SPECTRAL TRIPLES FOR FINITELY PRESENTED GROUPS INDEX 1

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Abstract. Using generalized Cayley graphs and Clifford algebras, we are able to give, for a large class of finitely presented groups, a uniform construction of spectral triples with D_+ of index 1.

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1. Introduction

In this paper, we define even θ-summable spectral triples for a large class of finitely presented groups such that D_+ is index 1. We just generalize the unbounded version of the construction of the Fredholm module for the free group given by Connes [1] and M. Pimsner-Voiculescu [5]. For so, we use the Clifford algebra in the same spirit that Julg-Valette do in [4]. We also use topics in geometric group theory as a generalized Cayley graph.

2000 *Mathematics Subject Classification.* Primary 46L87. Secondary 20F65.
Key words and phrases. non-commutative geometry; spectral triple; geometric group theory; Clifford algebra; Cayley graph; Dirac operator; finitely presented groups.

2. Basic definitions

Definition 2.1. *A spectral triple $(\mathcal{A}, H, D)$ is given by a unital $\star$-algebra $\mathcal{A}$ representated on the Hilbert space H, and an unbounded operator D, called the Dirac operator, such that:*

(1) *D is self-adjoint.*
(2) *$(D^2+I)^{-1}$ is compact.*
(3) *$\{a \in \mathcal{A} \mid [D,a] \in B(H)\}$ is dense in $\mathcal{A}$.*

See the article [6] of G. Skandalis, dedicated to A. Connes and spectral triple.

Definition 2.2. *A group Γ is finitely presented if it exists a finite generating set S and a finite set of relations R such that $\Gamma = \langle S \mid R \rangle$. We always take S equals to S^{-1} and the identity element $e \notin S$ (see [3] for more details).*

3. Geometric construction

Definition 3.1. *Let Γ_n be the set of irreducible n-blocks, defined by induction:*

- $\Gamma_0 = \Gamma$.
- $\Gamma_1 := \{\{g, gs\} \mid g \in \Gamma, s \in S\}$

An $(n+2)$-block is a finite set a of $(n+1)$-blocks such that:

$$\forall b \in a, \forall c \in b, \exists ! b' \in a \text{ such that } b \cap b' = \{c\}.$$

Let a, a' be n-blocks then the commutative and associative composition:

$$a.a' := a \triangle a' = (a \cup a') \backslash a \cap a'$$

gives also an n-block if it's non empty (we take $n \neq 0$).
*Let $n > 1$, an n-block a'' is called **irreducible** if $\forall a, a'$ n-blocks:*

$$\text{(1) } a'' = a.a' \Rightarrow card(a) \text{ or } card(a') \geq card(a'')$$

$$\text{(2) } \forall b \in a'', b \text{ is a irreducible } (n-1)\text{-block.}$$

- *Γ_{n+2} is the set of irreducible $(n+2)$-blocks.*

*Note that if $b \in \Gamma_n$, we call n the **dimension** of b.*

Definition 3.2. *An n-block is called **admissible** if it decomposes into irreducibles.*

Example 3.3. *Let $\mathbb{Z} = \langle s^{\pm 1} \mid \rangle$ then $a = \{e, s^{10}\}$ is an admissible 1-block because $a = \{e,s\}.\{s,s^2\}...\{s^9,s^{10}\}$; but, $b = \{\{e,s\},\{e,s^{-1}\},\{s^{-1},s\}\}$ is a non-admissible 2-block, because there is no irreducible 2-block in this case.*

Remark 3.4. *The graph with vertices Γ_0 and edges Γ_1 is the Cayley graph $\mathcal{G}$.*

Remark 3.5. *Let a be an n-block then $a.a = \emptyset$ and if $a = \{b_1, .., b_r\}$ then $b_i = b_1.b_2...b_{i-1}.b_{i+1}...b_r$ and $b_1.b_2...b_r = \emptyset$.*

Remark 3.6. *$\Gamma_{n+1} \neq \emptyset$ iff $\exists r > 1; a_1, ..., a_r \in \Gamma_n$ all distincts with $a_1...a_r = \emptyset$.*

Remark 3.7. *Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then $\exists N$ such that $\Gamma_N \neq \emptyset$ and $\forall n > N$, $\Gamma_n = \emptyset$. In fact $N \leq card(S)$*

Examples 3.8. *For $\mathbb{F}_r = \langle s_1^{\pm 1}, ..., s_r^{\pm 1} \mid \rangle$, we have $N = 1$.
For $\mathbb{Z}^r = \langle s_1^{\pm 1}, ..., s_r^{\pm 1} \mid s_i s_j s_i^{-1} s_j^{-1}, i, j = 1, ..., r \rangle$, we have $N = r$.
Here an n-block ($n \leq r$) is just an n-dimensional hypercube.*

Definition 3.9. *We define the action of Γ on Γ_n recursively:*

- *Γ acts on $\Gamma_0 = \Gamma$ as: $u_g : h \to g.h$ with $g, h \in \Gamma$.*
- *Action on Γ_{n+1}: $u_g : a \to g.a = \{g.b \mid b \in a\}$ with $g \in \Gamma$, $a \in \Gamma_{n+1}$.*

Note that the action is well-defined: $g.\Gamma_n = \Gamma_n$, $\forall g \in \Gamma$.

Definition 3.10. *Let a and b be blocks, then we say that $b \Subset a$ if $b = a$ or if $b \in a$ or if $\exists c \in a$ such that $b \Subset c$ (recursive definition).*

Definition 3.11. *Let $n > 1$ then an n-block c is **connected** if $\forall b \subset c$: 'b is an n-block' $\Rightarrow b = c$.*

Definition 3.12. *An n-block b is called **maximal** if there is no $(n+1)$-block c with $b \in c$. We note $\mathbf{\Gamma_{max}}$ the set of maximal irreducible blocks.*

Example 3.13. *Let $\Gamma = \mathbb{Z}^2 \star \mathbb{Z} = \langle s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1} \mid s_1 s_2 s_1^{-1} s_2^{-1} \rangle$, then $\{e, s_3\}$ is a maximal 1-block, $\{\{e, s_1\}, \{s_1, s_1 s_2\}, \{s_1 s_2, s_2\}, \{s_2, e\}\}$ is a maximal 2-block.*

Definition 3.14. *We define the **block lenght** $\ell(.)$ as follows: let b be a block, then $\ell(b)$ is the minimal number of irreducible blocks decomposing a connected admissible block c with $e \Subset c$ and, $b \Subset c$ or $b \cap c \neq \emptyset$.*

Definition 3.15. *Let b be a block, then a sequence $(c_1, ..., c_{\ell(b)})$ with $b \Subset c_1$, $e \Subset c_{\ell(b)}$, c_i irreducible and $c_i \cap c_{i+1} \neq \emptyset$ is called a **geodesic block-path**, from b to e beginning with c_1.*

Lemma 3.16. *There is a unique irreducible block $\beta(b)$ of minimal dimension, beginning a geodesic block-path from b to e.*

Proof. We prove by contradiction: let $\beta(b)$ and $\beta'(b)$ be two differents such blocks, then they are the same dimension n. But then there is an admissible connected block d of dimension $n+1$, with $\beta(b), \beta'(b) \in d$ and $e \Subset d$, such that d decomposes into strictly less than $\ell(b)$ irreducible blocks, contradiction. $\square$

Remark 3.17. *Consider the group Γ and its finite presentation $\langle S \mid R \rangle$, then we can complete the presentation as follows: let T be a finite subset of Γ with $T \cap S = \emptyset$, $T = T^{-1}$ and $e \notin T$, let $S' = T \cup S$ an amplified generating set and $R' = R \cup \{t = \bar{t} \mid \}$ where $\bar{t}$ is t considered as a generator. Then $\Gamma = \langle S' \mid R' \rangle$.*

Lemma 3.18. *We can choose T such that if we build the blocks with the completed presentation $\langle S' \mid R' \rangle$, then every irreducible blocks are triangular, i.e. $\forall b \in \Gamma_n$, card$(b) = n+1$. We call $\langle S' \mid R' \rangle$ a* ***triangularized presentation****.*

Example 3.19. *The complete triangularization: let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then Γ acts on Γ_{max} (def. 3.9, 3.12); there are only finitely many orbits $O_1, ..., O_r$; choose $b_i \in O_i$; let $E_i = \{g \in \Gamma \mid g \Subset b_i\}$; let $T_i = \{gh^{-1} \mid g, h \in E_i, gh^{-1} \notin S \cup \{e\}\}$. Then amplifying the generating set with $T = \bigcup T_i$, we obtain obviously a triangularization called the complete triangularization. Note that this process increases the maximal dimension of the blocks. Note that card(T) is finite because the group is finitely presented.*

4. Clifford algebra

We first quickly recall here the notion of Clifford algebra, for a more detailed exposition, see the course of A. Wassermann [7].

Definition 4.1. *For V a n-dimensional Hilbert space, define the exterior algebra $\Lambda(V)$ equals to $\oplus_{k=0}^{n} \Lambda^k(V)$ with $\Lambda^0(V) = \mathbb{C}\Omega$. We called Ω the vacuum vector. Recall that $v_1 \wedge v_2 = -v_2 \wedge v_1$ so that $v \wedge v = 0$.*
Note that dim$(\Lambda^k(V)) = C_n^k$ and dim$(\Lambda(V)) = 2^n$.

Definition 4.2. *Let α_v be the creation operator on $\Lambda(V)$ defined by:*

$$\alpha_v(v_1 \wedge ... \wedge v_r) = v \wedge v_1 \wedge ... \wedge v_r \text{ and } \alpha_v(\Omega) = v$$

Reminder 4.3. *The dual $\alpha_v^\star$ is called the annihilation operator, then:*
$\alpha_v^\star(v_1 \wedge ... \wedge v_r) = \sum_{i=0}^{r}(-1)^{i+1}(v, v_i) v_1 \wedge ... v_{i-1} \wedge v_{i+1} \wedge ... \wedge v_r$ *and* $\alpha_v^\star(\Omega) = 0$

Reminder 4.4. *Let $\gamma_v = \alpha_v + \alpha_v^\star$, then $\gamma_v = \gamma_v^\star$ and $\gamma_v\gamma_w + \gamma_w\gamma_v = 2(v,w)I$.*

Definition 4.5. *The operators γ_v generate the Clifford algebra Cliff(V).*
Note that the operators γ_v are bounded and that Cliff$(V).\Omega = \Lambda(V)$.

Remark 4.6. *V admits the orthonormal basis $(v_a)_{a \in I}$.*
We will write γ_a instead of γ_{v_a}, so that $[\gamma_a, \gamma_{a'}]_+ = 2\delta_{a,a'} I$.

Let Γ be a finitely presented group, with a triangularized presentation $\langle S \mid R \rangle$.

Definition 4.7. *For any irreducible block c, let $\Delta_c = \{b \in \bigcup \Gamma_n \mid \beta(b) = c\}$, with $\beta(b)$ defined on lemma 3.16.*

Definition 4.8. *Let b, c be blocks such that $b \Subset c \Subset \beta(b)$ then we write $b \propto c$. In this case, we see that $\beta(b) = \beta(c)$, so that $\propto$ is an order relation.*

Lemma 4.9. *For any irreducible block c with $\Delta_c \neq \emptyset$, Δ_c admits a unique minimal element c_{min} with respect to $\propto$. Let m be the dimension of c_{min};*

denote by I_c the set of blocks of dimension $m+1$ in Δ_c; then Δ_c is in one-to-one correspondence with the power set $\mathcal{P}(I_c)$; in particular, the cardinality of Δ_c is $2^{dim(c)-m}$ (see section 3 of Julg-Valette paper [4]).

Definition 4.10. *We naturally identify $\ell^2(\Delta_c)$ with the exterior algebra $\Lambda(\ell^2(I_c))$ on which operates the Clifford algebra Cliff($\ell^2(I_c)$) generated by γ_a, $a \in I_c$.*

5. Dirac operator

Definition 5.1. *We define the **n-block lenght** $\ell_n(.)$ as follows: let b be a block, then $\ell_n(b)$ is the minimal number of irreducible blocks decomposing a connected admissible n-dimensional block c with $e \Subset c$ and, $b \Subset c$ or $b \cap c \neq \emptyset$.*

Definition 5.2. *Let b be a block, then a sequence $(c_1, ..., c_{\ell_n(b)})$ with $b \Subset c_1$, $e \Subset c_{\ell(b)}$, $c_i \in \Gamma_n$ and $c_i \cap c_{i+1} \neq \emptyset$ is called a **geodesic n-block-path**, from b to e beginning with c_1.*

Definition 5.3. *For any irreducible block c with $\Delta_c \neq \emptyset$, let $n = dim(c_{min})+1$; for any $a \in I_c$ define $p_a(c)$ the number of geodesic n-block path from c_{min} to e beginning with a; let $p(c) = \sum_{a \in I_c} p_a(c)$; let $\lambda_a = \frac{p_a(c)}{p(c)} \ell_n(c_{min})$.*

Definition 5.4. *On $\ell^2(\Delta_c) = \Lambda(\ell^2(I_c))$, define the Dirac operator D_c by:*

$$D_c = \sum_{a \in I_c} \lambda_a.\gamma_a$$

Remark 5.5. $\Delta_e = \{e\}$, $\ell^2(\Delta_c) = \mathbb{C}e_1$, $I_e = \emptyset$ *and* $D_e = 0$.

Definition 5.6. *Consider then the Hilbert space:*

$$\mathcal{H} = \bigoplus_n \ell^2(\Gamma_n) = \bigoplus_c \ell^2(\Delta_c) = \bigoplus_c \Lambda(\ell^2(I_c))$$

$\mathbb{Z}_2$-graded by the decomposition into even and odd dimensional blocks:

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

Define the unbounded selfadjoint operators $\mathcal{D} = \bigoplus_c D_c$.

Lemma 5.7. $D^2 = \bigoplus_c D_c^2 = \sum_c (\sum_{a \in I_c} \lambda_a^2).p_c$ *with p_c, projection on $\ell^2(\Delta_c)$.*

Proposition 5.8. $\mathcal{D}_+ : \mathcal{H}^+ \to \mathcal{H}^-$ *is a Fredholm operator of index* 1.

Proposition 5.9. $(\mathcal{D}^2 + I)^{-1}$ *is compact.*
For $t > 0$, the operator $e^{-t\mathcal{D}^2}$ is trass-class.

Definition 5.10. *For any $g \in \Gamma$ and for any $s \in S$ define $p_s(g)$ the number of geodesic 1-block path from g to e beginning with $\{g, gs\}$; let $p(g) = \sum_{s \in S} p_s(g)$.*

Definition 5.11. *Let $\mathcal{C}$ be the class of finitely presented groups Γ admitting a triangularized finite presentation $\langle S \mid R \rangle$ such that $\forall g \in \Gamma$, $\exists K_g \in \mathbb{R}_+$ such that $\forall s \in S$ and $\forall h \in \Gamma$ (with $h, gh \neq e$):*

$$|\frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)}| \leq \frac{K_g}{\ell_1(h)}$$

Examples 5.12. *The class $\mathcal{C}$ is stable by direct or free product, it contains $\mathbb{Z}^n$, $\mathbb{F}_n$, the finite groups, and probably every amenable or automatic groups (containing the hyperbolic groups, see* [2]*).*

Proposition 5.13. *Let Γ of class $\mathcal{C}$, $\mathcal{A} = C^\star_r(\Gamma)$ and $\mathcal{D}$ as previously then: $\{a \in \mathcal{A} \mid [\mathcal{D}, a] \in B(\mathcal{H})\}$ is dense in $\mathcal{A}$.*

Theorem 5.14. *$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even θ-summable spectral triple and $\mathcal{D}_+$ is index 1. It then gives a non-trivial element for the K-homology of $\mathcal{A}$.*

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