \times H)$. Then $\mathfrak{f}(A)$ induces a Puig equivalence between A and B.*

Proof. This follows from **3.4**(iv) and **2.6**. □

4. Obtaining stable equivalences

This enables us to determine a stable equivalence of Morita type between the principal 2-block of the smallest Ree group $R(3)$ and the non-principal 2-block of Co_3 with defect group $C_2 \times C_2 \times C_2$ under consideration. The following hypothesis determines our standard setting which we fix here for future reference.

Hypothesis 4.1. Let G be the sporadic group Co_3, and let A be the block algebra of kG with defect group $P = C_2 \times C_2 \times C_2$, see [59, Co_3], [26, p.1879] and [53, p.494 §2]. Set $N = N_G(P)$, and let A_N be the Brauer correspondent of A in kN. Furthermore, let (P, e) be a maximal A-Brauer pair in G.

Let Q be a subgroup of P isomorphic to $C_2 \times C_2$, and R one which is cyclic of order 2. Let e_Q and f_Q be block idempotents of the block algebras of $kC_G(Q)$ and $kC_H(Q)$, respectively, such that $(Q, e_Q) \subseteq (P, e)$ and $(Q, f_Q) \subseteq (P, e)$, see [54, §10 p.346]. Similarly define e_R and f_R by replacing Q with R. We denote by F_{21} the Frobenius group of order 21, namely, $F_{21} \cong C_7 \rtimes C_3$, which is a maximal subgroup of $\mathrm{GL}_3(2)$. Also, let $R(3) \cong \mathrm{SL}_2(8) \rtimes C_3$ be the smallest Ree group, see [10, p.6].

We first collect information on the subgroups of Co_3 to consider.

Lemma 4.2. *Assume* **4.1**. *Then the following hold:*

(i) $N \cong (P \rtimes F_{21}) \times \mathfrak{S}_3 \cong \Big((P \rtimes C_7) \rtimes C_3\Big) \times \mathfrak{S}_3$.

(ii) *There is a maximal subgroup H of G such that $N \leqslant H \cong (\mathrm{SL}_2(8) \rtimes C_3) \times \mathfrak{S}_3$, and $P \rtimes C_7$ isomorphic to a Borel subgroup of $\mathrm{SL}_2(8)$.*

(iii) $C_G(P) = C_H(P) = C_N(P) \cong P \times \mathfrak{S}_3$.

(iv) *There exists a unique block algebra β of $k\mathfrak{S}_3$ such that β has defect zero, $\beta \cong \mathrm{Mat}_2(k)$ as k-algebras, and $ekC_G(P) \cong kP \otimes \beta$.*

(v) $N_G(P, e) = N$.

(vi) *The inertial quotient $N_G(P, e)/C_G(P)$ is isomorphic to F_{21}.*

(vii) *All elements of $P - \{1\}$ are conjugate in N. That is, any subgroup of P of order 2 is conjugate to R in N.*

(viii) $C_G(R) \cong R \times \mathsf{M}_{12}$ *and* $C_H(R) = C_N(R) \cong R \times \mathfrak{A}_4 \times \mathfrak{S}_3 \cong (P \rtimes C_3) \times \mathfrak{S}_3$.

(ix) *All subgroups of P of order 4 are conjugate in N. That is, any subgroup of P of order 4 is conjugate to Q in N.*

(x) $C_G(Q) \cong Q \times \mathfrak{S}_5$ *and* $C_H(Q) = C_N(Q) = C_H(P) \cong P \times \mathfrak{S}_3$.

(xi) *Let* $B = B_0(kR(3)) \otimes \beta$, *see* (iv) *for* β. *Then* B *is a block algebra of* kH *with the defect group* P, *the block* B *is the Brauer correspondent of* A *and of* A_N *in* H, *and we furthermore know that* B *and* $B_0(kR(3))$ *are Puig equivalent.*

Proof. This is verified easily using GAP [13], with the help of the smallest faithful permutation representation of G on 276 points, available in [58] in terms of so-called standard generators [57]. Since in [58] also representatives of the conjugacy classes of elements, as well as of the maximal subgroups of G are provided, all above-mentioned subgroups of G can be constructed explicitly.

To begin with, using the character table of G [10, p.135], it turns out that the conjugacy class 3C of G is a defect class of A. Hence P is a Sylow 2-subgroup of the centralizer $C_G(3C)$, where by [10, p.135] again we have $C_G(3C) \cong (\mathrm{SL}_2(8) \rtimes C_3) \times C_3$, while the normalizer $H = N_G(3C) \cong (\mathrm{SL}_2(8) \rtimes C_3) \times \mathfrak{S}_3$ is a maximal subgroup of G.

Using the data on subgroup fusions available in [4], it follows that the elements of $P - \{1\}$ belong to the $2B$ conjugacy class of G, hence [10, p.134] shows that $C_G(R) \cong R \times \mathsf{M}_{12}$, which is another maximal subgroup of G. Moreover, it follows that $C_G(Q) \cong C_2 \times C_{\mathsf{M}_{12}}(2A) \cong C_2 \times (C_2 \times \mathfrak{S}_5)$, where by [10, p.33] $C_2 \times \mathfrak{S}_5$ is a maximal subgroup of M_{12}. Finally, the structure of $C_H(P)$, $C_H(R)$, and $C_H(Q)$ follows from a consideration of the action of $F_{21} \leqslant \mathrm{GL}_3(2)$ on the defect group P.

(xi) This follows by **2.7**. □

Notation 4.3. We use the notation H, β and B as in **4.2**(ii), (iv) and (xi), respectively. We denote the unique simple $k\mathfrak{S}_3$-module in β by $2_{\mathfrak{S}_3}$.

It is now time to harvest what we have sown in our analysis of the 2-local structure of G. In **4.5**, we use our previous results to obtain a stable equivalence of Morita type between the blocks A and A_N via **2.3**. Similarly in **4.4**, we derive a stable equivalence between the blocks B and A_N, which together with the first yields the stable equivalence sought between A and B in **4.6**.

Lemma 4.4. *Let* $\mathfrak{f}_1$ *be the Green correspondence with respect to* $(H \times H, \Delta P, H \times N)$, *and set* $\mathfrak{N} = \mathfrak{f}_1(B)$. *Then* $\mathfrak{N}$ *induces a stable equivalence of Morita type between* B *and* A_N.

Proof. By **2.4**, $\mathfrak{N} | 1_B \cdot kH \cdot 1_{A_N}$. We know by **4.2**(viii) and **4.2**(x) that

$$C_H(Q) = C_N(Q) = P \times \mathfrak{S}_3 \quad \text{and} \quad C_H(R) = C_N(R) = (P \rtimes C_3) \times \mathfrak{S}_3.$$

Let $\mathbb{A}_Q$, $\mathbb{A}_R$, $\mathbb{B}_Q$ and $\mathbb{B}_R$ be the block algebras of $kC_H(Q)$, $kC_H(R)$, $kC_N(Q)$ and $kC_N(R)$, respectively, such that they have P as a defect group. Then

$$\mathbb{A}_Q = \mathbb{B}_Q = kP \otimes k\mathfrak{S}_3 \cdot \beta \cong \mathrm{Mat}_2(kP) \quad \text{and} \quad \mathbb{A}_R = \mathbb{B}_R = k[P \rtimes C_3] \otimes k\mathfrak{S}_3 \cdot \beta,$$

where the isomorphism is of k-algebras. Thus we obviously know that

$$\mathfrak{f}_Q(\mathbb{A}_Q) = \mathbb{A}_Q \quad \text{and} \quad \mathfrak{f}_R(\mathbb{A}_R) = \mathbb{A}_R,$$

where $\mathfrak{f}_Q$ and $\mathfrak{f}_R$ are the Green correspondences with respect to

$$(C_H(Q) \times C_H(Q),\ \Delta P,\ C_H(Q) \times C_N(Q)) \qquad \text{and} \qquad (C_H(R) \times C_H(R),\ \Delta P,\ C_H(R) \times C_N(R)),$$

respectively. Thus $\mathfrak{f}_Q(\mathbb{A}_Q)$ induces a Morita equivalence between $\mathbb{A}_Q$ and $\mathbb{B}_Q$, and $\mathfrak{f}_R(\mathbb{A}_R)$ induces a Morita equivalence between $\mathbb{A}_R$ and $\mathbb{B}_R$. Therefore we get the assertion by **2.3**. □

Lemma 4.5. *Let* $\mathfrak{f}_2$ *be the Green correspondence with respect to* $(G \times G, \Delta P, G \times N)$, *and set* $\mathfrak{M} = \mathfrak{f}_2(A)$. *Then we get*

(i) $\mathfrak{M} \mid 1_A \cdot kG \cdot 1_{A_N}$.

(ii) *The bimodule* $e_R \mathfrak{M}(\Delta R) f_R$ *induces a Morita equivalence between the block algebras* $kC_G(R)e_R$ *and* $kC_N(R)f_R$.

(iii) *The bimodule* $e_Q \mathfrak{M}(\Delta Q) f_Q$ *induces a Morita equivalence between the block algebras* $kC_G(Q)e_Q$ *and* $kC_N(Q)f_Q$.

(iv) $\mathfrak{M}$ *induces a stable equivalence of Morita type between* A *and* A_N.

Proof. (i) This follows from **4.2**(v) and **2.4**.

(ii) Let $\mathfrak{f}_R$ be the Green correspondence with respect to $(C_G(R) \times C_G(R), \Delta P, C_G(R) \times C_N(R))$. We get from (i) and **2.2** that $\mathfrak{f}_R\Big(e_R k C_G(R)\Big) = e_R \mathfrak{M}(\Delta R) f_R$. Hence we obtain the assertion by **3.5**.

(iii) Analogous to the proof of (ii) if we use **3.3** instead of **3.5**.

(iv) This follows by **3.5** and **3.3**, (i)–(iii) and **2.3**. □

Lemma 4.6. *There is an (A, B)-bimodule M which satisfies the following:*

1. *${}_A M_B$ is indecomposable,*
2. *$({}_A M_B, {}_B M^\vee{}_A)$ induces a stable equivalence of Morita type between A and B,*
3. *${}_A M_B \mid k_{\Delta P}\uparrow^{G\times H}$ and ${}_B M^\vee{}_A \mid k_{\Delta P}\uparrow^{H\times G}$,*
4. *the stable equivalence of Morita type induced by ${}_A M_B$ preserves vertices and sources,*
5. *for any indecomposable $X \in$ mod-A with vertex in $\mathfrak{A}(G, P, N)$, it holds $(X \otimes_A M)_B = f(X) \oplus (\text{proj})$, where f is the Green correspondence with respect to (G, P, H) (recall that $\mathfrak{A}(G, P, N) \subseteq \mathfrak{A}(G, P, H) \cap \mathfrak{A}(H, P, N)$ by* **2.8**(i)*).*

Proof. Let $\mathfrak{f}_2$ be the Green correspondence with respect to $(G \times G, \Delta P, G \times N)$, and set $\mathfrak{M} = \mathfrak{f}_2(A)$. Let f_2 be the Green correspondence with respect to (G, P, N). Moreover, let $\mathfrak{f}_1$ be the Green correspondence with respect to $(H \times H, \Delta P, H \times N)$, and set $\mathfrak{N} = \mathfrak{f}_1(B)$. Let f_1 be the Green correspondence with respect to (H, P, N). Then by **4.4** and **4.5** the bimodules $\mathfrak{N}$ and $\mathfrak{M}$ induce stable equivalences, so by **A.3**, and **2.5** there is a bimodule ${}_A M_B$ such that

$$(*) \qquad {}_A(\mathfrak{M} \otimes_{A_N} \mathfrak{N}^\vee)_B = {}_A M_B \oplus (\text{proj } (A, B)\text{-bimodule})$$

and (1)–(4) hold.

It remains to show (5). Take any indecomposable $X \in$ mod-A with a vertex which is in $\mathfrak{A}(G, P, N)$. Then it follows from $(*)$ that

$$X \otimes_A (\mathfrak{M} \otimes_{A_N} \mathfrak{N}^\vee) \;=\; X \otimes_A (M \oplus (\text{proj } (A, B)\text{-bimodule})).$$

On the other hand, by **2.8**(ii) we get

$$\begin{aligned}
(X \otimes_A \mathfrak{M}) \otimes_{A_N} \mathfrak{N}^\vee &= [f_2(X) \oplus (\text{proj } A_N\text{-module})] \otimes_{A_N} \mathfrak{N}^\vee \\
&= (f_2(X) \otimes_{A_N} \mathfrak{N}^\vee)_B \oplus ((\text{proj } A_N\text{-module}) \otimes_{A_N} \mathfrak{N}^\vee)_B \\
&= (f_2(X) \otimes_{A_N} \mathfrak{N}^\vee)_B \oplus (\text{proj } B\text{-module}) \\
&= ({f_1}^{-1}(f_2(X)))_B \oplus (\text{proj } B\text{-module}) \\
&= f(X) \oplus (\text{proj } B\text{-module})
\end{aligned}$$

□

5. Modules in A, B and A_N

In the previous section, we have shown that there is a stable equivalence of Morita type between the blocks A and B. As outlined in the introduction, our aim now is verify that this equivalence is in fact a Morita equivalence with the help of **2.1**. In other words, we need to show that the associated tensor functor takes simple modules to simple modules. Therefore in this intermediate section we collect all the necessary information on the simple modules and some indecomposable modules lying in the three blocks we consider.

In addition to the notation of our standard hypothesis **4.1**, we fix the following:

Lemma 5.1 (Suleiman-Wilson [53])**.** *The 2-decomposition matrix of A is given in Table 1, where $S_1, \cdots, S_5$ are non-isomorphic simple kG-modules in A whose degrees are* 73600*,* 896*,* 896*,* 19712*,* 131584*, respectively. The two simple modules S_2 and S_3 are dual to each other, while the remaining are self-dual. There are two pairs (χ_6, χ_7) and (χ_{18}, χ_{19}) of complex conjugate characters. All other χ's are real-valued.*

Proof. See [53, §6]. □

degree	[10, p.135]	S_1	S_2	$S_3 = S_2^*$	S_4	S_5
73 600	χ_{29}	1	.	.	.	.
896	χ_6	.	1	.	.	.
896	$\chi_7 = \chi_6^*$	.	.	1	.	.
93 312	χ_{32}	1	.	.	1	.
20 608	χ_{18}	.	1	.	1	.
20 608	$\chi_{19} = \chi_{18}^*$	.	.	1	1	.
226 688	χ_{38}	1	1	1	1	1
246 400	χ_{39}	1	1	1	2	1

TABLE 1. The 2-modular decomposition matrix of Co_3.

Remark 5.2. The 2-blocks of Co_3 have been studied before by several other people, see [12, p.193 Table 6], [26, §7 p.1879] and [27, Theorems 3.10 and 3.11].

Notation 5.3. We use the notation $\chi_{29}, \chi_6, \chi_7, \chi_{32}, \chi_{18}, \chi_{19}, \chi_{38}, \chi_{39}$, and $S_1, \cdots, S_5$ as in **5.1**.

Lemma 5.4. *All simple kG-modules $S_1, \cdots, S_5$ in A have P as a vertex.*

Proof. See [16, 3.7.Corollary]. □

Lemma 5.5. *We get the following:*

(i) $A_N = k[P \rtimes F_{21}] \otimes \beta \cong \mathrm{Mat}_2(k[P \rtimes F_{21}])$*, as k-algebras.*

(ii) *We can write* $\mathrm{Irr}(F_{21}) = \{k, 1, 1^*, 3, 3^*\}$.

(iii) *We can write*

$$\mathrm{IBr}(A_N) = \{\widetilde{2}_0 = k_{P\rtimes F_{21}} \otimes 2_{\mathfrak{S}_3},\ \widetilde{2} = 1 \otimes 2_{\mathfrak{S}_3},\\ \widetilde{2}^* = 1^* \otimes 2_{\mathfrak{S}_3},\ \widetilde{6} = 3 \otimes 2_{\mathfrak{S}_3},\ \widetilde{6}^* = 3^* \otimes 2_{\mathfrak{S}_3}\}.$$

Note that there exists a unique simple $\widetilde{2}_0$ which is self-dual.

(iv) *The trivial source A_N-modules with vertex P are precisely the simple A_N-modules.*

Proof. (i)–(iii) are easy by **4.2** and the definition of A_N.

(iv) This follows from (iii) and the Green correspondence [40, Chap.4 Problem 10]. □

Lemma 5.6. *Set $\mathfrak{R} = R(3) \cong \mathrm{SL}_2(8) \rtimes C_3$. We get the following:*

(i) *For the principal block of $k\mathfrak{R}$ we have*

$$\mathrm{Irr}(B_0(k\mathfrak{R})) = \{1_{\mathfrak{R}}, \chi_1, \chi_1^*, \chi_{7a}, \chi_{7b}, \chi_{7c}, \chi_{21}, \chi_{27}\},$$

and

$$\mathrm{IBr}(B_0(k\mathfrak{R})) = \{k_{\mathfrak{R}}, 1, 1^*, 6, 12\},$$

where the indices give the degrees (dimensions). The simples $k_{\mathfrak{R}}, 6, 12$ are self-dual, and the simples $k_{\mathfrak{R}}, 1, 1^*$ are trivial source $k\mathfrak{R}$-modules.

(ii) *For the block B we have*

$$\mathrm{Irr}(B) = \{\chi_{2a}, \chi_2, \chi_2^*, \chi_{14a}, \chi_{14b}, \chi_{14c}, \chi_{42}, \chi_{54}\},$$

and

$$\mathrm{IBr}(B) = \{2_0 = k_{\mathfrak{R}} \otimes 2_{\mathfrak{S}_3}, 2 = 1 \otimes 2_{\mathfrak{S}_3},\\ 2^* = 1^* \otimes 2_{\mathfrak{S}_3},\ 12 = 6 \otimes 2_{\mathfrak{S}_3},\ 24 = 12 \otimes 2_{\mathfrak{S}_3}\},$$

where the indices give the degrees (dimensions). The simple kH-modules $2_0, 2, 2^*$ in B are trivial source modules, the simple kH-modules 2_0, 12, 24 are self-dual, and all the simples in B have P as their vertices.

Proof. (i) It follows from [10, p.6], and [14, $L_2(8).3$ (mod 2)] or [59, $L_2(8).3$ (mod 2)], see **4.2**(xi). Clearly, $k_{\mathfrak{R}}, 1, 1^*$ are trivial source $k\mathfrak{R}$-modules.

(ii) $2_{\mathfrak{S}_3}$ is a trivial source $k\mathfrak{S}_3$-module. Therefore the simples $2_0, 2, 2^*$ are trivial source kH-modules, by (i) and **4.2**(xi). Finally, use [16, 3.7.Corollary]. □

Notation 5.7. We use the notation $\mathfrak{R}$, $\chi_{2a}, \chi_2, \chi_2^*, \chi_{14a}, \chi_{14b}, \chi_{14c}, \chi_{42}, \chi_{54}$, $\widetilde{2}_0, \widetilde{2}, \widetilde{2}^*, \widetilde{6}, \widetilde{6}^*$ and $2_0, 2, 2^*, 12, 24$ as in **5.5** and **5.6**.

Lemma 5.8 (Landrock-Michler [29])**.** *The radical and socle series of projective indecomposable kH-modules in B are the following:*

$$\boxed{\begin{array}{c} 2_0 \\ 12 \\ 2_0\ 2\ 2^*\ 24 \\ 12\ 12 \\ 2_0\ 2\ 2^*\ 24 \\ 12 \\ 2_0 \end{array}}, \quad \boxed{\begin{array}{c} 2 \\ 12 \\ 2_0\ 2\ 2^*\ 24 \\ 12\ 12 \\ 2_0\ 2\ 2^*\ 24 \\ 12 \\ 2 \end{array}} \quad \boxed{\begin{array}{c} 2^* \\ 12 \\ 2_0\ 2\ 2^*\ 24 \\ 12\ 12 \\ 2_0\ 2\ 2^*\ 24 \\ 12 \\ 2^* \end{array}} \quad \boxed{\begin{array}{c} 12 \\ 2_0\ 2\ 2^*\ 24 \\ 12\ 12\ 12 \\ 2_0\ 2_0\ 2\ 2\ 2^*\ 2^*\ 24\ 24 \\ 12\ 12\ 12 \\ 2_0\ 2\ 2^*\ 24 \\ 12 \end{array}} \quad \boxed{\begin{array}{c} 24 \\ 12 \\ 2_0\ 2\ 2^* \\ 12 \\ 2_0\ 2\ 2^* \\ 12 \\ 24 \end{array}}$$

Proof. This follows from [29, Theorem 3.9, Theorem 4.1] and **5.6**. □

Lemma 5.9. *Recall that R is a subgroup of P with $R \cong C_2$, see* **4.1**.

(i) *The Scott module* $\mathrm{Scott}(\mathfrak{R}, R)$ *has the radical and socle series*

$$\boxed{\begin{array}{c} k \\ 6 \\ 1\ 1^*\ 12 \\ 6 \\ k \end{array}} \leftrightarrow 1_{\mathfrak{R}} + \chi_{27}.$$

(ii) *A kH-module* $\mathrm{Scott}(\mathfrak{R}, R) \otimes 2_{\mathfrak{S}_3}$ *has the radical and socle series*

$$\boxed{\begin{array}{c} 2_0 \\ 12 \\ 2\ 2^*\ 24 \\ 12 \\ 2_0 \end{array}} \leftrightarrow \chi_{2a} + \chi_{54}.$$

Proof. By **5.6**(ii), it suffices to prove (i). [10, p.6] says that $\mathfrak{R}$ has a maximal subgroup M such that $M = C_9 \rtimes C_6$, $|\mathfrak{R} : M| = 28$ and $1_M{\uparrow}^{\mathfrak{R}} = 1_{\mathfrak{R}} + \chi_{27}$. Set $X = k_M{\uparrow}^{\mathfrak{R}}$. Then $X = 2 \times k + 1 + 1^* + 2 \times 6 + 12$ as composition factors by [14, $L_3(8)$.3 (mod 2)] and [59, $L_3(8)$.3 (mod 2)]. It holds by [40, 4 Thm.8.9(i)] that $[X, X]^{\mathfrak{R}} = 2$, $[X, k]^{\mathfrak{R}} = [k, X]^{\mathfrak{R}} = 1$ Thus, $X/\mathrm{rad}(X) \cong \mathrm{soc}(X) \cong k_{\mathfrak{R}}$. Now, it follows from [29, Theorem 4.1] that $P(k_{\mathfrak{R}})$ has the following radical and socle series:

$$P(k_{\mathfrak{R}}) = \boxed{\begin{array}{c} k \\ 6 \\ k\ 1\ 1^*\ 12 \\ 6\ 6 \\ k\ 1\ 1^*\ 12 \\ 6 \\ k \end{array}}.$$

Since there is an epimorphism $P(k_{\mathfrak{R}}) \twoheadrightarrow X$, we infer $\mathrm{soc}(X) < \mathrm{soc}^2(X) < \mathrm{rad}^2(X) < \mathrm{rad}(X)$ and $\mathrm{rad}(X)/\mathrm{rad}^2(X) \cong \mathrm{soc}^2(X)/\mathrm{soc}(X) \cong 6$. Thus X has the radical and socle series as asserted. By the definition of X, it holds that $X = \mathrm{Scott}(\mathfrak{R}, C_2)$, see [40, Chap.4 Theorem 8.4 and Corollary 8.5]. □

Lemma 5.10. *Recall that Q is a subgroup of P with $Q \cong C_2 \times C_2$, see* **4.1**. *Set $U = \mathrm{Scott}(\mathfrak{R}, Q)$.*

(i) *We have $U \leftrightarrow 1_{\mathfrak{R}} + \chi_{7a} + 2 \times \chi_{27}$, and $U = 4 \times k_{\mathfrak{R}} + 2 \times 1 + 2 \times 1^* + 5 \times 6 + 2 \times 12$ as composition factors.*

(ii) *Set $V = U \otimes 2_{\mathfrak{S}_3}$. Then V is a trivial source kH-module in B with vertex Q, $V \leftrightarrow \chi_{2a} + \chi_{14a} + 2 \times \chi_{54}$, and $V = 4 \times 2_0 + 2 \times 2 + 2 \times 2^* + 5 \times 12 + 2 \times 24$, as composition factors.*

Proof. (i) We know that $\mathfrak{R}$ has a subgroup $\mathfrak{A}_4$, see [10, p.6]. Clearly, $\mathrm{Irr}(\mathfrak{A}_4) = \{1_{\mathfrak{A}_4}, \psi_1, \psi_2 = \psi_1^*, \psi_3\}$ where ψ_3 has degree 3. It follows from computations with GAP [13] that

$$1_{\mathfrak{A}_4}{\uparrow}^{\mathfrak{R}} \cdot 1_{B_0(k\mathfrak{R})} = 1_{\mathfrak{R}} + \chi_{7a} + \chi_{21} + 3 \times \chi_{27}, \tag{1}$$

$$\psi_1{\uparrow}^{\mathfrak{R}} \cdot 1_{B_0(k\mathfrak{R})} = \chi_1 + \chi_{7b} + \chi_{21} + 3 \times \chi_{27}, \tag{2}$$

$$\psi_{1^*}{\uparrow}^{\mathfrak{R}} \cdot 1_{B_0(k\mathfrak{R})} = \chi_{1^*} + \chi_{7c} + \chi_{21} + 3 \times \chi_{27}. \tag{3}$$

Let $X = k_{\mathfrak{A}_4}{\uparrow}^{\mathfrak{R}} \cdot 1_{B_0(k\mathfrak{R})}$. First, we want to claim that $P(12) \mid X$, where $P(12)$ is the projective cover 12.

Set $S = \mathrm{SL}_2(8)$. By Clifford theory, we have $12 \downarrow_S = 4_1 \oplus 4_2 \oplus 4_3$, where 4_1, 4_2, 4_3 are non-isomorphic simple kS-modules in $B_0(kS)$ of dimension 4, see [14, $L_2(8)$ (mod 2)] and [59, $L_2(8)$ (mod 2)]. Let V_1 be the tautological kS-module, which is simple of dimension 2, and let V_2 and V_3 be its images under the action of the Frobenius automorphism of $\mathbb{F}_8$. Then the V_i are pairwise non-isomorphic, and by [1, p.220] we may assume that

$$4_1 = V_1 \otimes V_2, \quad 4_2 = V_2 \otimes V_3, \quad 4_3 = V_3 \otimes V_1.$$

Set $g_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in S$ for all $a \in \mathbb{F}_8$. We may assume that $P = \{g_a \mid a \in \mathbb{F}_8\} \leq S$, namely, P is a Sylow 2-subgroup of S with $P \cong C_2 \times C_2 \times C_2$, and that $Q = \{g_0, g_1, g_\alpha, g_{1+\alpha}\}$, where $\alpha \in \mathbb{F}_8^*$ is a fixed primitive root, hence $Q \cong C_2 \times C_2$. Now the action of $g_0 + g_1 + g_\alpha + g_{1+\alpha} = (1 + g_1)(1 + g_\alpha) \in kQ$ is easily described in terms of Kronecker products of matrices, and it turns out that this element does not annihilate any of the kQ-modules 4_i. Therefore $4_i{\downarrow}_Q$ has a projective indecomposable summand, and thus we infer that $4_i{\downarrow}_Q = P(k_Q)$.

We conclude $12{\downarrow}_Q = 12{\downarrow}_S{\downarrow}_Q = (4_1 \oplus 4_2 \oplus 4_3){\downarrow}_Q \cong 3 \times P(k_Q)$, and it follows from [50, Theorem 3] that

$$\begin{aligned} 3 &= [P(k_Q) \mid 12{\downarrow}_Q]^Q = [P(12) \mid k_Q{\uparrow}^{\mathfrak{R}}]^{\mathfrak{R}} = [P(12) \mid k_Q{\uparrow}^{\mathfrak{A}_4}{\uparrow}^{\mathfrak{R}}]^{\mathfrak{R}} \\ &= [P(12) \mid (k_{\mathfrak{A}_4} \oplus 1_{\mathfrak{A}_4} \oplus 1^*_{\mathfrak{A}_4}){\uparrow}^{\mathfrak{R}}]^{\mathfrak{R}} = [P(12) \mid (k_{\mathfrak{A}_4}{\uparrow}^{\mathfrak{R}} \oplus 1_{\mathfrak{A}_4}{\uparrow}^{\mathfrak{R}} \oplus 1^*_{\mathfrak{A}_4}{\uparrow}^{\mathfrak{R}})]^{\mathfrak{R}}. \end{aligned}$$

Suppose that $P(12) \nmid k_{\mathfrak{A}_4}{\uparrow}^{\mathfrak{R}}$. Then $3 \times P(12) \mid (1_{\mathfrak{A}_4}{\uparrow}^{\mathfrak{R}} \oplus 1^*_{\mathfrak{A}_4}{\uparrow}^{\mathfrak{R}})$. Since $P(12) \leftrightarrow \chi_{21} + \chi_{27}$ by [14, $L_2(8).3$ (mod 2)] and [59, $L_2(8).3$ (mod 2)], we know by (2) and (3) that $3 \times \chi_{21} + 3 \times \chi_{27}$ is contained in $(\chi_1 + \chi_{7b} + \chi_{21} + 3 \times \chi_{27}) + (\chi_1^* + \chi_{7c} + \chi_{21} + 3 \times \chi_{27})$, which contradicts the multiplicity of χ_{21}.

Therefore $P(12) \mid k_{\mathfrak{A}_4}{\uparrow}^{\mathfrak{R}}$. Since $P(12) \leftrightarrow \chi_{21} + \chi_{27}$ as seen above, it follows from (1) that

$$k_{\mathfrak{A}_4}{\uparrow}^{\mathfrak{R}} \cdot 1_{B_0(k\mathfrak{R})} \;=\; X \oplus P(12)$$

for a $k\mathfrak{R}$-module X such that

$$X \leftrightarrow 1_{\mathfrak{R}} + \chi_{7a} + 2 \times \chi_{27}.$$

Now, let $U = \mathrm{Scott}(\mathfrak{R}, Q)$, and hence $U|X$ since Q is a Sylow 2-subgroup of $\mathfrak{A}_4$, see [40, Chap.4 Corollary 8.5]. By the definition of Scott modules and [40, 4 Thm.8.9(i)], we know $(\chi_{\widehat{U}}, 1_{\mathfrak{R}})^{\mathfrak{R}} = 1$. Clearly, $\chi_{\widehat{U}} \neq 1_{\mathfrak{R}}$ since $Q \lneqq P$. Since P is a Sylow 2-subgroup of $\mathfrak{R}$, it follows from [40, Chap.4, Theorem 7.5] that $\dim_k(U)$ is even. This means that $\chi_{\widehat{U}} \neq 1_{\mathfrak{R}} + 2 \times \chi_{27}$ and that $\chi_{\widehat{U}} \neq 1_{\mathfrak{R}} + \chi_{7a} + \chi_{27}$. If $\chi_{\widehat{U}} = 1_{\mathfrak{R}} + \chi_{7a}$ then $\chi_{\widehat{U}}(2A) = 1 + (-1) = 0$ by [10, p.6], contradicting [28, II Lemma 12.6] since $2A \in Q$. Suppose that $\chi_{\widehat{U}} = 1_{\mathfrak{R}} + \chi_{27}$. Then since U is a trivial source $k\mathfrak{R}$-module, we get that U has the same radical and socle series of $\mathrm{Scott}(\mathfrak{R}, R)$ just by the same method as in **5.9**. Since $[U, \mathrm{Scott}(\mathfrak{R}, R)]^{\mathfrak{R}} = 2$ by [40, 4 Thm.8.9(i)], we have $U \cong \mathrm{Scott}(\mathfrak{R}, R)$, and hence $Q \cong R$ by [40, Chap.4, Corollary 8.5], again a contradiction.

Therefore we know that $\chi_{\widehat{U}} = 1_{\mathfrak{R}} + \chi_{7a} + 2 \times \chi_{27}$ and $U = X$, so that $U = 4 \times k_{\mathfrak{R}} + 2 \times 1 + 2 \times 1^* + 5 \times 6 + 2 \times 12$ as composition factors.

(ii) This follows from (i) and **4.2**(xi). □

Remark 5.11. We will not need the precise structure of $U = \mathrm{Scott}(\mathfrak{R}, Q)$. Still we would like to remark that using the table of marks library of GAP [13], and the facilities available in the MeatAxe [49] and its extensions, U can actually be constructed and analysed explicitly.

In particular, it turns out that U has Loewy length 5, but its radical and socle series do not coincide; they are

$$\boxed{\begin{array}{c} k\ 6 \\ k\ 1\ 1^*\ 12 \\ 6\ 6\ 6 \\ k\ k\ 1\ 1^*\ 12 \\ 6 \end{array}} \qquad \text{and} \qquad \boxed{\begin{array}{c} 6 \\ k\ k\ 1\ 1^*\ 12 \\ 6\ 6\ 6 \\ k\ 1\ 1^*\ 12 \\ k\ 6 \end{array}},$$

respectively.

6. Images of simples in A via Green correspondence

In this section we prove that the crucial hypothesis of **2.1** is fulfilled for the stable equivalence of Morita type we have established in **4.6**. Namely, we show that simple modules in A are taken to simple modules in B. For the first four simples this is almost immediate, as this amounts to determining the Green correspondents with respect to (G, P, H), and these are easily determined theoretically and computationally. The image of the last simple A-module however, is more difficult to determine, and we make use of our knowledge on the modules of the blocks A and B we have gained in Section 5.

Notation 6.1. We use the notation ${}_AM_B$, f, f_1 and f_2 as in **4.6**. Let F : mod-$A \to$ mod-B denote the functor giving the stable equivalence of Morita type of **4.6**, namely, in the notation of **4.6** we have $F(X) = X \otimes_A M$ for each $X \in$ mod-A.

Lemma 6.2. *The following hold:*

(i) $S_4 = 22 \otimes S_2$, *where* 22 *is a simple* kG*-module in* $B_0(kG)$.

(ii) *We have*

$$22{\downarrow_H} = (6 \otimes k_{\mathfrak{S}_3}) \oplus (\text{proj}), \quad S_2{\downarrow_H} = 2 \oplus 110 \oplus (\text{proj}) \quad \text{and} \quad (6 \otimes k_{\mathfrak{S}_3}) \otimes 2 = 12,$$

where $6 \otimes k_{\mathfrak{S}_3}$ *is a simple* kH*-module in* $B_0(kH) = B_0(kR(3)) \otimes B_0(k\mathfrak{S}_3)$, *and* 110 *is an indecomposable* kH*-module in* $B_0(kH)$, *hence* $S_2{\downarrow_H}{\cdot}1_B = 2$ *and* $S_2^*{\downarrow_H}{\cdot}1_B = 2^*$.

(iii) $12 \mid S_4{\downarrow_H}$.

Proof. (i) This is obtained by [53, p.502], see [59, Co_3 (mod 2)], and a direct computation with Brauer characters in GAP [13].

(ii) By [14, $L_3(8).3$ (mod 2)] or [59, $L_3(8).3$ (mod 2)], except for the principal 2-block $B_0(k[R(3)])$ of $kR(3) = k[\mathrm{SL}_2(8) \rtimes C_3]$ there are only three 2-blocks of defect zero, consisting of the extensions of the Steinberg character of $\mathrm{SL}_2(8)$ to $R(3)$. Hence it is easy to write down the block idempotents of $kR(3)$, and similarly those of $k\mathfrak{S}_3$. Thus, H being a small group of order 9 072, using GAP [13] the block idempotents of kH can be explicitly evaluated in a given representation. This yields the block components, which are then further analysed using the MeatAxe [49] and its extensions.

(iii) It follows from (i) and (ii) that

$$\begin{aligned} S_4{\downarrow_H} &= (22 \otimes S_2){\downarrow_H} = 22{\downarrow_H} \otimes S_2{\downarrow_H} \\ &= \Big((6 \otimes k_{\mathfrak{S}_3}) \oplus (\text{proj})\Big) \otimes \Big(2 \oplus 110 \oplus (\text{proj})\Big) \\ &= ((6 \otimes k_{\mathfrak{S}_3}) \otimes 2) \oplus (\text{other}) = 12 \oplus (\text{other}). \end{aligned}$$

□

Lemma 6.3. *We have* $f(S_2) = 2$, $f(S_2^*) = 2^*$, $f(S_4) = 12$, *and hence that* $F(S_2) = 2$, $F(S_2^*) = 2^*$ *and* $F(S_4) = 12$.

Proof. By **6.2**(ii) the Green correspondents of S_2 and S_2^* are immediate. By **5.4** all simple A-modules have vertex $P \in \mathfrak{A}(G, P, H)$, and by **6.2**(ii) the direct summands of $(6 \otimes k_{\mathfrak{S}_3}) \otimes 110$ lie in the principal block. Therefore by **6.2**(iii) and **5.6**(ii) the simple module 12 is the unique summand of $S_4{\downarrow_H}$ in B with vertex P. Hence $f(S_4) = 12$. By **4.6**(5) and **2.1**(i) the functor F maps any simple A-module to its Green correspondent in B, and so the claim follows. □

Lemma 6.4. *The simples S_2 and S_2^* are trivial source kG-modules with $S_2 \leftrightarrow \chi_6$ and $S_2^* \leftrightarrow \chi_6^*$.*

Proof. We know by **5.6**(ii) that 2 and 2^* are trivial source kH-modules. Hence, by the definition of Green correspondence, **6.3** and **5.1**, we get the assertion. □

Lemma 6.5. *The simple kG-module S_1 in A is a trivial source module with $S_1 \leftrightarrow \chi_{29}$.*

Proof. It follows from [10, p.143] that G has a maximal subgroup L with $L = 2^{\cdot}S_6(2)$. Then using GAP [13], we know that $1_L{\uparrow}^G{\cdot}1_A = \chi_{29}$. Hence the assertion follows by **5.1**. □

Lemma 6.6. *It is $f(S_1) = 2_0$, and hence $F(S_1) = 2_0$.*

Proof. First, let f_1' be the Green correspondence with respect to $(R(3), P, P \rtimes F_{21})$. Clearly, $f_1'(k_{R(3)}) = k_{P\rtimes F_{21}}$. Since f_1 is the Green correspondence with respect to $(H, P, N) = (R(3) \times \mathfrak{S}_3, P, (P \rtimes F_{21}) \times \mathfrak{S}_3)$, we know that $f_1(k_{R(3)} \otimes 2_{\mathfrak{S}_3}) = k_{P\rtimes F_{21}} \otimes 2_{\mathfrak{S}_3}$, namely, $f_1(2_0) = \widetilde{2}_0$.

By **2.8**(ii), $f_1 \circ f = f_2$. Thus it follows from **5.4**, **6.5** and **2.8**(iii) that $f_1 \circ f(S_1)$ is a trivial source kN-module in A_N with vertex P. Hence **5.5**(iv) implies that

$$f_1 \circ f(S_1) \in \{\widetilde{2}_0, \widetilde{2}, \widetilde{2}^*, \widetilde{6}, \widetilde{6}^*\}.$$

Then since S_1 is self-dual by **5.1**, we know that $f_1 \circ f(S_1)$ is also self-dual. Therefore $f_1 \circ f(S_1) = \widetilde{2}_0$, giving $f_1 \circ f(S_1) = f_1(2_0)$. This implies that $f(S_1) = 2_0$. Hence we get the assertion from **4.6**(5) and **2.1**(i). □

Lemma 6.7. *The following hold:*

(i) $\mathrm{Ext}_A^1(S_1, S_2) = \mathrm{Ext}_A^1(S_1, S_2^*) = \mathrm{Ext}_A^1(S_2, S_1) = \mathrm{Ext}_A^1(S_2^*, S_1) = 0$.

(ii) $\mathrm{Ext}_A^1(S_2, S_2^*) = \mathrm{Ext}_A^1(S_2^*, S_2) = 0$.

(iii) $\dim_k[\mathrm{Ext}_A^1(S_1, S_4)] = \dim_k[\mathrm{Ext}_A^1(S_4, S_1)] = 1$.

Proof. By **6.6** and **6.3** we know the simple images of the simple modules given under the stable equivalence F of **6.1**. Hence the results are immediate by looking at the B-PIMs in **5.8** and using **A.1**. □

Lemma 6.8. *All composition factors of $F(S_5)/\mathrm{rad}(F(S_5))$ and $\mathrm{soc}(F(S_5))$ are isomorphic to the simple module 24.*

Proof. Take any simple kH-module T in B such that $T \not\cong 24$. Then we know by **5.6**, **6.3** and **6.6** that $T = F(S_i)$ for $i \in \{1, 2, 3, 4\}$, where $S_3 = S_2^*$. Then it follows from [28, II Lemma 2.7 and Corollary 2.8] and **6.1** that $\mathrm{Hom}_B(F(S_5), T) = \underline{\mathrm{Hom}}_B(F(S_5), T) = \underline{\mathrm{Hom}}_B(F(S_5), F(S_i)) \cong \underline{\mathrm{Hom}}_A(S_5, S_i) = \mathrm{Hom}_A(S_5, S_i) = 0$. Thus we get the assertion for the head of $F(S_5)$. The assertion for the socle follows by the same argument and considering $\mathrm{Hom}_B(T, F(S_5))$ instead. □

We can now finally prove that also the image of the last remaining simple A–module S_5 under F is a simple B-module.

Lemma 6.9. *It is $F(S_5) = 24$.*

Proof. By [10, p.134], G has a maximal subgroup $\mathfrak{U} = \mathrm{U}_3(5) \rtimes \mathfrak{S}_3$. Set $X = k_{\mathfrak{U}}{\uparrow}^G{\cdot}1_A$. By calculations in GAP [13] we know that $1_{\mathfrak{U}}{\uparrow}^G{\cdot}1_A = \chi_{29} + \chi_{39}$, so that

$$X \leftrightarrow \chi_{29} + \chi_{39}. \tag{4}$$

Hence, by **5.1**

$$X = 2 \times S_1 + S_2 + S_2^* + 2 \times S_4 + S_5 \text{ as composition factors.} \tag{5}$$

Since S_1, S_2 and S_2^* are trivial source kG-modules by **6.5** and **6.4**, it follows from (4), **5.1** and [40, 4 Thm.8.9(i)] that

$$[S_1, X]^G = [X, S_1]^G = 1, \qquad [S_2, X]^G = [X, S_2]^G = [S_2^*, X]^G = [X, S_2^*]^G = 0.$$

If $[S_5, X]^G \neq 0$ or $[X, S_5]^G \neq 0$, then the self-duality of X and S_5 implies that $S_5 \mid X$, and hence S_5 is a trivial source kG-module, so that S_5 is liftable to $\mathcal{O}$ by [40, 4 Thm.8.9(iii)], which contradicts to **5.1**. Hence

$$[S_5, X]^G = [X, S_5]^G = 0.$$

Assume $[S_4, X]^G \neq 0$ or $[X, S_4]^G \neq 0$. Then again the self-dualities of X and S_4 in **5.1** say that both are non-zero. Thus we have endomorphisms ψ_1, ψ_2 and ψ_3 of X such that $\psi_1 = id_X$, $\mathrm{Im}(\psi_2) \cong S_1$ and $\mathrm{Im}(\psi_3) \cong S_4$. This means $[X, X]^G \geqslant 3$. But [40, 4 Thm.8.9(i)] and (4) yield that $[X, X]^G = 2$, a contradiction. Thus $[S_4, X]^G = [X, S_4]^G = 0$. These imply that

$$X/\mathrm{rad}(X) \cong \mathrm{soc}(X) \cong S_1. \tag{6}$$

Hence X is indecomposable. Set $X_0 = \mathrm{rad}(X)/\mathrm{soc}(X)$, the heart of X. Thus (5) implies

$$X_0 = S_2 + S_2^* + 2 \times S_4 + S_5, \ \text{as composition factors.} \tag{7}$$

By **6.7**(i), it holds

$$[X_0, S_2]^G = [X_0, S_2^*]^G = [S_2, X_0]^G = [S_2^*, X_0]^G = 0.$$

Moreover, **6.7**(iii) yields that $X_0/\mathrm{rad}(X_0) \mid (S_4 \oplus S_5)$. These imply that the radical and socle series of X is one of the following:

$$X = \boxed{\begin{array}{c} S_1 \\ S_4 \\ S_2\ S_2^*\ S_5 \\ S_4 \\ S_1 \end{array}}\,, \quad \boxed{\begin{array}{c} S_1 \\ \boxed{\begin{array}{c} S_4 \\ S_2\ S_2^* \\ S_4 \end{array}} \oplus \ S_5 \\ S_1 \end{array}}\,, \quad \boxed{\begin{array}{c} S_1 \\ S_4 \\ S_2 \\ S_5 \\ S_2^* \\ S_4 \\ S_1 \end{array}} \quad \text{or} \quad \boxed{\begin{array}{c} S_1 \\ S_4 \\ S_2^* \\ S_5 \\ S_2 \\ S_4 \\ S_1 \end{array}}\,. \tag{8}$$

Now, it follows from **6.1**, [28, II Lemma 2.7 and Corollary 2.8], **6.3** and (6) that

$$\begin{aligned} \mathrm{Hom}_B(F(X), 2) &= \underline{\mathrm{Hom}}_B(F(X), 2) = \underline{\mathrm{Hom}}_B(F(X), F(S_2)) \\ &\cong \underline{\mathrm{Hom}}_A(X, S_2) = \mathrm{Hom}_A(X, S_2) = 0. \end{aligned}$$

Hence $[F(X), 2)]^B = 0$. Similarly we obtain $[F(X), 2^*]^B = 0$ and $[F(X), 12]^B = 0$ and $[F(X), 2_0]^B = 1$. Similar for $\mathrm{soc}(F(X))$, too. Thus, by **5.6**, we know that

$$F(X)/\mathrm{rad}(F(X)) \cong 2_0 \oplus (r \times 24) \ \text{and} \ \mathrm{soc}(F(X)) \cong 2_0 \oplus (r' \times 24) \tag{9}$$

for some $r, r' \geqslant 0$. By **6.1**, we have

$$F(X) = Y \oplus (\text{proj } B\text{-module}) \tag{10}$$

for a non-projective indecomposable kH-module Y in B. Thus, by **6.6** and **A.1** we have

$$2_0 \Big| Y/\mathrm{rad}(Y) \quad \text{and} \quad 2_0 \Big| \mathrm{soc}(Y). \tag{11}$$

Recall that $2_0 = k_{\mathfrak{R}} \otimes 2_{\mathfrak{S}_3}$ in **5.6**(ii). Since B and $B_0(k\mathfrak{R})$ are Puig equivalent by **4.2**(xi), and Y is a trivial source module by **4.6**, it follows that $Y \cong \mathrm{Scott}(\mathfrak{R}, S) \otimes 2_{\mathfrak{S}_3}$ for a subgroup S of P. Clearly $S \neq 1$ since Y is non-projective indecomposable. If $S = P$ then (11) yields $Y = 2_0$, so that $F(X) = 2_0 \oplus (\text{proj})$ and $F(S_1) = 2_0$ by **6.6**. This is a contradiction since X is non-projective indecomposable and non-simple. Thus $S \cong Q$ or $S \cong R$.

Suppose that $S \cong Q$, namely $Y \cong \mathrm{Scott}(\mathfrak{R}, Q) \otimes 2_{\mathfrak{S}_3}$. Then it follows by **5.10**(ii) that

$$Y \leftrightarrow \chi_{2a} + \chi_{14a} + 2 \times \chi_{54},$$

and we have

$$Y = 4 \times 2_0 + 2 \times 2 + 2 \times 2^* + 5 \times 12 + 2 \times 24, \ \text{as composition factors.} \tag{12}$$

We know by **6.6** and **6.3** that

$$F(S_1) = 2_0, \ F(S_4) = 12, \ F(S_2) = 2, \ F(S_2^*) = 2^*.$$

Thus it follows by (6), (8) and **A.1** that we can *strip off* $2 \times S_1$, $2 \times S_4$, S_2, and S_2^* from the top of X and from the bottom of X, and also $2 \times 2_0$, 2×12, 2, and 2^* from the top of

Y and from the bottom of Y sequentially, by looking at (8) and (12). Consequently by **2.1**(i), we have $F(S_5) = Z$ for an indecomposable kH-module Z in B such that $Z = 2 \times 2_0 + 2 + 2^* + 3 \times 12 + 2 \times 24$ as composition factors. Then **6.8** yields $Z/\mathrm{rad}(Z) \cong \mathrm{soc}(Z) \cong 24$ and $\mathrm{rad}(Z)/\mathrm{soc}(Z) = 2_0 + 2 + 2^* + 3 \times 12$ as composition factors, which contradicts **5.8**.

Therefore $S \cong R$ and $Y \cong \mathrm{Scott}(\mathfrak{R}, R) \otimes 2_{\mathfrak{S}_3}$. Hence we get by **5.9**(ii) that

$$(13) \qquad F(X) \;=\; Y \oplus (\mathrm{proj}), \qquad Y \;=\; \boxed{\begin{array}{c} 2_0 \\ 12 \\ 2\ 2^*\ 24 \\ 12 \\ 2_0 \end{array}}.$$

Thus by the same *stripping off* method taken above, we can subsequently strip off $2 \times S_1$, $2 \times S_4$, S_2, and S_2^* from the top of X and the bottom of X, and also $2 \times 2_0$, 2×12, 2, and 2^* from the top of Y and the bottom of Y, by looking at (8) and (13). Hence we arrive at $F(S_5) = 24 \oplus (\mathrm{proj})$, so that **2.1** yields $F(S_5) = 24$. □

Remark 6.10. We know by **1.7** that the block A of G and the principal 2-block $B_0(kR(3))$ of $R(3)$ are Puig equivalent. Let X be the same as in the proof of **6.9**. Thus it follows from **5.9**(i)-(ii) and the proof of **6.9** that the radical and socle series of X is actually the first one in (8) in the proof of **6.9**, and that X is a trivial source kG-module in A with vertex C_2.

7. Proof of the main results

Proof of 1.5. First of all, consider the blocks A and B over k, namely, A and B are block algebras of kG and kH, respectively. Hence M is a (kG, kH)-bimodule. We know by **4.6**(ii) and **6.1** that the functor F defined by M realises a stable equivalence of Morita type between A and B. It follows from **5.1**, **6.3**, **6.6** and **6.9** that, for any simple kG-module S in A, $F(S)$ is a simple kH-module in B. Hence, **2.1**(ii) yields that ${}_AM_B$ realizes a Morita equivalence between A and B. Since M is a ΔP-projective trivial source $k[G \times H]$-module, the Morita equivalence is a Puig equivalence by [44, Remark 7.5] or [33, Theorem 4.1] (note that this was independently observed by L. Scott). Moreover, by [40, 4 Thm.8.9(i)], the Morita equivalence lifts from k to $\mathcal{O}$; see also [54, (38.8)Proposition] or [43, 7.8.Lemma]. □

Proof of Corollary 1.7. This follows by **1.5**, **1.6** and **2.7**. □

Proof of Theorem 1.3. This follows from **1.7**, **2.7** and **4.2** (i). □

Appendix A. Properties of the stable equivalences considered

In this appendix we collect some fundamental properties of the stable equivalences which are found throughout this paper, and in particular of the stable equivalence F of **6.1**. For the large part, these properties are used at several steps in this paper, but they are also of independent interest, as a reference providing collection with proofs is desirable. Also, in this section, we aim to supply more general hypotheses for clarity.

The first fundamental property to be shown, is the fact that the Heller operator commutes with applying a stable equivalence in the following sense. Moreover, we give the following "stripping off"-method, which enables us to reduce the problem of determining the image of a module under a stable equivalence to determining the images of its head and socle components; the proof of **6.9** bears testimony of the utility of this lemma.

Lemma A.1. *Let A and B be finite dimensional k-algebras for a field k such that A and B are both self-injective. Let F be a covariant functor such that*

(1) *F is exact.*
(2) *If X is a projective A-module, then $F(X)$ is a projective B-module,*
(3) *F induces a stable equivalence from* mod-A *to* mod-B.

Then the following holds:

(i) *For any positive integer n and for any A-modules $X, Y \in$ mod-A, we have that*

$$\mathrm{Ext}^n_A(X,Y) \cong \mathrm{Ext}^n_B(F(X),F(Y))$$

as k-spaces.

(ii) *Let X be a projective-free A-module, and write $F(X) = Y \oplus (\mathrm{proj})$ for a projective-free B-module Y. Let S be a simple A-submodule of X, and set $T = F(S)$. Now, if T is a simple B-module, then we may assume that Y contains T and that*

$$F(X/S) = Y/T \oplus (\mathrm{proj}).$$

(iii) *Similarly, let X be a projective-free A-module, and write $F(X) = Y \oplus (\mathrm{proj})$ for a projective-free B-module Y. Let X' be an A-submodule of X such that X/X' is simple, and set $T = F(X/X')$. Now, if T is a simple B-module, then we may assume that T is an epimorphic image of Y and that*

$$\mathrm{Ker}(F(X) \twoheadrightarrow T) = \mathrm{Ker}(Y \twoheadrightarrow T) \oplus (\mathrm{proj}).$$

Proof. (i) Note first that the short exact sequence $0 \to \Omega X \xrightarrow{i} P(X) \to X \to 0$, where $P(X)$ is the projective cover of X, gives rise to the short exact sequence $0 \to F(\Omega X) \to F(P(X)) \to F(X) \to 0$ of B-modules by (1). Hypothesis (2) implies $F(P(X)) = P(F(X)) \oplus P$ and $F(\Omega X) = \Omega(F(X)) \oplus P$ for a projective B-module P. Thus we have $F(\Omega X) \cong \Omega(F(X))$ in $\underline{\mathrm{mod}}$-B, and hence $F(\Omega^n X) \cong \Omega^n(F(X))$ in $\underline{\mathrm{mod}}$-B for all $n \geqslant 1$.

Moreover, letting $i^* : \mathrm{Hom}_A(P(X),Y) \to \mathrm{Hom}_A(\Omega X, Y)$ be the canonical k-map Then we have $\mathrm{Ext}^1_A(X,Y) \cong \mathrm{Hom}_A(\Omega X,Y)/\mathrm{Im}(i^*)$ as k-spaces. A being self-injective, it is well-known that $\mathrm{Im}(i^*)$ is the set of projective A-homomorphisms from ΩX to Y. We thus know that $\mathrm{Ext}^1_A(X,Y) \cong \underline{\mathrm{Hom}}_A(\Omega X, Y)$, and similarly $\mathrm{Ext}^n_A(X,Y) \cong \underline{\mathrm{Hom}}_A(\Omega^n X, Y)$ for all $n \geqslant 1$.

Then for all $n \geqslant 1$ we have

$$\begin{aligned} \mathrm{Ext}^n_A(X,Y) &\cong \underline{\mathrm{Hom}}_A(\Omega^n X, Y) \\ &\cong \underline{\mathrm{Hom}}_B(F(\Omega^n X), F(Y)) \qquad \text{by } (3) \\ &\cong \underline{\mathrm{Hom}}_B(\Omega^n(F(X)), F(Y)) \\ &\cong \mathrm{Ext}^n_B(F(X),F(Y)). \end{aligned}$$

(ii) Using [28, II Lemma 2.7 and Corollary 2.8] we get

$$0 \neq \mathrm{Hom}_A(S,X) = \underline{\mathrm{Hom}}_A(S,X) = \underline{\mathrm{Hom}}_B(F(S),F(X)) = \underline{\mathrm{Hom}}_B(T,Y) = \mathrm{Hom}_B(T,Y).$$

Now the assertions follow from [21, 1.11.Lemma]; see also [21, 3.25.Lemma and 3.26.Lemma]. Note that statement and proof of [21, 1.11.Lemma] remain valid in our more general setting of self-injective algebras.

(iii) is similar. □

Next, we want to show that the stable equivalence of Morita type also commutes with taking the contragredient module if A and B are blocks of group algebras. This is made precise in **A.2**(iv), but first we place ourselves into a more general context.

Lemma A.2. *Let A and B be finite dimensional k-algebras for a field k.*

(i) *Assume that $X \in$ mod-A, and $M \in A$-mod-B, and that ${}_AM$ is projective. Then the correspondence*

$$\Phi : {}_B(M^\vee \otimes_A X^{\circledast}) \to {}_B[(X \otimes_A M)^{\circledast}]$$

defined by

$$\Big[\Phi(\psi \otimes_A \theta)\Big](x \otimes_A m) = \theta\Big(x \cdot \psi(m)\Big)$$

for $\psi \in M^\vee$, $\theta \in X^{\circledast}$ and $m \in M$, is an isomorphism of left B-modules.

(ii) *Assume that $Y \in A$-mod, and $N \in B$-mod-A, and that N_A is projective. Then the correspondence*

$$\Theta : (Y^{\circledast} \otimes_A N^\vee)_B \to [(N \otimes_A Y)^{\circledast}]_B$$

defined by

$$\Big[\Theta(\theta \otimes_A \psi)\Big](n \otimes_A y) = \theta\Big(\psi(n) \cdot y\Big)$$

for $\psi \in N^\vee$, $\theta \in Y^\circledast$ and $n \in N$, is an isomorphism of right B-modules.

(iii) *If A moreover is a symmetric algebra, with symmetrising form $t \in \mathrm{Hom}_k(A,k)$, then as (B,A)-bimodules we have*

$$_B(M^\vee)_A \cong {}_B(M^\circledast)_A \quad \text{via the correspondence} \quad t_* : f \mapsto t \circ f.$$

Thus we have an isomorphism of left B-modules

$$\Psi : {}_B(M^\circledast \otimes_A X^\circledast) \xrightarrow{\approx} {}_B(M^\vee \otimes_A X^\circledast) \xrightarrow{\Phi} {}_B(X \otimes_A M)^\circledast$$

given by

$$t_*(\psi) \otimes_A \theta \mapsto \psi \otimes_A \theta \mapsto \Phi(\psi \otimes_A \theta).$$

(iv) *If finally A and B are block algebras of finite groups, and M is self-dual, namely, $M^* \cong M$ as (A,B)-bimodules, then as right B-modules we have*

$$(X^* \otimes_A M)_B \ \cong \ [(X \otimes_A M)^*]_B.$$

Proof. We prove (i), the proof of (ii) is entirely similar. It is easy to see that Φ is well-defined and a homomorphism of left B-modules. Next, we want to claim that Φ is onto. Since $_AM$ is projective, it follows from [7, Theorem 1.7(v)] that a map $\tau_M = \tau_{M,M} : M^\vee \otimes_A M \to \mathrm{Hom}_A({}_AM, {}_AM)$ defined by $[\tau_M(\psi \otimes_A m)](m') = \psi(m')\cdot m$ for $\psi \in M^\vee$ and $m, m' \in M$, is onto. Since the identity map of M exists, there are a positive integer ℓ; $\psi_1, \cdots, \psi_\ell \in M^\vee$ and $m_1, \cdots, m_\ell \in M$ such that

$$\sum_{i=1}^{\ell} \psi_i(m)\cdot m_i = m \qquad \text{for} \quad m \in M. \qquad (*)$$

Now take any $\sigma \in (X \otimes_A M)^\circledast$. Then we can define a k-linear map $f_\sigma : M \to X^\circledast$ by $[f_\sigma(m)](x) = \sigma(x \otimes_A m)$ for $m \in M$ and $x \in X$. Take any $a \in A$. Then,

$$[f_\sigma(am)](x) = \sigma(x \otimes_A am) = \sigma(xa \otimes_A m).$$

On the other hand, $[a\cdot f_\sigma(m)](x) = [f_\sigma(m)](xa) = \sigma(xa \otimes_A m)$ by the definitions of f_σ and $_A(X^\circledast)$. This means that f is a left A-module homomorphism. Now by $(*)$, we have an element $\alpha = \sum_{i=1}^{\ell} \psi_i \otimes_A f_\sigma(m_i) \in M^\vee \otimes_A X^\circledast$. Then, for $x \in X$ and $m \in M$, it is straightforward to check that $\Big[\Phi(\alpha)\Big](x \otimes_A m) = \sigma(x \otimes_A m)$. Thus, $\Phi(\alpha) = \sigma$, which means that Φ is onto. Now we have the following five epimorphisms of k-spaces:

$$\begin{aligned} M^\vee \otimes_A X^\circledast &\overset{\Phi}{\twoheadrightarrow} (X \otimes_A M)^\circledast \overset{\approx}{\twoheadrightarrow} X \otimes_A M \qquad \text{as } k\text{-spaces} \\ &\overset{\approx}{\twoheadrightarrow} (X^\circledast)^\circledast \otimes_A (M^\vee)^\vee \qquad \text{(at least) as } k\text{-spaces} \\ &\overset{\Theta}{\twoheadrightarrow} (M^\vee \otimes_A X^\circledast)^\circledast \overset{\approx}{\twoheadrightarrow} M^\vee \otimes_A X^\circledast \qquad \text{as } k\text{-spaces.} \end{aligned}$$

Since all modules above are of finite k-dimension, all the five epimorphisms above have to be isomorphisms of k-spaces.

(iii) It is easy to see that t_* is a homomorphism of (B,A)-bimodules, and that t_* is injective. Hence the first assertion follows from [7, Proposition 2.7]. The second assertion now follows from this together with (i).

(iv) follows easily from (iii). □

Finally, a fundamental property of the stable equivalences obtained through **2.3** (see also **2.4**) is that it preserves vertices and sources, and takes indecomposable modules to their Green correspondents.

Lemma A.3. *Let H be a proper subgroup of G, and let A and B be block algebras of kG and kH, respectively. Now, let M and M' be finitely generated (A,B)- and (B,A)-bimodules, respectively, which satisfy the following:*

(1) *$_AM_B \mid 1_A\cdot kG\cdot 1_B$ and $_BM'_A \mid 1_B\cdot kG\cdot 1_A$.*

(2) *The pair (M, M') induces a stable equivalence between* mod-A *and* mod-B.

Then we get the following:

(i) *Assume that X is a non-projective indecomposable kG-module in A with vertex Q. Then there exists a non-projective indecomposable kH-module Y in B, unique up to isomorphism, such that $(X \otimes_A M)_B = Y \oplus (\mathrm{proj})$, and Q^g is a vertex of Y for some element $g \in G$ (and hence $Q^g \subseteq H$). Since Q^g is also a vertex of X, this means that X and Y have the same vertices.*

(ii) *Assume that Y is a non-projective indecomposable kH-module in B with vertex Q. Then there exists a non-projective indecomposable kG-module X in A, unique up to isomorphism, such that $(Y \otimes_B M')_A = X \oplus (\mathrm{proj})$, and Q is a vertex of X.*

(iii) *Let X, Y and $Q \leqslant H$ be the as in* (i). *Then there is an indecomposable kQ-module L such that L is a source of both X and Y. This means that X and Y have the same sources.*

(iv) *Let X, Y and $Q \leqslant H$ be the same as in* (ii). *Then there is an indecomposable kQ-module L such that L is a source of both X and Y. This means that X and Y have the same sources.*

(v) *Let X, Y, Q and L be the same as in* (iii). *In addition, suppose that A and B have a common defect group P (and hence $P \subseteq H$) and that $H \geqslant N_G(P)$. Let f be the Green correspondence with respect to (G, P, H). If $Q \in \mathfrak{A} = \mathfrak{A}(G, P, H)$, then we have $(X \otimes_A M)_B = f(X) \oplus (\mathrm{proj})$.*

(vi) *Let X, Y, Q and L be the same as in* (ii). *Furthermore, as in* (v)*, assume that P is a common defect group of A and B, and that $H \geqslant N_G(P)$, and let f and $\mathfrak{A}$ be the same as in* (v). *Now, if $Q \in \mathfrak{A}$, then we have $(Y \otimes_B M')_A = f^{-1}(Y) \oplus (\mathrm{proj})$.*

Proof. (i) Clearly, $X \mid X{\downarrow_Q}{\uparrow^G}$. By (2) there exists a non-projective indecomposable kH-module Y in B, unique up to isomorphism, such that $(X \otimes_A M)_B = Y \oplus (\mathrm{proj})$. Hence,

$$Y \mid X \otimes_A M = X \otimes_{kG} M \mid X \otimes_{kG} kG_{kH} = X{\downarrow_H} \mid X{\downarrow_Q}{\uparrow^G}{\downarrow_H} = \bigoplus_{g \in [Q \backslash G/H]} (X{\downarrow_Q})^g{\downarrow_{Q^g \cap H}}{\uparrow^H}.$$

The last equality follows from Mackey Decomposition. Since Y_{kH} is indecomposable, the Krull-Schmidt Theorem yields $Y \mid (X{\downarrow_Q})^g{\downarrow_{Q^g \cap H}}{\uparrow^H}$ for some $g \in G$. That is, Y is $(Q^g \cap H)$-projective, so that there is a vertex R of Y such that $R \leqslant Q^g \cap H$. Since $Y \mid Y{\downarrow_R}{\uparrow^H}$, it holds as above that

$$X \mid \otimes_B M' = Y \otimes_{kH} M' \mid Y \otimes_{kH} kG_{kG} = Y{\uparrow^G} \mid (Y{\downarrow_R}{\uparrow^H}){\uparrow^G} = Y{\downarrow_R}{\uparrow^G}.$$

Hence, X is R-projective, so that there is a vertex S of X with $S \subseteq R$. Since Q is also a vertex of X, we have $S = Q^{g'}$ for some $g' \in G$. Namely, $Q^{g'} \subseteq R$. This implies that $Q^{g'} = S \subseteq R \subseteq Q^g \cap H \subseteq Q^g$, and hence $Q^{g'} = R = Q^g \cap H = Q^g$. This yields that $Q^g \subseteq H$.

(ii) Similar to (i).

(iii) By the assumption, Q is a common vertex of X and Y. Let L_{kQ} be a source of Y_{kH}. Then by the proof of (i), $X \mid Y{\uparrow^G} \mid L{\uparrow^H}{\uparrow^G} = L{\uparrow^G}$. Hence, $X \mid L{\uparrow^G}$. Since X has vertex Q and L is an indecomposable kQ-module, it follows that L is a source of X, too.

(iv) This follows from (iii).

(v) Let $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{A}$ be those with respect to (G, P, H) as in [40, Chap.4 §4]. Now, let X be an indecomposable kG-module in A such that a vertex of X is in $\mathfrak{A}$. Thus, we can assume that $Q \in \mathfrak{A}$. If X is projective then Q is trivial, so that the trivial group is not contained in $\mathfrak{X}$ by the definition of $\mathfrak{A}$, a contradiction, since $H \neq G$.

Hence, X is non-projective. Thus, we get by (i) and (ii) that there is a non-projective indecomposable kH-module Y in B such that $X \otimes_A M = Y \oplus (\mathrm{proj}\ B\text{-mod})$ and that Y also has Q as its vertex. On the other hand, we know $(X \otimes_A M) \mid X_{kH} = f(X) \oplus (\mathfrak{Y}\text{-proj}\ B\text{-mod})$. This implies that $f(X) \oplus (\mathfrak{Y}\text{-proj}\ B\text{-mod}) = Y \oplus (\mathrm{proj}\ B\text{-mod}) \oplus V$ for a kH-module V.

Assume that Y is $\mathfrak{Y}$-projective. Since Q is a vertex of Y, we have $Q \in_H \mathfrak{Y}$. Hence, we get by [40, Chap.4 Lemma 4.1(ii)] that $Q \in \mathfrak{X}$. Then we have $Q \notin \mathfrak{A}$, a contradiction. Therefore, by the Krull-Schmidt Theorem, we have $Y \cong f(X)$.

(vi) We get this exactly as in (iii) just by replacing X, M, and f by Y, M', and f^{-1}, respectively. □

Acknowledgements

We are grateful to Burkhard Külshammer for pointing out [56] to us. A part of this work was done while the first author was staying in RWTH Aachen University in 2009 and 2010. He is grateful to Gerhard Hiss for his kind hospitality. For this research the first author was partially supported by the Japan Society for Promotion of Science (JSPS), Grant-in-Aid for Scientific Research (C)20540008, 2008–2010. The first author still remembers that in January 1985, in Bad Honnef Bonn, Germany, there was a conference and he was informed by Peter Landrock about the non-principal 2-block of Co_3 with defect group $C_2 \times C_2 \times C_2$, which was interesting because the block is non-principal and the inertial index is 21.

References

[1] J.L. Alperin, Projective modules for $SL(2, 2^n)$, J. Pure Appl. Algebra **15** (1979), 219–234.
[2] J.L. Alperin and M. Broué, Local methods in block theory, Ann. of Math. **110** (1979), 143–157.
[3] J.L. Alperin, M. Linckelmann and R. Rouquier, Source algebras and source modules, J. Algebra **239** (2001), 262–271.
[4] T. Breuer, GAP package CTblLib — The GAP Character Table Library, Version 1.1.3, http://www.gap-system.org/Packages/ctbllib.html, 2004.
[5] M. Broué, Isométries parfaites, types de blocs, catégories dérivées, Astérisque **181–182** (1990), 61–92.
[6] M. Broué, Equivalences of blocks of group algebras, In: Finite Dimensional Algebras and Related Topics, Dlab, V., Scott, L.L. (eds.) pp.1–26 Kluwer Acad. Pub., Dordrecht, 1994.
[7] M. Broué, Higman's criterion revisited, Michigan Math. J. **58** (2009), 125–179.
[8] M. Broué and L. Puig, Characters and local structure in G-algebras, J. Algebra **73** (1980), 306–371.
[9] M. Broué and L. Puig, A Frobenius theorem for blocks, Invent. Math. **56** (1980), 117–128.
[10] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, Atlas of Finite Groups, Clarendon Press, Oxford, 1985.
[11] C.W. Curtis, I. Reiner, Methods of Representation Theory with Applications to Finite Groups and Orders, Vol.1, Wiley, New York, 1981.
[12] D. Fendel, A characterization of Conway's group .3, J. Algebra **24** (1973), 159–196.
[13] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.4.12, http://www.gap-system.org, 2008.
[14] C. Jansen, K. Lux, R. Parker and R. Wilson, An Atlas of Brauer Characters, Clarendon Press, Oxford, 1995.
[15] R. Kessar, S. Koshitani and M. Linckelmann, Alperin's weight conjecture for 2-blocks with elementary abelian defect groups of order 8, preprint, 2010.
[16] R. Knörr, On the vertices of irreducible modules, Ann. of Math. **110** (1979), 487–499.
[17] S. König and A. Zimmermann, Derived Equivalences for Group Rings, Lecture Notes in Math., Vol.**1685**, Springer, Berlin, 1998.
[18] S. Koshitani, The principal 2-block of finite groups with abelian Sylow 2-subgroups, Tsukuba J. Math. **4** (1980), 21–66.
[19] S. Koshitani and N. Kunugi, Broué's conjecture holds for principal 3-blocks with elementary abelian defect group of order 9, J. Algebra **248** (2002), 575–604.
[20] S. Koshitani, N. Kunugi and K. Waki, Broué's conjecture for non-principal 3-blocks of finite groups, J. Pure Appl. Algebra **173** (2002), 177–211.
[21] S. Koshitani, N. Kunugi and K. Waki, Broué's abelian defect group conjecture for the Held group and the sporadic Suzuki group, J. Algebra **279** (2004), 638–666.
[22] S. Koshitani, N. Kunugi and K. Waki, Broué's abelian defect group conjecture holds for the Janko simple group J_4, J. Pure Appl. Algebra **212** (2008), 1438–1456.
[23] S. Koshitani and M. Linckelmann, The indecomposability of a certain bimodule given by the Brauer construction, J. Algebra **285** (2005), 726-729.
[24] S. Koshitani and J. Müller, Broué's abelian defect group conjecture holds for the Harada-Norton sporadic simple group HN, J. Algebra **324** (2010), 394–429.
[25] B. Külshammer, Some indecomposable modules and their vertices, J. Pure Appl. Algebra **86** (1993), 65–73.
[26] P. Landrock, The non-principal 2-blocks of sporadic simple groups, Commun. Algebra **6** (1978), 1865–1891.
[27] P. Landrock, On the number of irreducible characters in a 2-block, J. Algebra **68** (1981), 426–442.
[28] P. Landrock, Finite Group Algebras and Their Modules, London Math. Society Lecture Note Series, Vol.**84**, London Math. Soc., Cambridge, 1983.
[29] P. Landrock and G.O. Michler, Principal 2-blocks of the simple groups of Ree type, Trans. Amer. Math. Soc. **260** (1980), 83–111.
[30] M. Linckelmann, Derived equivalence for cyclic blocks over a P-adic ring. Math. Z. 207 (1991), no. 2, 293–304.
[31] M. Linckelmann, Stable equivalences of Morita type for self-injective algebras and p-groups, Math. Z. **223** (1996), 87–100.

[32] M. Linckelmann, On derived equivalences and local structure of blocks of finite groups, Turkish J. Math. **22** (1998), 93–107.
[33] M. Linckelmann, On splendid derived and stable equivalences between blocks of finite groups, J. Algebra **242** (2001), 819–843.
[34] M. Linckelmann, Trivial source bimodule rings for blocks and p-permutation equivalences, Trans. Amer. Math. Soc. **361** (2009), 1279–1316.
[35] K. Lux and J. Müller, M. Ringe, Peakword condensation and submodule lattices: an application of the MeatAxe, J. Symb. Comput. **17** (1994), 529–544.
[36] K. Lux and M. Szőke, Computing decompositions of modules over finite-dimensional algebras, Experiment. Math. **16** (2007), 1–6.
[37] K. Lux and M. Szőke, Computing homomorphism spaces between modules over finite dimensional algebras, Experiment. Math. **12** (2003), 91–98.
[38] K. Lux and M. Wiegelmann, Determination of socle series using the condensation method, in: Computational algebra and number theory, Milwaukee, 1996, J. Symb. Comput. **31** (2001), 163–178.
[39] J. Müller and M. Schaps, The Broué conjecture for the faithful 3-blocks of $4.M_{22}$, J. Algebra **319** (2008), 3588–3602.
[40] H. Nagao and Y. Tsushima, Representations of Finite Groups, Academic Press, New York, 1988.
[41] T. Okuyama, Module correspondence in finite groups, Hokkaido Math. J. **10** (1981), 299–318.
[42] T. Okuyama, Some examples of derived equivalent blocks of finite groups, preprint (1997).
[43] L. Puig, Nilpotent blocks and their source algebras, Invent. Math. **93** (1988), 77–116.
[44] L. Puig, On the Local Structure of Morita and Rickard Equivalences between Brauer Blocks, Birkhäuser Verlag, Basel, 1999.
[45] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. (2) 43 (1991), no. 1, 37–48.
[46] J. Rickard, Splendid equivalences: derived categories and permutation modules, Proc. London Math. Soc. **(3) 72** (1996), 331–358.
[47] J. Rickard, Triangulated categories in the modular representation theory of finite groups, in: Derived Equivalences for Group Rings, edited by S. König and A. Zimmermann, Lecture Notes in Math., Vol.**1685**, Springer, Berlin, 1998, pp.177–198.
[48] J. Rickard, Some recent advances in modular representation theory, in: Algebras and Modules, edited by I. Reiten, S.O. Smalø and Ø. Solberg, Canadian Math. Soc. Conference Proc. **23** (1998), pp.157–178.
[49] M. Ringe, The C-MeatAxe — Computing with Modular Representations, Version 2.4.13, http://www.math.rwth-aachen.de/homes/MTX, 2009.
[50] G.R. Robinson, On projective summands of induced modules, J. Algebra **122** (1989), 106–111.
[51] R. Rouquier, The derived category of blocks with cyclic defect groups, in: Derived Equivalences for Group Rings, edited by S. König and A. Zimmermann, Lecture Notes in Math., Vol.**1685**, Springer, Berlin, 1998, pp.199–220
[52] R. Rouquier, Block theory via stable and Rickard equivalences. in; Modular representation theory of finite groups, edited by M.J. Collins et al., Walter de Gruyter, Berlin, 2001, pp.101–146.
[53] I.A.I. Suleiman and R.A. Wilson, The 2-modular characters of Conway's third group Co_3, J. Symbolic Computation **24** (1997), 493–506.
[54] J. Thévenaz, G-Algebras and Modular Representation Theory, Clarendon Press, Oxford, 1995.
[55] A. Watanabe, Notes on p-blocks of characters of finite groups, J. Algebra **136** (1991), 109–116.
[56] W. Willems, A note on Brauer's induction theorem, J. Algebra **58** (1979), 523–526.
[57] R. Wilson, Standard generators for sporadic simple groups, J. Algebra **184** (1996), 505–515.
[58] R. Wilson, R. Parker, S. Nickerson, J. Bray and T. Breuer, GAP package AtlasRep — A GAP Interface to the Atlas of Group Representations, Version 1.4.0, http://www.gap-system.org/Packages/atlasrep.html, 2008.
[59] R. Wilson, J. Thackray, R. Parker, F. Noeske, J. Müller, F. Lübeck, C. Jansen, G. Hiss and T. Breuer, The Modular Atlas Project, http://www.math.rwth-aachen.de/~MOC.
[60] R. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray and R. Abbott, Atlas of Finite Group Representations, http://brauer.maths.qmul.ac.uk/Atlas/v3.