SIMPLE RELATIONS IN THE CREMONA GROUP

JÉRÉMY BLANC

Let k be any fixed algebraically closed field. The *Cremona group* $\mathrm{Bir}(\mathbb{P}^2)$ is the group of birational transformations of the projective plane $\mathbb{P}^2 = \mathbb{P}^2_k$.

The classical Noether-Castelnuovo Theorem says that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by the group $\mathrm{Aut}(\mathbb{P}^2) \cong \mathrm{PGL}(3,k)$, that we will denote by $\mathbf{A}$, and by the *standard quadratic transformation*

$$\sigma\colon (X:Y:Z) \dashrightarrow (YZ:XZ:XY).$$

For a proof which is valid over any algebraically closed field (in particular in any characteristic), see for example [Sha, Chapter V, §5, Theorem 2, page 100].

A presentation of $\mathrm{Bir}(\mathbb{P}^2)$ was given in [Giz]. The generators are all the quadratic transformations of the plane (among them, all elements of the form $a_1\sigma a_2$, where $a_1, a_2 \in \mathbf{A}$), and the relations are all those of the form $q_1q_2q_3 = 1$ where q_i is a quadratic map. The proof is quite long and uses many sophisticated tools of algebraic geometry, such as cell complexes associated to rational surfaces.

Another presentation was given in [Isk2] (and announced in [Isk1]). The surface taken here is $\mathbb{P}^1\times\mathbb{P}^1$, and the generators used are the group $\mathrm{Aut}(\mathbb{P}^1\times\mathbb{P}^1)$ and the de Jonquières group $\mathbf{J}$ of birational maps of $\mathbb{P}^1\times\mathbb{P}^1$ which preserve the first projection (see below). There is only one relation in the amalgamated product of these two groups, which is $(\rho\tau)^3 = \sigma$, where $\rho = (x, y) \mapsto (x, x/y)$ and $\tau = (x, y) \mapsto (y, x)$ in local coordinates. The proof is much shorter than the one of [Giz], and the number of relations is also much smaller, but everything is now on $\mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$. There is also some gap in the proof (observed by S. Lamy): the author implicitly uses relations of the form $(\rho'\tau)^3 = \sigma'$ where ρ' has base-points infinitely near, without proving that they are generated by the first one (a fact not so hard to prove).

In this short note, we give a new presentation of the Cremona group, which are as simple as the one of [Isk2], but stays on $\mathbb{P}^2$. The proof is also very short, and is in fact strongly inspired from the one of [Isk2]. We take care of infinitely near points, and translate the idea of Iskovskikh from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$, where it becomes simpler. We only use classical tools of plane birational geometry (base-points and blow-ups), as mathematicians of the XIX^{th} century did, and as in [Isk2].

The *de Jonquières group*, that we will denote by $\mathbf{J}$, is the subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ consisting of elements which preserve the pencil of lines passing through $p_1 = (1 : 0 : 0)$. This group can be viewed in local coordinates $x = X/Z$ and $y = Y/Z$ as

$$\mathbf{J} = \left\{ \Big(x,y\Big) \dashrightarrow \left(\frac{ax+b}{cx+d}, \frac{\alpha(x)y+\beta(x)}{\gamma(x)y+\delta(x)}\right) \;\middle|\; \begin{array}{l} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,k), \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PGL}(2,k(x)) \end{array} \right\}.$$

It is thus naturally isomorphic to $\mathrm{PGL}(2,k(x)) \rtimes \mathrm{PGL}(2,k)$, where $\mathrm{PGL}(2,k) = \mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathrm{PGL}(2,k(x))$ via its action on $k(x) = k(\mathbb{P}^1)$.

Since $\sigma \in \mathbf{J}$, the group $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\mathbf{A}$ and $\mathbf{J}$. The aim of this note is to prove the following result:

Theorem 1. *The Cremona group* $\mathrm{Bir}(\mathbb{P}^2)$ *is the amalgamated product of* $\mathbf{A} = \mathrm{Aut}(\mathbb{P}^2)$ *and* $\mathbf{J}$ *along their intersection, divided by one relation, which is*

$$\sigma\tau = \tau\sigma,$$

where $\tau \in \mathbf{A}$ *is given by* $\tau = (X : Y : Z) \mapsto (Y : X : Z)$.

Since $\sigma\tau = \tau\sigma$ is easy to verify, it suffices to prove that no other relation holds. We prove this after proving the following simple lemma.

Lemma 1. *If* $\theta \in \mathbf{J}$ *is a quadratic map having* $p_1 = (1 : 0 : 0)$ *and* q *as base-points, where* q *is a proper point of* $\mathbb{P}^2 \backslash \{p_1\}$, *and* $\nu \in \mathbf{A}$ *exchanges* p_1 *and* q, *the map* $\theta' = \nu\theta\nu^{-1}$ *belongs to* $\mathbf{J}$ *and the relation*

$$\nu\theta^{-1} = (\theta')^{-1}\nu$$

is generated by the relation $\sigma\tau = \tau\sigma$ *in the amalgamated product of* $\mathbf{A}$ *and* $\mathbf{J}$.

Proof of Lemma 1. The relations $\theta' = \nu\theta\nu^{-1}$ and $\nu\theta^{-1} = (\theta')^{-1}\nu$ are clearly equivalent. In particular, the result is invariant under conjugation of both θ and ν by an element of $\mathbf{A} \cap \mathbf{J}$. Choosing an element in $\mathbf{A} \cap \mathbf{J}$ which sends q onto $p_2 = (0 : 1 : 0)$, we can assume that $q = p_2$. Then ν is equal to $a\tau$, where $\tau = (X : Y : Z) \mapsto (Y : X : Z)$ and a is an element of $\mathbf{A} \cap \mathbf{J}$ which fixes p_2. We can thus assume that $\nu = \tau$. We study two cases separately, depending on the number of proper base-points of θ.

(a) Suppose that θ has exactly three proper base-points, which means that $\theta = a_1\sigma a_2$ for some $a_1, a_2 \in \mathbf{A} \cap \mathbf{J}$. This yields the following equality in the amalgamated product:

$$\tau\theta\tau^{-1} = \tau a_1\sigma a_2\tau^{-1} = (\tau a_1\tau^{-1})(\tau\sigma\tau^{-1})(\tau a_2\tau^{-1}).$$

This implies that $\tau\theta\tau^{-1}$ is equal to an element of $\mathbf{J}$ modulo the relation $\sigma\tau = \tau\sigma$, and yields the result.

(b) Suppose now that θ has only two proper base-points, p_1, p_2, and that its third base-point, is infinitely near to p_i for some $i \in \{1, 2\}$. This means that $\theta = a_1\nu_i a_2$ for some $a_1, a_2 \in \mathbf{A} \cap \mathbf{J}$, where ν_1, ν_2 are the following quadratic involutions:

$$\begin{array}{lll} \nu_1 \colon & (X : Y : Z) \dashrightarrow & (XY : Z^2 : YZ), \\ \nu_2 \colon & (X : Y : Z) \dashrightarrow & (Z^2 : XY : XZ). \end{array}$$

Denoting by $\rho_1, \rho_2 \in \mathbf{A} \cap \mathbf{J}$ the maps

$$\begin{array}{lll} \rho_1 \colon & (X : Y : Z) \dashrightarrow & (X : Z - Y : Z), \\ \rho_2 \colon & (X : Y : Z) \dashrightarrow & (Z - X : Y : Z), \end{array}$$

we have $\nu_i = \rho_i\sigma\rho_i\sigma\rho_i$ in $\mathbf{J}$. As above, this yields the following equality:

$$\tau\theta\tau^{-1} = (\tau a_1\tau^{-1})(\tau\rho_i\tau^{-1})(\tau\sigma\tau^{-1})(\tau\rho_i\tau^{-1})(\tau\sigma\tau^{-1})(\tau\rho_i\tau^{-1})(\tau a_2\tau^{-1}).$$

Using $\sigma\tau = \tau\sigma$ and the fact that $\tau\rho_i\tau^{-1} = \rho_j$ in $\mathbf{A}$, with $j = 3 - i$, we obtain

$$\tau\sigma\tau^{-1} = (\tau a_1\tau^{-1})(\rho_j\sigma\rho_j\sigma\rho_j)(\tau a_2\tau^{-1}) = (\tau a_1\tau^{-1})\nu_j(\tau a_2\tau^{-1}).$$

So $\tau\theta\tau^{-1}$ is again equal to an element of $\mathbf{J}$ modulo the relation $\sigma\tau = \tau\sigma$. □

Proof of Theorem 1. Taking an element f in the amalgamated product $\mathbf{A} \star_{\mathbf{A}\cap\mathbf{J}} \mathbf{J}$ which corresponds to the identity map of $\mathrm{Bir}(\mathbb{P}^2)$, we have to prove that f is the identity in the amalgamated product, modulo the relation $\sigma\tau = \tau\sigma$.

We write $f = j_r a_r \dots j_1 a_1$ where $a_i \in \mathbf{A}$, $j_i \in \mathbf{J}$ for $i = 1, \dots, n$ (maybe trivial).

We denote by Λ_0 the linear system of lines of the plane and for $i = 1, \dots, n$, we denote by Λ_i the linear system $j_i a_i \dots j_1 a_1(\Lambda_0)$, and by d_i its degree. We define

$$D = \max\Big\{d_i \Big|\ i = 1, \dots, r\Big\}, n = \max\Big\{i\ \Big| d_i = D\Big\} \text{ and } k = \sum_{i=1}^{m}\Big(\deg(j_i) - 1\Big).$$

When $D = 1$, each j_i belongs to $\mathbf{A}$, and the word is equal to an element of $\mathbf{A}$ in the amalgamated product; since $\mathbf{A}$ embeds into $\mathrm{Bir}(\mathbb{P}^2)$, this case is clear. We can thus assume that $D > 1$ and prove the result by induction on the pairs (D, k), ordered lexicographically.

If j_n belongs to $\mathbf{A}$, we replace $a_{n+1} j_n a_n$ by its product in $\mathbf{A}$; this does not change the pair (D, k) but decreases n by 1. If j_{n+1} belongs to $\mathbf{A}$, a similar replacement decreases r by 1 without changing the pair (D, k). We can thus assume that $j_n, j_{n+1} \in \mathbf{J}\backslash\mathbf{A}$ and that $a_{n+1} \in \mathbf{A}\backslash\mathbf{J}$, which means that $a_{n+1}(p_1) \neq p_1$ (recall that $p_1 = (1 : 0 : 0)$ is the base-point of the pencil associated to $\mathbf{J}$).

The system $\Lambda_{n+1} = j_{n+1} a_{n+1}(\Lambda_n)$ has degree $d_{n+1} < d_n = D$, and $\Lambda_{n-1} = (a_n)^{-1}(j_n)^{-1}(\Lambda_n)$ has degree $d_{n-1} \leq d_n$. The maps $j_{n+1}, j_n \in J\backslash A$ have degree D_R and D_L respectively, for some integers $D_R, D_L \geq 2$. The points $l_0 = (a_{n+1})^{-1}(p_1) \neq p_1$ and $r_0 = p_1$ are base-points of respectively $j_{n+1} a_{n+1}$ and $(a_n)^{-1}(j_n)^{-1}$ of multiplicity $D_L - 1$ and $D_R - 1$. Writing $l_1, \dots, l_{2D_L-2}$ and $r_1, \dots, r_{2D_R-2}$ the other base-points of these two maps, the linear systems Λ_{n+1} and Λ_{n-1} have respectively degree

$$\begin{array}{rcccl} d_{n+1} & = & D_L \cdot d_n - (D_L - 1) \cdot m(l_0) - \sum_i^{2D_L-2} m(l_i) & > & d_n, \\ d_{n-1} & = & D_R \cdot d_n - (D_R - 1) \cdot m(r_0) - \sum_i^{2L_R-2} m(r_i) & \geq & d_n, \end{array}$$

where $m(q) \geq 0$ is the multiplicity of a point q as a base-point of Λ_n. We order the points $l_1, \dots, l_{2D_L-2}$ so that $m(l_i) \geq m(l_{i+1})$ for each $i \geq 1$ and that if l_i is infinitely near to l_j then $i > j$, and we do the same for $r_1, \dots, r_{2D_R-2}$. With this order and the above inequalities, we find

$$\begin{array}{rcl} m(l_0) + m(l_1) + m(l_2) & > & d_n, \\ m(r_0) + m(r_1) + m(r_2) & \geq & d_n. \end{array} \tag{1}$$

(a) Suppose that $m(l_0) \geq m(l_1)$ and $m(r_0) \geq m(r_1)$. We choose a point q in the set $\{l_1, l_2, r_1, r_2\}\backslash\{l_0, r_0\}$ with the maximal multiplicity $m(q)$, and so that q is a proper point of the plane or infinitely near to l_0 or r_0 (which are distinct proper points of the plane). We now prove that

$$m(l_0) + m(r_0) + m(q) > d_n. \tag{2}$$

If $l_1 = r_0$, $m(q) \geq m(l_2)$ and $m(l_0) + m(r_0) + m(q) \geq m(l_0) + m(l_1) + m(l_2) > d_n$ by (1). If $l_1 \neq r_0$, $m(q) \geq m(l_1) \geq m(l_2)$ so $m(l_0) + m(q) > 2d_n/3$. Since $m(r_0) \geq m(r_1) \geq m(r_2)$, we have $m(r_0) \geq d_n/3$, and the inequality (2) is clear.

Because of Inequality (2), the points l_0, r_0 and q are not aligned, and there exists a quadratic map $\theta \in \mathbf{J}$ with base-points l_0, r_0, q (recall that $r_0 = p_1$ is the point associated to the pencil of $\mathbf{J}$). Moreover, the degree of $\theta(\Lambda_n)$ is $2d_n - m(l_0) - m(r_0) - m(q) < d_n$. Recall that $a_{n+1} \in \mathbf{A}$ sends l_0 onto $r_0 = p_1$. Choosing $\nu \in \mathbf{A}\cap\mathbf{J}$ which sends $a_{n+1}(r_0)$ onto l_0 and replacing respectively a_{n+1} and j_{n+1} by νa_{n+1} and

$j_{n+1}\nu^{-1}$, we can assume that a_{n+1} exchanges l_0 and r_0 . Using Lemma 1, we write $\theta' = a_{n+1}\theta(a_{n+1})^{-1} \in \mathbf{J}$ and obtain the following equality modulo the relation $\sigma\tau = \tau\sigma$:

$$j_{n+1}a_{n+1}j_n = j_{n+1}a_{n+1}\theta^{-1}(\theta j_n) = (j_{n+1}(\theta')^{-1})a_{n+1}(\theta j_n),$$

and both $(j_{n+1}(\theta')^{-1})$ and (θj_n) belong to $\mathbf{J}$, but $a_{n+1} \in \mathbf{A}$. Since $\theta(\Lambda_n)$ has degree $< d_n$, this rewriting decreases the pair (D, k).

(b) Suppose now that we are in a "bad case" where $m(l_0) < m(l_1)$ or $m(r_0) < m(r_1)$. We now prove that it is possible to change the writing of f in the amalgamated product (modulo the relation) without changing (D, k) but reversing the inequalities; we will thus be able to go back to the "good case" already studied in (a) to conclude.

Assume first that $m(r_1) > m(r_0)$. This implies that r_1 is a proper point of the plane, and that there exists a quadratic map $\theta \in \mathbf{J}$ with base-points $p_1 = r_0, r_1, r_2$. Since these three points are base-points of $(j_n)^{-1}$, the degree of $\theta j_n \in \mathbf{J}$ is equal to the degree of $j_n \in \mathbf{J}$ minus 1.

Taking $\nu \in \mathbf{A}$ which exchanges r_0 and r_1, and applying Lemma 1 we write $\theta' = \nu\theta\nu^{-1} \in \mathbf{J}$ and obtain the following equality modulo the relation $\sigma\tau = \tau\sigma$:

$$a_{n+1}j_n = (a_{n+1}\nu^{-1})\nu\theta^{-1}(\theta j_n) = (a_{n+1}\nu^{-1})(\theta')^{-1}\nu(\theta j_n),$$

and both θ' and (θj_n) belong to $\mathbf{J}$, but $(a_{n+1}\nu^{-1})$ and ν belong to $\mathbf{A}$. This rewriting replaces

$$\begin{array}{ccc} (j_1, \ldots, j_{n-1}, j_n, j_{n+1}, \ldots, j_r) & \text{with} & (j_1, \ldots, j_{n-1}, \theta j_n, (\theta')^{-1}, j_{n+1}, \ldots, j_r), \\ (\Lambda_0, \ldots, \Lambda_{n-1}, \Lambda_n, \Lambda_{n+1}, \ldots, \Lambda_r) & \text{with} & (\Lambda_0, \ldots, \Lambda_{n-1}, \theta(\Lambda_n), \nu(\Lambda_n), \Lambda_{n+1}, \ldots, \Lambda_r). \end{array}$$

The degree of $\theta(\Lambda_n)$ is equal to $2d_n - m(r_0) - m(r_1) - m(r_2) \leq d_n$, and the degree of $\nu(\Lambda_n)$ is d_n. The new sequence has thus the same D, n is replaced with $n+1$, and k stays the same since $\deg((\theta')^{-1}) - 1 + \deg(\theta j_n) - 1 = 2 - 1 + \deg(\theta j_n) - 1 = \deg(j_n) - 1$. The system Λ_n being replaced with $\nu(\Lambda_n)$, where $\nu \in \mathbf{A}$ exchanges r_0 and r_1, the multiplicity of r_0 as a base-point of $\nu(\Lambda_n)$ is now the biggest among the base-points of θ'. In the new sequence, we have $m(r_0) > m(r_1)$ instead of $m(r_1) > m(r_0)$.

If $m(l_1) > m(l_0)$, the same kind of replacement exchanges the points l_0 and l_1.

We can thus go back to case (a) after having made one or two replacements. This achieves the proof. □

References

[Giz] M. Kh. Gizatullin, *Defining relations for the Cremona group of the plane*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 909-970; English transl. in Math. USSR Izv. 21 (1983).

[Isk1] V. A. Iskovskikh, *Generators and relations in a two-dimensional Cremona group.* Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1983, no. 5, 43–48. English transl. in Moscow Univ. Math. Bull. 38 (1983), no. 5, 56–63.

[Isk2] V. A. Iskovskikh, *Proof of a theorem on relations in the two-dimensional Cremona group.* Uspekhi Mat. Nauk **40** (1985), no. 5 (245), 255–256. English transl. in Russian Math. Surveys **40** (1985), no. 5, 231–232

[Sha] I.R. Shafarevich, *Algebraic surfaces*, Proc. Steklov Inst. Math. **75** (1967).

E-mail address: `Jeremy.Blanc@unibas.ch`