Symmetries of the transfer operator for $\Gamma_0(N)$ and a character deformation of the Selberg zeta function for $\Gamma_0(4)$

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Abstract. The transfer operator for $\Gamma_0(N)$ and trivial character χ_0 possesses a finite group of symmetries generated by permutation matrices P with $P^2 = id$. Every such symmetry leads to a factorization of the Selberg zeta function in terms of Fredholm determinants of a reduced transfer operator. These symmetries are related to the group of automorphisms in $GL(2,\mathbb{Z})$ of the Maass wave forms of $\Gamma_0(N)$. For the group $\Gamma_0(4)$ and Selberg's character χ_α there exists just one non-trivial symmetry operator P. The eigenfunctions of the corresponding reduced transfer operator with eigenvalue $\lambda = \pm 1$ are related to Maass forms even respectively odd under a corresponding automorphism. It then follows from a result of Sarnak and Phillips that the zeros of the Selberg function determined by the eigenvalues $\lambda = -1$ of the reduced transfer operator stay on the critical line under the deformation of the character. From numerical results we expect that on the other hand all the zeros corresponding to the eigenvalue $\lambda = +1$ leave this line for α turning away from zero.

Contents

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1. Introduction

In the transfer operator approach to Selberg's zeta function for a Fuchsian group Γ this function gets expressed in terms of the Fredholm determinant of this operator which is constructed from the symbolic dynamics of the geodesic flow on the corresponding surface of constant negative curvature. Even if this approach has been carried out up to now only for certain groups like the modular subgroups of finite index [2],[3],[4], or the Hecke triangle groups [16], [14],[15] it has lead for instance to new points of view on this function [22] or the theory of period functions [12]. Another application of this method is a precise numerical calculation of the Selberg zeta function [20], which seems to be impossible by other means at the moment. In this paper we discuss the transfer operator approach to Selberg's zeta function for Hecke congruence subgroups with character, of special interest being the behaviour of its zeros for $\Gamma_0(4)$ under the singular deformation of Selberg's character [19].

As found numerically by M. Fraczek in [9], certain symmetries of the transfer operator for these groups play thereby an important role. These symmetries lead to a factorization of the Selberg zeta function as known for the full modular group $SL(2,\mathbb{Z})$. There it corresponds to the involution $Ju(z) = u(-z^*)$ of the Maass forms u for this group [8], [12]. Obviously the corresponding element $j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL(2,\mathbb{Z})$ generates the normalizer group of $SL(2,\mathbb{Z})$ in $GL(2,\mathbb{Z})$. It tuns out that also the symmetries of the transfer operator for $\Gamma_0(N)$ correspond to automorphisms of the Maass forms from its normalizer group in $GL(2,\mathbb{Z})$.

For the group $\Gamma_0(4)$ with a character χ_α introduced by Selberg in [19] and discussed also by Phillips and Sarnak in [18], there is only one such non-trivial symmetry of the transfer operator. It corresponds to the generator of $\Gamma_0(4)$'s normalizer group in $GL(2,\mathbb{Z})$ leaving invariant the character χ_α. Results of Sarnak and Phillips imply that the zeros on the critical line of one factor of Selberg's function stay on this line under the deformation of the character, and hence the corresponding

Maass wave forms for the trivial character remain Maass wave forms. Numerical results [9] on the other hand imply, that the zeros on the critical line of the second factor of this function should all leave this line when the deformation is turned on. A detailed discussion of these numerical results and their partial proofs is in preparation [1].
The paper is organized as follows: in Section 2 we recall briefly the form of the transfer operator $\mathbf{L}_{\beta,\rho_\pi} = \begin{pmatrix} 0 & \mathcal{L}^+_{\beta,\pi} \\ \mathcal{L}^-_{\beta,\pi} & 0 \end{pmatrix}$ for a general finite index subgroup Γ of the modular group $SL(2,\mathbb{Z})$ and unitary representation π and introduce the symmetries $\tilde{P} = \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}$ of this operator defined by permutation matrices P. Any such symmetry leads to a factorization of the Selberg zeta function in terms of the Fredholm determinants of the reduced transfer operator $P\mathcal{L}^+_{\beta,\pi}$. The eigenfunctions with eigenvalue $\lambda = \pm 1$ of this reduced transfer operator then fulfill certain functional equations. In Section 3 we discuss the generators $J_{n,-}$ of the group of automorphisms in $GL(2,\mathbb{Z})$ of the Maass forms u for $\Gamma = \Gamma_0(N)$ and $\pi = \chi_0$ the trivial character. We introduce their period functions $\underline{\psi}$ and derive a formula for the period function $J_{n,-}\underline{\psi}$ of the Maass form $J_{n,-}u$. In Section 4 we introduce Selberg's character χ_α and the non-trivial automorphism $J_{2,-}$ of the Maass forms for $\Gamma_0(4)$. We derive again a formula for the period function $J_{2,-}\underline{\psi}$ of the Maass form $J_{2,-}u$ leading to a permutation matrix $P_{2,-}$ which defines a symmetry $\tilde{P}_{2,-}$ of the transfer operator $\mathbf{L}_{\beta,\rho_{\chi_\alpha}}$. From this we conclude that the eigenfunctions with eigenvalue $\lambda = \pm 1$ of the operator $P_{2,-}\mathcal{L}^+_{\beta,\pi}$ correspond to Maass forms even respectively odd under the involution $J_{2,-}$. Former results of Phillips and Sarnak then imply that the zero's of the Selberg function on the critical line corresponding to the eigenfunctions with eigenvalue $\lambda = -1$ of this operator stay on this line under the deformation of the character.

2. The transfer operator and Selberg's zeta function for Hecke congruence subgroups $\Gamma_0(N)$

The starting point of the transfer operator approach to Selberg's zeta function for a subgroup Γ of the modular group $SL(2,\mathbb{Z})$ of index $\mu = [\mathrm{SL}(2,()2,\mathbb{Z}) : \Gamma] < \infty$ is the geodesic flow $\Phi_t : SM_\Gamma \to SM_\Gamma$ on the unit tangent bundle SM_Γ of the corresponding surface $M_\Gamma = \Gamma \setminus \mathbb{H}$ of constant negative curvature. Here $\mathbb{H} = \{z = x + iy : y > 0\}$ denotes the hyperebolic plane with hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ on which the group Γ acts via Möbius transformations $gz = \frac{az+b}{cz+d}$ if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In the present paper we are mostly working with the Hecke congruence subgroup

$$\Gamma_0(N) = \{g \in SL(2,\mathbb{Z}) : g = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}\}$$

with index $\mu_N = N \prod_{p|N}(1+\frac{1}{p})$, where p is a prime number. If $\rho : \Gamma \to \operatorname{end}(\mathbb{C}^d)$ is a unitary representation of Γ then Selberg's zeta function $Z_{\Gamma,\rho}$ is defined as

$$Z_{\Gamma,\rho}(\beta) = \prod_{\gamma} \prod_{k=o}^{\infty} \det\left(1 - \rho(g_\gamma)\exp((k+\beta)l_\gamma)\right), \tag{2.0.1}$$

where l_γ denotes the period of the prime periodic orbit γ of Φ_t and $g_\gamma \in \Gamma$ is hyperbolic with $g_\gamma(\gamma) = \gamma$. In the dynamical approach to this function it gets expressed in terms of the so called transfer operator well known from D. Ruelle's thermodynamic formalism approach to dynamical systems. For general modular groups Γ with finite index μ and finite dimensional representation π this operator $\mathbf{L}_{\beta,\pi} : B \to B$ was determined in [2],[3] as

$$\mathbf{L}_{\beta,\pi} = \begin{pmatrix} 0 & \mathcal{L}^+_{\beta,\rho_\pi} \\ \mathcal{L}^-_{\beta,\rho_\pi} & 0 \end{pmatrix}, \tag{2.0.2}$$

where $B = B(D,\mathbb{C}^{d\mu}) + B(D,\mathbb{C}^{d\mu})$ is the Banach space of holomorphic functions on the disc $D = \{z :| z-1 |< \frac{3}{2}\}$, and ρ_π denotes the representation of $SL(2,\mathbb{Z})$ induced from the representation π of Γ whereas $\mathcal{L}^{\pm}_{\beta,\rho_\pi}$ is given for $\Re\beta > \frac{1}{2}$ by

$$\left(\mathcal{L}^{\pm}_{\beta,\rho_\pi}\underline{f}\right)(z) = \sum_{n=1}^{\infty} \frac{1}{(z+n)^{2\beta}} \rho_\pi(ST^{\pm n})\underline{f}(\frac{1}{z+n}). \tag{2.0.3}$$

In the following we restrict ourselves to one dimensional unitary representations π, hence unitary characters, which we denote as usual by χ. In this case the following Theorem was proved in [3].

Theorem 2.0.1. *The transfer operator* $\mathbf{L}_{\beta,\chi} : B \to B$ *with* $\mathbf{L}_{\beta,\chi} = \begin{pmatrix} 0 & \mathcal{L}^+_{\beta,\chi} \\ \mathcal{L}^-_{\beta,\chi} & 0 \end{pmatrix}$ *and* $\left(\mathcal{L}^{\pm}_{\beta,\chi}\underline{f}\right)(z) = \sum_{n=1}^{\infty} \frac{1}{(z+n)^{2\beta}}\rho_\chi(ST^{\pm n})\underline{f}(\frac{1}{z+n})$ *extends to a meromorphic family of nuclear operators of order zero in the entire complex* β *plane with possible poles at* $\beta_k = \frac{1-k}{2}$, $k = 0,1,2,\ldots$. *The Selberg zeta function* $Z_{\Gamma,\chi}$ *for the modular group* Γ *and character* χ *can be expressed as* $Z_{\Gamma,\chi}(\beta) = \det(1-\mathbf{L}_{\beta,\chi}) = \det(1-\mathcal{L}^+_{\beta,\chi}\mathcal{L}^-_{\beta,\chi}) = \det(1-\mathcal{L}^-_{\beta,\chi}\mathcal{L}^+_{\beta,\chi})$.

This shows that the zero's of Selberg's function are given by those β-values for which $\lambda = 1$ belongs to the spectrum $\sigma(\mathbf{L}_{\beta,\chi})$ respectively $\sigma(\mathcal{L}^-_{\beta,\chi}\mathcal{L}^+_{\beta,\chi}) = \sigma(\mathcal{L}^+_{\beta,\chi}\mathcal{L}^-_{\beta,\chi})$. From Selberg's trace formula one knows that there are two kinds of such zeros: the trivial zeros at $\beta = -k$, $k = 1,2,\ldots$, and the so called spectral zeros. They correspond either to eigenvalues $\lambda = \beta(1-\beta)$ of the automorphic Laplacian with $\Re\beta = \frac{1}{2}$ or $\frac{1}{2} \leq \beta \leq 1$ respectively to resonances of the Laplacian, that means poles of the scattering determinant with $\Re\beta < \frac{1}{2}$ and $\Im\beta > 0$ [21][10]. For arithmetic groups like the congruence subgroups with trivial or congruent character χ one knows that these resonances are on the line $\Re\beta = \frac{1}{4}$, corresponding to the nontrivial zeros $\zeta_R(2\beta) = 0$ of Riemann's zeta function ζ_R when assuming his hypothesis, respectively on the line $\Re\beta = 0$. For general Fuchsian groups and congruence subgroups with non-congruent character however these resonances can be anywhere in the halfplane $\Re\beta < \frac{1}{2}$.

2.1. Symmetries of the transfer operator for $\Gamma_0(N)$. It turns out that there exists for any N a finite number h_N of $\mu_N \times \mu_N$ permutation matrices P with $P^2 = id_{\mu_N}$ such that the matrix $\tilde{P} = \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}$ commutes with the transfer operator $\mathbf{L}_{\beta,\chi}$ and hence

$$P\,\mathcal{L}^+_{\beta,\chi} = \mathcal{L}^-_{\beta,\chi} P. \tag{2.1.1}$$

Thereby $P = (P_{i\,j})_{1\leq i,j\leq \mu_N}$ acts in the Banach space $B(D, \mathbb{C}^{\mu_N})$ as $(P\underline{f})_i(z) = \sum_{j=1}^{\mu_N} P_{i\,j} f_j(z)$ if $\underline{f}(z) = (f_i(z))_{1\leq i\leq \mu_N}$. We call such a matrix $\tilde{P}$ a symmetry of the transfer operator. As an example consider the group $\Gamma_0(4)$ and Selberg"s character $\chi_\alpha, 0 \leq \alpha \leq 1$, which will be described later. Its transfer operator $\mathbf{L}_{\beta,\chi_\alpha}$ has the following form

$$\begin{aligned}
\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{+1} &= \sum_{q=0}^{\infty} f_{-3}|_{2\beta}\tilde{S}T^{1+4q} + f_{-4}|_{2\beta}\tilde{S}T^{2+4q} + f_{-5}|_{2\beta}\tilde{S}T^{3+4q} \\
&+ f_{-2}|_{2\beta}\tilde{S}T^{4+4q} \\
\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{+2} &= \sum_{q=0}^{\infty} e^{2\pi i(1+4q)\alpha} f_{-1}|_{2\beta}\tilde{S}T^{1+4q} + e^{2\pi i(2+4q)\alpha} f_{-1}|_{2\beta}\tilde{S}T^{2+4q} \\
&+ e^{2\pi i(3+4q)\alpha} f_{-1}|_{2\beta}\tilde{S}T^{3+4q} + e^{2\pi i(4+4q)\alpha} f_{-1}|_{2\beta}\tilde{S}T^{4+4q} \\
\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{+3} &= \sum_{q=0}^{\infty} e^{-2\pi i\alpha} f_{-2}|_{2\beta}\tilde{S}T^{1+4q} + e^{-2\pi i\alpha} f_{-3}|_{2\beta}\tilde{S}T^{2+4q} \\
&+ e^{-2\pi i\alpha} f_{-4}|_{2\beta}\tilde{S}T^{3+4q} + e^{-2\pi i\alpha} f_{-5}|_{2\beta}\tilde{S}T^{4+4q} \\
\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{+4} &= \sum_{q=0}^{\infty} e^{-2\pi i\alpha(1+4q)} f_{-6}|_{2\beta}\tilde{S}T^{1+4q} + e^{-2\pi i\alpha(2+4q)} f_{-6}|_{2\beta}\tilde{S}T^{2+4q} \\
&+ e^{-2\pi i\alpha(3+4q)} f_{-6}|_{2\beta}\tilde{S}T^{3+4q} + e^{-2\pi i\alpha(4+4q)} f_{-6}|_{2\beta}\tilde{S}T^{4+4q} \\
\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{+5} &= \sum_{q=0}^{\infty} e^{2\pi i\alpha} f_{-4}|_{2\beta}\tilde{S}T^{1+4q} + e^{2\pi i\alpha} f_{-5}|_{2\beta}\tilde{S}T^{2+4q} \\
&+ e^{2\pi i\alpha} f_{-2}|_{2\beta}\tilde{S}T^{3+4q} + e^{2\pi i\alpha} f_{-3}|_{2\beta}\tilde{S}T^{4+4q}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{+6} &= \sum_{q=0}^{\infty} f_{-5}|_{2\beta}\tilde{S}T^{1+4q} + f_{-2}|_{2\beta}\tilde{S}T^{2+4q} + f_{-3}|_{2\beta}\tilde{S}T^{3+4q} \\
&+ f_{-4}|_{2\beta}\tilde{S}T^{4+4q} \\
\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{-1} &= \sum_{q=0}^{\infty} f_{+5}|_{2\beta}\tilde{S}T^{1+4q} + f_{+4}|_{2\beta}\tilde{S}T^{2+4q} + f_{+3}|_{2\beta}\tilde{S}T^{3+4q} \\
&+ f_{+2}|_{2\beta}\tilde{S}T^{4+4q} \\
\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{-2} &= \sum_{q=0}^{\infty} e^{-2\pi i\alpha(1+4q)} f_{+1}|_{2\beta}\tilde{S}T^{1+4q} + e^{-2\pi i\alpha(2+4q)} f_{+1}|_{2\beta}\tilde{S}T^{2+4q} \\
&+ e^{-2\pi i\alpha(3+4q)} f_{+1}|_{2\beta}\tilde{S}T^{3+4q} + e^{-2\pi i\alpha(4+4q)} f_{+1}|_{2\beta}\tilde{S}T^{4+4q} \\
\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{-3} &= \sum_{q=0}^{\infty} e^{-2\pi i\alpha} f_{+4}|_{2\beta}\tilde{S}T^{1+4q} + e^{-2\pi i\alpha} f_{+3}|_{2\beta}\tilde{S}T^{2+4q} \\
&+ e^{-2\pi i\alpha} f_{+2}|_{2\beta}\tilde{S}T^{3+4q} + e^{-2\pi i\alpha} f_{+5}|_{2\beta}\tilde{S}T^{4+4q} \\
\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{-4} &= \sum_{q=0}^{\infty} e^{2\pi i\alpha(1+4q)} f_{+6}|_{2\beta}\tilde{S}T^{1+4q} + e^{2\pi i\alpha(2+4q)} f_{+6}|_{2\beta}\tilde{S}T^{2+4q} \\
&+ e^{2\pi i\alpha(3+4q)} f_{+6}|_{2\beta}\tilde{S}T^{3+4q} + e^{2\pi i\alpha(4+4q)} f_{+6}|_{2\beta}\tilde{S}T^{4+4q}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{-5} &= \sum_{q=0}^{\infty} e^{2\pi i\alpha} f_{+2}|_{2\beta}\tilde{S}T^{1+4q} + e^{2\pi i\alpha} f_{+5}|_{2\beta}\tilde{S}T^{2+4q} \\
&+ e^{2\pi i\alpha} f_{+4}|_{2\beta}\tilde{S}T^{3+4q} + e^{2\pi i\alpha} f_{+3}|_{2\beta}\tilde{S}T^{4+4q} \\
\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{-6} &= \sum_{q=0}^{\infty} f_{+3}|_{2\beta}\tilde{S}T^{1+4q} + f_{+2}|_{2\beta}\tilde{S}T^{2+4q} + f_{+5}|_{2\beta}\tilde{S}T^{3+4q} \\
&+ f_{+4}|_{2\beta}\tilde{S}T^{4+4q}
\end{aligned}$$

where $\tilde{f} \in B(D, \mathbb{C}^{d\mu}) \bigoplus B(D, \mathbb{C}^{d\mu})$ is given by $\tilde{f} = (\underline{f}_+, \underline{f}_-)$ and $\underline{f}_\pm = (f_{\pm i})_{1\le i\le 6}$. The induced representation ρ_χ of the character χ on $\Gamma_0(4)$ is defined in terms of the coset decomposition of $SL(2,\mathbb{Z})$

$$SL(2,\mathbb{Z}) = \bigcup_{i=1}^{6} \Gamma_0(4) R_i \tag{2.1.2}$$

as

$$\rho_\chi(g)_{i\,j} = \delta_{\Gamma_0(4)}(R_i g R_j^{-1})\chi(R_i g R_j^{-1}), \quad 1 \le i,j \le 6. \tag{2.1.3}$$

Thereby we have chosen the following representatives $R_i \in SL(2,\mathbb{Z})$ of the cosets $\Gamma_0(4)\, R_i$

$$R_1 = id_2,\ R_i = ST^{i-2}, 2 \le i \le 5 \quad \text{and} \quad R_6 = ST^2 S. \tag{2.1.4}$$

It turns out that the two permutation matrices $P_i, i = 1,2$ corresponding to the permutations σ_i with

$$\sigma_1 = \frac{1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6}{1 \;\; 2 \;\; 5 \;\; 4 \;\; 3 \;\; 6} \tag{2.1.5}$$

(2.1.6) $$\sigma_2 = \frac{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6}{6 \quad 4 \quad 3 \quad 2 \quad 5 \quad 1}$$

fulfill equation (2.1.1) for $\alpha = 0$ and hence the corresponding matrices $\tilde{P}_i$, $i = 1, 2$ commute with the transfer operator $\mathbf{L}_{\beta,\chi_0}$ where χ_0 is the trivial character. The matrix $\tilde{P}_2$ on the other hand commutes even with the operator $\mathbf{L}_{\beta,\chi_\alpha}$ for all α. Indeed the matrix $\rho_{\chi_0}(S)$ is given by the permutation σ_S where

(2.1.7) $$\sigma_S = \frac{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6}{2 \quad 1 \quad 5 \quad 6 \quad 3 \quad 4}$$

and an easy calculation shows that $P_i\rho_{\chi_0}(S) = \rho_{\chi_0}(S)P_i$, $i = 1, 2$. The matrix $\rho_{\chi_0}(T)$ on the other hand is given by the permutation σ_T with

(2.1.8) $$\sigma_T = \frac{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6}{5 \quad 2 \quad 1 \quad 4 \quad 6 \quad 3}.$$

One then checks that $P_i\rho_{\chi_0}(T) = \rho_{\chi_0}(T^{-1})P_i$, $i = 1, 2$. Therefore $P_i\rho_{\chi_0}(ST^n) = \rho_{\chi_0}(ST^{-n})P_i$ for all $n \in \mathbb{N}$ and $i = 1, 2$. For the character chi_α analogous relations hold for P_2.
For the trivial character χ_0 one can determine for the group $\Gamma_0(N)$ the number h_N of matrices P_i with the above properties and hence defining symmetries of the transfer operator as follows:

Theorem 2.1.1. *For the Hecke congruence subgroup $\Gamma_0(N)$ and trivial character $\chi_0 \equiv 1$ there exist h_N matrices $\tilde{P} = \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}$ commuting with the transfer operator $\mathbf{L}_{\beta,\chi_0}$ where P is a $\mu_N \times \mu_N$ permutation matrix with $P^2 = \mathbf{1}_{\mu_N}$ and $P\rho_{\chi_0}(S) = \rho_{\chi_0}(S)P$ respectively $P\rho_{\chi_0}(T) = \rho_{\chi_0}(T^{-1})P$ and hence*
$$P\,\mathcal{L}^+_{\beta,\chi_0} = \mathcal{L}^-_{\beta,\chi_0}P.$$
Thereby $h_N = max\{k : k \mid 24 \text{ and } k^2 \mid N\}$. The permutation matrices P are determined by the h_N generators j of the normalizer group $\mathcal{N}_N$ of $\Gamma_0(N)$ in $GL(2,\mathbb{Z})$. The Selberg zeta function Z_{Γ,χ_0} can be written as
$$Z_{\Gamma,\chi_0} = \det(1 - P\,\mathcal{L}^+_{\beta,\chi_0})\det(1 + P\,\mathcal{L}^+_{\beta,\chi_0}).$$

.

Remark 2.1.2. For $\Gamma_0(4)$ obviously $h_N = 2$ and there exist according to Theorem 2.1.1 two such permutation matrices P_1 and P_2 which indeed are given by the aforementioned permutations σ_i, $i = 1, 2$. Since $P_1\,P_2 = P_2\,P_1$ we find
$$P_1\,P_2\,P_1\,\mathcal{L}^+_{\beta,\chi_0} = P_1\,P_2\,\mathcal{L}^-_{\beta,\chi_0}\,P_1 = P_1\,\mathcal{L}^+_{\beta,\chi_0}P_2\,P_1 = P_1\,\mathcal{L}^+_{\beta,\chi_0}P_1\,P_2$$
and the operators $P_1\,P_2$ and $P_1\,\mathcal{L}^+_{\beta,\chi_0}$ commute. Since $(P_1\,P_2)^2 = id_{\mu_N}$ this operator has only the eigenvalues $\lambda = \pm 1$ and the Banach space $B(D,\mathbb{C}^{\mu_N})$ decomposes as $B(D,\mathbb{C}^{\mu_N}) = B(D,\mathbb{C}^{\mu_N})_+ \oplus B(D,\mathbb{C}^{\mu_N})_-$ with $P_1\,P_2\underline{f}_\pm = \pm\underline{f}_\pm$ for $\underline{f}_\pm \in B(D,\mathbb{C}^{\mu_N})_\pm$. Denote by

$$P_1\,\mathcal{L}^+_{\beta,\chi_0 \pm} : B(D,\mathbb{C}^{\mu_N})_\pm \to B(D,\mathbb{C}^{\mu_N})_\pm$$

the restriction of the operator $P_1\,\mathcal{L}^+_{\beta,\chi_0}$ to the subspace $B(D,\mathbb{C}^{\mu_N})_\pm$. Then $\det(1 \pm P_1\,\mathcal{L}^+_{\beta,\chi_0}) = \det(1 \pm P_1\,\mathcal{L}^+_{\beta,\chi_0 +})\det(1 \pm P_1\,\mathcal{L}^+_{\beta,\chi_0 -})$. The Selberg zeta function hence factorizes in this case as

$$\begin{aligned} Z_{\Gamma,\chi_0} &= \det(1 - P_1\,\mathcal{L}^+_{\beta,\chi_0 +})\det(1 - P_1\,\mathcal{L}^+_{\beta,\chi_0 -}) \\ &\times \det(1 + P_1\,\mathcal{L}^+_{\beta,\chi_0 +})\det(1 + P_1\,\mathcal{L}^+_{\beta,\chi_0 -}) \end{aligned}$$

To prove Theorem 2.1.1 we relate the matrices P to the generating automorphisms in $GL(2,\mathbb{Z})$ of the Maass wave forms for $\Gamma_0(N)$ and can determine this way the explicit form of these matrices P. For this we derive in a first step a Lewis type functional equation for the eigenfunctions of the operator $P\,\mathcal{L}^+_{\beta,\chi}$ with eigenvalue $\lambda = \pm 1$.

2.2. A Lewis type functional equation. Consider any finite index modular subgroup Γ and any unitary character $\chi : \Gamma \to \mathbb{C}^*$ respectively the induced representation ρ_χ of $SL(2,\mathbb{Z})$. Assume there exists a symmetry $\tilde{P} = \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}$ with P a permutation matrix with the properties analogous to Theorem 2.1.1, and commuting with the transfer operator $\mathbf{L}_{\beta,\chi} = \begin{pmatrix} 0 & \mathcal{L}^+_{\beta,\rho_\chi} \\ \mathcal{L}^-_{\beta,\rho_\chi} & 0 \end{pmatrix}$ of Γ. If $\underline{f}$ is an eigenfunction of the operator $P\,\mathcal{L}^+_{\beta,\chi}$ with eigenvalue $\lambda = \pm 1$ then one can show

Proposition 2.2.1. *If $P\,\mathcal{L}^+_{\beta,\chi}\underline{f}(\zeta) = \lambda \underline{f}(\zeta)$ with $\lambda = \pm 1$ then the function $\underline{\Psi}(\zeta) := P\rho_\chi(T^{-1}S)P\underline{f}(\zeta - 1)$ fulfills the functional equations*

$$\underline{\Psi}(\zeta) = \lambda \zeta^{-2\beta} P\,\rho_\chi(S)\,\underline{\Psi}(\frac{1}{\zeta}), \tag{2.2.1}$$

respectively

$$\underline{\Psi}(\zeta) - \rho_\chi(T^{-1})\,\underline{\Psi}(\zeta+1) - (\zeta+1)^{-2\beta}\,\rho_\chi(T'^{-1})\,\underline{\Psi}(\frac{\zeta}{\zeta+1}) = \underline{0}, \tag{2.2.2}$$

where $T' = ST^{-1}S$. On the other hand every solution $\underline{\Psi}$ of equations (2.2.1) and (2.2.2) holomorphic in the cut β-plane $(-\infty, 0]$ with $\Psi_i(z) = o(z^{-\min\{1,2\Re s\}})$ as $z \downarrow 0$, respectively $\Psi_i(z) = o(z^{-\min\{0,2\Re s-1\}})$ as $z \to \infty$, determines an eigenfunction $\underline{f}$ with eigenvalue $\lambda = \pm 1$ of the operator $P\,\mathcal{L}^+_{\beta,\chi}$.

Proof. Let $\Re\beta > \frac{1}{2}$. If $P\mathcal{L}^+_\beta \underline{f}(\zeta) = \lambda \underline{f}(\zeta)$, $\lambda = \pm 1$ then obviously $P\rho_\chi(STS)PP\mathcal{L}^+_\beta \underline{f}(\zeta+1) = \lambda P\rho_\chi(STS)P\underline{f}(\zeta+1)$. Subtracting the two equations leads to

$$\lambda \underline{f}(\zeta) - \lambda P\rho_\chi(STS)P\underline{f}(\zeta+1) - (\zeta+1)^{-2\beta}P\rho_\chi(ST)\underline{f}(\frac{1}{\zeta+1}) = \underline{0},$$

and hence the function $\underline{\psi}(\zeta) := P\underline{f}(\zeta - 1)$ fulfills the equation

$$\underline{\psi}(\zeta) - \rho_\chi(STS)\underline{\psi}(\zeta+1) - \lambda\zeta^{-2\beta}\rho_\chi(ST)P\underline{\psi}(\frac{\zeta+1}{\zeta}) = \underline{0}. \tag{2.2.3}$$

Replacing there ζ by $\frac{1}{\zeta}$ and multiplying the resulting equation by $\zeta^{-2\beta}\rho_\chi(STS)P\rho_\chi(T^{-1}S)$ gives

$$\zeta^{-2\beta}\rho_\chi(STS)P\rho_\chi(T^{-1}S)\underline{\psi}(\frac{1}{\zeta}) - \zeta^{-2\beta}\rho_\chi(STS)P\rho_\chi(S)\underline{\psi}(\frac{\zeta+1}{\zeta}) -$$
$$-\lambda\rho_\chi(STS)\underline{\psi}(\zeta+1) = \underline{0}.$$

Since $\rho_\chi(S)P = P\rho_\chi(S)$ one finds, comparing with equation (2.2.3),

$$\underline{\psi}(\zeta) = \lambda\zeta^{-2\beta}\rho_\chi(STS)P\rho_\chi(T^{-1}S)\underline{\psi}(\frac{1}{\zeta}).$$

Hence the function $\tilde{\underline{\psi}} := \rho_\chi(T^{-1}S)\underline{\psi}$ fulfills equation (2.2.1). The same equation is then fulfilled also by the function

$$\underline{\Psi}(\zeta) := P\tilde{\underline{\psi}}(\zeta) = P\,\rho_\chi(T^{-1}S)\,P\,\underline{f}(\zeta-1), \tag{2.2.4}$$

that is

$$\underline{\Psi}(\zeta) = \lambda\zeta^{-2\beta}P\rho_\chi(S)\underline{\Psi}(\frac{1}{\zeta}). \tag{2.2.5}$$

Inserting finally $\underline{\psi}(\zeta) = \rho_\chi(ST)P\underline{\Psi}(\zeta)$ into equation (2.2.3) and using (2.2.1) leads to the equation

$$\underline{\Psi}(\zeta) - P\rho_\chi(T)P\underline{\Psi}(\zeta+1) - (\zeta+1)^{-2\beta}P\rho_\chi(T')P\underline{\Psi}(\frac{\zeta}{\zeta+1}) = \underline{0}.$$

But by assumption $P\rho_\chi(T)P = \rho_\chi(T^{-1})$, hence $P\rho_\chi(T')P = \rho_\chi(T'^{-1})$ and therefore

$$\underline{\Psi}(\zeta) - \rho_\chi(T^{-1})\underline{\Psi}(\zeta+1) - (\zeta+1)^{-2\beta}\rho_\chi(T'^{-1})\underline{\Psi}(\frac{\zeta}{\zeta+1}) = \underline{0}. \tag{2.2.6}$$

Hence for $\Re\beta > \frac{1}{2}$ the first part of the proposition holds. By analytic continuation in β one proves the general case.
To prove the second part we follow the arguments of Deitmar and Hilgert in [7] (see their Lemma 4.1): if $\underline{\Psi}(\zeta)$ is a solution of the Lewis equation (2.2.2) with $\beta \notin \mathbb{Z}$ then $\underline{\Psi}$ has the following asymptotic expansions:

$$\underline{\Psi}(\zeta) \sim_{\zeta\to 0} \zeta^{2\beta}Q_0(\frac{1}{\zeta}) + \sum_{l=-1}^{\infty}\underline{C}_l^*\zeta^l,$$

$$\underline{\Psi}(\zeta) \sim_{\zeta\to\infty} Q_\infty(\zeta) + \sum_{l=-1}^{\infty}\underline{C}_l^{*\prime}\zeta^{-l-2\beta},$$

where $Q_0, Q_\infty : \mathbb{C} \to \mathbb{C}^\mu$ are smooth functions with $Q_0(\zeta+1) = \rho_\chi(T')Q_0(\zeta)$ respectively $Q_\infty(\zeta+1) = \rho_\chi(T)Q_\infty(\zeta)$ and the constants $\underline{C}_l^*$ and $\underline{C}_l^{*\prime}$ are determined by the Taylor coefficients $\underline{C}_m = \frac{1}{m!}\underline{\Psi}^{(m)}(1)$.

The functions Q_0 and Q_∞ are defined as follows for general β with $-2\Re\beta < M \in \mathbb{N}$:

$$\begin{aligned} Q_0(\zeta) &:= \zeta^{-2\beta}\underline{\Psi}(\frac{1}{\zeta}) - \sum_{m=0}^{M} \zeta_{\rho_\chi}(m+2\beta, z)\underline{C}_m \\ &- \sum_{n=0}^{\infty}(n+\zeta)^{-2\beta}\rho_\chi(T'^{-n}T^{-1})\left(\underline{\Psi}(1+\frac{1}{n+\zeta}) - \sum_{m=0}^{M}\frac{\underline{C}_m}{(n+\zeta)^m)}\right) \end{aligned}$$

respectively

$$\begin{aligned} Q_\infty(\zeta) &:= \underline{\Psi}(\zeta) - \sum_{m=0}^{M} \zeta'_{\rho_\chi}(m+2\beta, \zeta+1)\underline{C}_m \\ &- \sum_{n=0}^{\infty}(n+\zeta)^{-2\beta}\rho_\chi(T^{-(n-1)}T'^{-1})\left(\underline{\Psi}(1-\frac{1}{n+\zeta}) - \sum_{m=0}^{M}\frac{\underline{C}_m}{(n+\zeta)^m)}\right). \end{aligned}$$

Thereby

$$\zeta_{\rho_\chi}(a,\zeta) = \frac{1}{N^a}\sum_{k=0}^{N-1}\rho_\chi(T'^{-k}T^{-1})\zeta(a, \frac{k+\zeta}{N})$$

and

$$\zeta'_{\rho_\chi}(a,\zeta) = \frac{1}{N^a}\sum_{k=0}^{N-1}\rho_\chi(T^{-k}T'^{-1})\zeta_H(a, \frac{k+\zeta}{N})$$

with $\zeta_H(a,\zeta)$ the Hurwitz zeta function. According to Remark 4.2 in ([7]) any solution $\underline{\Psi}$ of equation (2.2.2) with $\underline{\Psi}(\zeta) = \underline{o}(\zeta^{-\min\{1,2\beta\}})$ for $\zeta \to 0$ fulfills the equation

$$\underline{\Psi}(\zeta) = \zeta^{-2\beta}\sum_{n=0}^{\infty}(n+\zeta^{-1})^{-2\beta}\rho_\chi(T'^{-n}T^{-1})\underline{\Psi}(1+\frac{1}{n+\zeta^{-1}})$$

and moreover $\underline{C}^*_{-1} = 0$. But if $\underline{\Psi}(\zeta)$ fulfills also the equation (2.2.1) then one finds

$$\lambda\zeta^{-2\beta}P\rho_\chi(S)\underline{\Psi}(\frac{1}{\zeta}) = \zeta^{-2\beta}\sum_{n=0}^{\infty}(n+\zeta^{-1})^{-2\beta}\rho_\chi(T'^{-n}T^{-1})\underline{\Psi}(1+\frac{1}{n+\zeta^{-1}})$$

and hence

$$\lambda P\rho_\chi(S)\underline{\Psi}(\zeta+1) = \sum_{n=1}^{\infty}(n+\zeta)^{-2\beta}\rho_\chi(T'^{-(n-1)}T^{-1})\underline{\Psi}(1+\frac{1}{n+\zeta}). \tag{2.2.7}$$

According to equation (2.2.4) $\underline{\Psi}(\zeta+1) = P\rho_\chi(T^{-1}S)P\underline{f}(\zeta)$ and hence we get

$$\lambda\rho_\chi(ST^{-1}S)P\underline{f}(\zeta) = \sum_{n=1}^{\infty}(n+\zeta)^{-2\beta}\rho_\chi(T'^{-(n-1)}T^{-1})P\rho_\chi(T^{-1}S)P\underline{f}(\frac{1}{\zeta+n}).$$

Inserting $T'^{-(n-1)} = ST^{(n-1)}S$ one arrives at

$$\lambda\underline{f}(\zeta) = \sum_{n=1}^{\infty}(n+\zeta)^{-2\beta}P\rho_\chi(ST^n)\rho_\chi(ST^{-1})P\rho_\chi(T^{-1}S)P\underline{f}(\frac{1}{\zeta+n}).$$

Since $\rho_\chi(ST^{-1})P = P\rho_\chi(ST)$ we get finally

$$\lambda \underline{f}(\zeta) = \sum_{n=1}^{\infty} \frac{1}{(n+\zeta)^{2\beta}} P\rho_\chi(ST^n)\underline{f}(\frac{1}{n+\zeta}).$$

Hence any solution $\underline{\Psi}$ of the Lewis equations (2.2.1) and (2.2.2) with the asymptotics at the cut $\zeta = 0$ determines an eigenfunction $\underline{f}$ of the transfer operator $P\mathcal{L}^+_{\beta,\chi}$ with eigenvalue $\lambda = \pm 1$.

□

3. Automorphism of the Maass forms and their period functions for $\Gamma_0(N)$

The Maassforms $u = u(z)$ of a cofinite Fuchsian group Γ and unitary character χ are real analytic functions $u : \mathbb{H} \to \mathbb{C}$ with

- $\Delta\, u(z) = \lambda\, u(z)$,
- $u(gz) = \chi(g)\, u(z)$ for all $g \in \Gamma$,
- $u(g_j z) = O(y^C$ as $y \to \infty$ for some constant $C \in \mathbb{R}$ and all cusps $z_j = g_j(i\infty)$ of Γ.

The cusp forms are those forms which decay exponentially fast at the cusps. If $u \in L_2(M_\Gamma)$ we call u a Maass wave form.

Definition 3.0.1. An element $j \in GL(2,\mathbb{Z})$ defines an automorphism J of the Maass wave form u for the group Γ and character χ if $J\,u$ with $Ju(z) := u(jz)$ is a Maass form for Γ and character χ.

Obviously j defines an automorphism J iff j is a normalizer of the group Γ and the character χ is invariant under j, that is $\chi(j\,g\,j^{-1}) = \chi(g)$ for all $g \in \Gamma$. Thereby $jz = \frac{az^*+b}{cz^*+d}$ if $\det g = ad - bd = -1$. We have to show that the function $Ju(z) = u(jz)$ has at most polynomial growth at the cusps $z_i = \tau_i(i\infty)$ of Γ, where $\tau_i \in SL(2,\mathbb{Z})$. If $\det j = -1$, one has $u(j\tau_i(z)) = u(j\tau_i j_{0,-} j_{0,-}(z))$ where $j_{0,-} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $j\tau_i j_{0,-} \in SL(2,\mathbb{Z})$ and hence $j\tau_i j_{0,-} = \gamma_i R_i$ for some $\gamma_i \in \Gamma$ and some representative R_i of the cosets $\Gamma \setminus SL(2,\mathbb{Z})$. But $R_i = \eta\tau_{\sigma(i)}$ for some $\eta \in \Gamma$ and some index $\sigma(i)$. Hence $u(j\tau_i(z)) = u\left(\tau_{\sigma(i)}(-z^*)\right)$ which is at most of polynomial growth at the cusps. The same argument applies if $\det j = 1$. It shows also that Ju is a Maass wave form or a cusp form if u is one.

3.1. The group of automorphisms of Maass forms for $\Gamma_0(N)$ and trivial character χ_0.

We restrict ourselves now to the case $\Gamma = \Gamma_0(N)$ and assume $\chi = \chi_0$. Denote by $\mathcal{N}_N$ the normalizer group $\{\Gamma_0(N)\, j : j$ normalizer of $\Gamma_0(N)$ in $GL(2,\mathbb{Z})\}$. Using results by Lehner and Newman [11] respectively Conway and Norton [6] we find

Proposition 3.1.1. *For $h_N = \max\{r : r \mid 24$ and $r^2 \mid N\}$ and $k_N := \frac{N}{h_N}$ the normalizer group $\mathcal{N}_N$is given by $\mathcal{N}_N = \{\Gamma_0(N)\, j_{n,\pm},\, j_{n,\pm} = \begin{pmatrix} 1 & 0 \\ nk_N & \pm 1 \end{pmatrix}, 0 \leq n \leq h_N - 1\}$*

Proof. Using the fact that the divisors k of 24 are exactly the numbers for which $a \cdot d = 1 \mod k$ implies $a = d \mod k$ one shows that the normalizer group of $\Gamma_0(N)$ in $SL(2,\mathbb{Z})$ is $\Gamma_0(N) \setminus \Gamma_0(\frac{N}{\nu})$ [11] with $\nu = 2^{\min\{3,[\frac{\epsilon_2}{2}]\}} \cdot 3^{\min\{1,[\frac{\epsilon_3}{2}]\}}$ and $\epsilon_2 = \max\{l : 2^l \mid N\}$ respectively $\epsilon_3 = \max\{l : 3^l \mid N\}$. But obviously $\nu = h_N$ and $[\Gamma_0(k_N) : \Gamma_0(N)] = h_N$ and hence $\mathcal{N}_N = \Gamma_0(N) \setminus (\Gamma_0(k_N) \bigcup \Gamma_0(k_N) j_{0,-})$. Since $j_{n,\pm} \neq j_{m,\pm} \mod \Gamma_0(N)$ for $n \neq m$, this group has just the $2h_N$ elements $\Gamma_0(N) j_{n,\pm}$, $0 \leq n \leq h_N - 1$. The normalizer group $\mathcal{N}_N$ is therefore generated by the h_N generators $\{\Gamma_0(N) j_{n,-},\, 0 \leq n \leq h_N - 1\}$. □

3.2. The period functions of $\Gamma_0(N)$ and character χ.

For u a Maass form with $\Delta u = \beta(1-\beta)u$ and $\Gamma_0(N)\backslash SL(2,\mathbb{Z}) = \{\Gamma_0(N)R_i,\, 1 \leq i \leq \mu_N\}$ its vector valued period function $\underline{u}$ is defined by

$$\underline{u} = (u_i(z))_{1 \leq i \leq \mu_N} \text{ where } u_i(z) = u(R_i z) \tag{3.2.1}$$

Then one has as shown for instance in [17]:

- $\underline{u}(gz) = \rho_\chi(g)\underline{u}(z)$ for all $g \in SL(2,\mathbb{Z})$ and ρ_χ the representation of $SL(2,\mathbb{Z})$ induced from the character χ on $\Gamma_0(N)$
- $\Delta u_i(z) = \beta(1-\beta)u_i(z),\ 1 \leq i \leq \mu_N$.

Given next two eigenfunctions $u = u(z)$ and $v = v(z)$ of the hyperbolic Laplacian with identical eigenvalue $\lambda = \beta(1-\beta)$, one knows [12] that the 1- form $\eta = \eta(u,v)$ with

$$\eta(u,v)(z) := v(z)\partial_y u(z) - u(z)\partial_y v(z) dx + [u(z)\partial_x v(z) - v(z)\partial_x u(z)]dy$$

is closed. If $u = u(z)$ is a Maass wave form for $\Gamma_0(N)$ with eigenvalue $\lambda = \beta(1-\beta)$ and $R_\zeta(z) = \frac{y}{((\zeta-x)^2+y^2)^2}$ denotes the Poisson kernel, the vector valued period function $\underline{\psi} = (\psi_j(\zeta))_{1 \leq j \leq \mu_N}$ is defined as

$$\psi_j(\zeta) := \int_0^\infty \eta(u_j, R_\zeta^\beta)(z). \tag{3.2.2}$$

The following result has been shown for trivial character χ_0 by Mühlenbruch in [17]. His proof can be extended however immediately to the case of a nontrivial character χ.

Proposition 3.2.1. *The period function $\underline{\psi} = \underline{\psi}(\zeta)$ of a Maass wave form $u = u(z)$ for $\Gamma_0(N)$ and unitary character χ is holomorphic in the cut ζ-plane $\mathbb{C} \setminus (-\infty, 0]$ and fullfills there the Lewis functional equation (2.2.2)*

$$\underline{\psi}(\zeta) - \rho_\chi(T^{-1})\underline{\psi}(\zeta+1) - (\zeta+1)^{-2\beta}\rho_\chi(T'^{-1})\underline{\psi}(\frac{\zeta}{\zeta+1}) = \underline{0},$$

where ρ_χ denotes the representation of $SL(2,\mathbb{Z})$ induced from the character χ of $\Gamma_0(N)$.

On the other hand it follows from the work of Deitmar and Hilgert in [7] that the solutions of the above equation holomorphic in the cut ζ-plane with certain asymptotic behaviour at the cut 0 and at ∞ are in one-to-one correspondence with the Maass wave forms. Their paper treats only the trivial character but it can be extended also to the case of nontrivial character χ. Since the function $\underline{\Psi}(\zeta) = P\rho_\chi(T^{-1}S)P\underline{f}(\zeta-1)$ with $\underline{f}$ an eigenfunction of the operator $P\mathcal{L}^+_{\beta,\chi}$ with eigenvalue $\lambda = \pm 1$ is such a solution of equation (2.2.2), these eigenfunctions are in one-to-one correspondence with the Maass wave forms. As in the case of the full modular group $SL(2,\mathbb{Z})$ treated in [5] respectively in [12] one can extend this result to arbitrary Maass forms, that is also to the real analytic Eisenstein series for $\Gamma_0(N)$ and unitary character χ

3.3. Automorphisms of the period functions. We have seen that the group of automorphisms in $GL(2,\mathbb{Z})$ of the Maass forms u of $\Gamma_0(N)$ and trivial character χ_0 is generated by the matrices $j_{n,-} = \begin{pmatrix} 1 & 0 \\ nk_N & -1 \end{pmatrix}$, $0 \leq n \leq \mu_N - 1$. Denote by $J_{n,-}u$ the Maass form $J_{n,-}u(z) := u(j_{n,-}z)$ and by $J_{n,-}\underline{\psi}$ its period function. Then one shows

Theorem 3.3.1. *The period function $J_{n,-}\underline{\psi} = (J_{n,-}\psi_j(\zeta))_{1\leq j\leq \mu_N}$ is given by*

$$J_{n,-}\psi_j(\zeta) = \zeta^{-2\beta}\psi_{\lambda_{n_-}\circ\sigma\circ\delta(j)}(\frac{1}{\zeta}), \tag{3.3.1}$$

where the permutations λ_{n_-}, σ and δ are determined through the coset representatives R_j of $\Gamma_0(N)\setminus SL(2,\mathbb{Z})$ as follows:

$$j_{n,+}R_j = \theta_j R_{\lambda_{n,-}(j)},\ j_{0,-}R_j j_{0,-} = \gamma_j R_{\sigma(j)} \text{ and } R_j S = \eta_j R_{\delta(j)}$$

with $\theta_j, \gamma_j, \eta_j \in \Gamma_0(N)$ for $1 \leq j \leq \mu_N$

Proof. For $u = u(z)$ a Maass form for $\Gamma_0(N)$ and trivial character χ_0 and $\underline{u} = \underline{u}(z)$ its vector valued Maass form consider the Maass forms $J_{n,\pm}u(z) = u(j_{n,\pm}z)$ respectively $J_{n,\pm}\underline{u}(z) = (J_{n,\pm}u_j(z))_{1\leq j\leq\mu_N}$ with $J_{n,\pm}u_j(z) = u(j_{n,\pm}R_j z)$. Since $j_{n,+}R_j = \theta_j R_{\lambda_{n,-}(j)}$ for some uniquely defined $\theta_j \in \Gamma_0(N)$ and permutation $\lambda_{n,-}$ of $\{1, 2, \ldots, \mu_N\}$ one gets for $J_{n,+}u_j$

$$J_{n,+}u_j(z) = u(R_{\lambda_{n,-}(j)}z) = u_{\lambda_{n,-}(j)}(z). \tag{3.3.2}$$

For $J_{n,+}u_j(-z^*) = u(j_{n,+}R_j(-z^*) = u(j_{n,+}R_j j_{0,-}z)$ on the other hand one finds

$$J_{n,+}u_j(-z^*) = u(j_{n,-}j_{0,-}R_j j_{0,-}z) = u(j_{n,-}R_{\sigma(j)}z)$$

since $j_{0,-}R_j j_{0,-} = \gamma_j R_{\sigma(j)}$ for some unique $\gamma_j \in \Gamma_0(N)$ and permutation σ of $\{1, 2, \dots, \mu_N\}$. Hence

$$J_{n,+}u_j(-z^*) = J_{n,-}u_{\sigma(j)}(z). \tag{3.3.3}$$

Consider next $J_{n,+}u_j(Sz) = J_{n,+}u(R_j Sz)$. Since $R_j S = \eta_j R_{\delta(j)}$ for unique $\eta_j \in \Gamma_0(N)$ and permutation δ of $\{1, 2, \dots, \mu_N\}$, one has

$$J_{n,+}u_j(Sz) = J_{n,+}u(R_{\delta(j)}z) = J_{n,+}u_{\delta(j)}(z).$$

Hence by equation (3.3.2)

$$J_{n,+}u_j(Sz) = u_{\lambda_{n,-}\circ\delta(j)}(z). \tag{3.3.4}$$

On the other hand one gets for $J_{n,+}u_j(S(-z^*)) = J_{n,+}u_j(-Sz^*)$ by using equation (3.3.3):

$$J_{n,+}u_j(S(-z^*)) = J_{n,-}u_{\sigma(j)}(Sz) = u(j_{n,-}R_{\sigma(j)}Sz),$$

and therefore

$$J_{n,+}u_j(-Sz^*) = u(j_{n,-}\eta_{\sigma(j)}R_{\delta\circ\sigma(j)}(z)) = J_{n,-}u_{\delta\circ\sigma(j)}(z).$$

But $\sigma \circ \delta = \delta \circ \sigma$ and therefore

$$J_{n,+}u_j(S(-z^*)) = J_{n,-}u_{\sigma\circ\delta(j)}(z). \tag{3.3.5}$$

Define next

$$v_{\pm,j}(z) := J_{n,+}u_j(z) \pm J_{n,+}u_j(-z^*).$$

Then by equations (3.3.2) and (3.3.3) one has

$$v_{\pm,j}(z) = u_{\lambda(j)}(z) \pm J_{n,-}u_{\sigma(j)}(z)$$

and hence, if $\Delta u(z) = \beta(1-\beta)u(z)$,

$$\Delta v_{\pm,j}(z) = \beta(1-\beta)v_{\pm,j}(z), \tag{3.3.6}$$

respectively

$$v_{\pm,j}(-z^*) = \pm v_{\pm,j}(z) \tag{3.3.7}$$

Equations (3.3.4) and (3.3.5) on the other hand show

$$v_{\pm,j}(Sz) = v_{\pm,\delta(j)}(z). \tag{3.3.8}$$

Set $\underline{\psi}'_{\pm}(\zeta) := \int\limits_0^{i\infty} \eta(\underline{v}_{\pm}, R_\zeta^\beta)(z)$. Then, since $v_{\pm,j}(-z^*) = \pm v_{\pm,j}(z)$ one finds [12]

$$\psi'_{+,j}(\zeta) = 2\beta \int\limits_0^\infty \frac{t^\beta v_{+,j}(it)}{(\zeta^2 + t^2)^{\beta+1}}\, dt, \tag{3.3.9}$$

respectively

$$\psi'_{-,j}(\zeta) = -\int\limits_0^\infty \frac{t^\beta \partial_x v_{-,j}(it)}{(\zeta^2 + t^2)^{\beta}}\, dt. \tag{3.3.10}$$

Using next the identity (4.0.23) one easily shows

$$\psi'_{\pm,j}(\zeta) = \pm\zeta^{-2\beta}\psi'_{\pm,\delta(j)}(\frac{1}{\zeta}). \tag{3.3.11}$$

But $v_{\pm,j}(z) = u_{\lambda(j)}(z) \pm J_{n,-}u_{\sigma(j)}(z)$ and hence

$$\psi'_{\pm,j}(\zeta) = \psi_{\lambda_{n,-}(j)}(\zeta) \pm J_{n,-}\psi_{\sigma(j)}(\zeta).$$

Therefore

(3.3.12)

$$\psi_{\lambda_{n,-}(j)}(\zeta) \pm J_{n,-}\psi_{\sigma(j)}(\zeta) = \pm\zeta^{-2\beta}\left(\psi_{\lambda_{n,-}\circ\delta(j)}(\frac{1}{\zeta}) \pm J_{n,-}\psi_{\sigma\circ\delta(j)}(\frac{1}{\zeta})\right)$$

Adding these two equations leads finally to

$$\psi_{\lambda_{n,-}(j)}(\zeta) = \zeta^{-2\beta}J_{n,-}\psi_{\sigma\circ\delta(j)}(\frac{1}{\zeta}), \tag{3.3.13}$$

and therefore to the equation

$$J_{n,-}\psi_j(\zeta) = \zeta^{-2\beta}\psi_{\lambda_{n,-}\circ\sigma\circ\delta(j)}(\frac{1}{\zeta}), \tag{3.3.14}$$

which was to be proven. □

Remark 3.3.2. As can be seen from their action on the coset representatives R_j the permutation δ commutes with the permutations $\lambda_{n,-}$ and σ. Furthermore one has $\sigma^2 = \delta^2 = (\lambda_{n,-}\circ\sigma)^2 = id$ where id denotes the identy permutation. This shows also that the automorphisms $J_{n,-}$ are involutions both of the Maass forms and the period functions, a special case of these involutions for all groups $\Gamma_0(N)$ being $J_{0,-}u(z) = u(-z^*)$.

Denote by $Q_{n,-}$, $0 \le n \le h_N - 1$, the $\mu_N \times \mu_N$ permutation matrix corresponding to the permutation $\lambda_{n,-}\circ\sigma\circ\delta$. Then the following Theorem holds:

Theorem 3.3.3. *The permutation matrices* $P_{n,-} := \rho_{\chi_0}(S)Q_{n,-}$, $0 \le n \le h_N - 1$, *define symmetries* $\tilde{P}_{n,-} = \begin{pmatrix} 0 & P_{n,-} \\ P_{n,-} & 0 \end{pmatrix}$ *for the transfer operator* $\mathbf{L}_{\beta,\chi_0} = \begin{pmatrix} 0 & \mathcal{L}^+_{\beta,\chi_0} \\ \mathcal{L}^+_{\beta,\chi_0} & 0 \end{pmatrix}$ *for* $\Gamma_0(N)$ *and trivial character* $\chi_0 \equiv 1$ *with* $P^2_{n,-} = id_{\mu_N}$ *and* $P_{n,-}\rho_{\chi_0}(S) = \rho_{\chi_0}(S)P_{n,-}$ *respectively* $P_{n,-}\rho_{\chi_0}(T) = \rho_{\chi_0}(T^{-1})P_{n,-}$ *and therefore* $P_{n,-}\mathcal{L}^+_{\beta,\chi_0} = \mathcal{L}^+_{\beta,\chi_0}P_{n,-}$. *The permutation matrix* $P_{n,-}$ *is determined by the permutation* $\lambda_{n,-}\circ\sigma$ *and hence by the coset representatives* $j_{n,-}R_jj_{0,-}$.

Proof. Since the matrix $P_{n,-}\rho_{\chi_0}(S)$ is determined by the coset representatives $j_{n,-}R_jSj_{0,-}$ whereas $\rho_{\chi_0}(S)P_{n,-}$ is determined by the coset representatives $j_{n,-}R_jj_{0,-}S$ and $j_{0,-}S = Sj_{0,-}$ we get $P_{n,-}\rho_{\chi_0}(S) = \rho_{\chi_0}(S)P_{n,-}$. On the other hand $Tj_{0,-} = j_{0,-}T^{-1}$ and therefore $P_{n,-}\rho_{\chi_0}(T) = \rho_{\chi_0}(T^{-1})P_{n,-}$ and hence the Theorem is proven. □

Obviously Theorem 2.1.1 follows now from Theorem 3.3.3. For the automorphisms $j_{n,+} = j_{n,-}j_{0,-}$ one gets the symmetry $\tilde{P}_{n,+} = \begin{pmatrix} P_{n,+} & 0 \\ 0 & P_{n,+} \end{pmatrix}$ with $P_{n,+}$ the permutation matrix corresponding to the permutation $\lambda_{n,-} \circ \sigma \circ \lambda_{0,-} \circ \sigma$ determined by the coset representatives $j_{n,+}R_j$.

Remark 3.3.4. The symmetry $P_{0,-}$ is given by $\rho_{\chi_0}(SM)$ where $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and ρ_{χ_0} denotes the representation of $GL(2, \mathbb{Z})$ induced from the trivial character χ_0 of $\Gamma_0(N)$. The transfer operator $\mathcal{L}_\beta^{MM}$ of Manin and Marcolli for $\Gamma_0(N)$ introduced in [13] turns out to coincide with the operator $\rho_{\chi_0}(S)P_{0,-}\mathcal{L}^+_{\beta,\chi_0}\rho_{\chi_0}(S)$ and appears as a special case of our operators $P_{n,-}\mathcal{L}^+_{\beta,\chi_0}$.

Corollary 3.3.5. *The permutation matrices $P_{n,-}$, $0 \le n \le h_N - 1$, generate a finite group consisting of the permutation matrices $\{P_{n,\pm},\ 0 \le n \le h_N - 1\}$ and isomorphic to the normalizer group mathcal$(N)_N$ of $\Gamma_0(N)$ in $GL(2, \mathbb{Z})$. The symmetries $\{\tilde{P}_{n,\pm},\ 0 \le n \le h_N - 1\}$ of the transfer operator $\mathbf{L}_{\beta,\chi_0}$ for $\Gamma_0(N)$ and trivial character χ_0 define a finite group isomorphic to the group $\mathcal{N}_N$.*

4. Selberg's character χ_α for $\Gamma_0(4)$

The group $\Gamma_0(4)$ is freely generated by the two elements $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$. Hence any $g \in \Gamma_0(4)$ can be written as $g = \prod_{i=1}^{N_g} T^{m_i} B^{n_i}$. If $\Omega(g) = \sum_{i=1}^{N_g} m_i$ then Selberg's character χ_α [19] is defined as

$$\chi_\alpha(g) = \exp(2\pi i \alpha \Omega(g)),\ 0 \le \alpha \le 1. \tag{4.0.15}$$

Denote by z_i, $1 \le i \le 3$ the inequivalent cusps of $\Gamma_0(4)$ and by T_i the generators of their stabilizer groups Γ_{z_i} with $T_i z_i = z_i$. They can be taken as $z_1 = i\infty, z_2 = 0, z_3 = -\frac{1}{2}$ and $T_1 = T, T_2 = B, T_3 = T^{-1}B^{-1}$. The character χ_α is singular in the cusp z_i iff $\chi_\alpha(T_i) = 1$. Otherwise the character is non-singular in z_i. It is well known that the multiplicity $\kappa(\chi_\alpha)$ of the continuous spectrum of the automorphic Laplacian Δ with character χ_α is given by $\kappa(\chi_\alpha) = \#\{i : \chi_\alpha(T_i) = 1\}$. Therefore $\kappa(\chi_\alpha) = 3$ for $\alpha = 0$ whereas $\kappa(\chi_\alpha) = 1$ for $\alpha \neq 0$ and hence the multiplicity of the continuos spectrum of the Laplacian changes from 3 to 1 when the trivial character is deformed to χ_α with $\alpha \neq 0$. It is known [18] that the character χ_α is congruent (or arithmetic) iff $\alpha \in \{k\frac{1}{8},\ 0 \le k \le 4\}$. Since the Selberg zeta function given in (2.0.1) has the property $Z_{\Gamma_0(4),\chi_\alpha} = Z_{\Gamma_0(4),\chi_{-\alpha}}$ and obviously $\chi_\alpha = \chi_{\alpha+1}$ we can restrict the deformation parameter α to the range $0 \le \alpha \le \frac{1}{2}$.

Lemma 4.0.1. *The Selberg character χ_α is invariant under the map $j_{2,-}z = \frac{z^*}{2z^*-1}$ and $J_{2,-}u(z) := u(j_{2,-}z)$ is a Maass form for $\Gamma_0(4)$ and character χ_α if $u = u(z)$ is such a Maass form.*

Proof. We only have to show that χ_α is invariant under the map $j_{2,-}z = \frac{z^*}{2z^*-1}$. For $g = T$ we find $j_{2,-}Tj_{2,-} = TB$ and hence

$$\chi_\alpha(j_{2,-}Tj_{2,-}) = \chi_\alpha(TB) = \chi_\alpha(T),$$

whereas for $g = B$ one finds $j_{2,-}Bj_{2,-} = B^{-1}$ and hence

$$\chi_\alpha(j_{2,-}Bj_{2,-}) = \chi_\alpha(B^{-1}) = \chi_\alpha(B).$$

Therefore $\chi_\alpha(j_{2,-}gj_{2,-}) = \chi_\alpha(g)$ for all $g \in \Gamma_0(4)$. □

For $u = u(z)$ a Maass form for $\Gamma_0(4)$ and character χ_α and $\underline{\psi} = (\psi_j(\zeta))_{1\le j\le 6}$ its period function denote by J_-u the Maass form $J_-u(z) := u(j_{2,-}z)$ respectively by $J_-\underline{\psi} = (J_-\psi_j(\zeta))_{1\le j\le 6}$ its period function. Then one shows

Theorem 4.0.2. *The period function $J_-\underline{\psi}$ of the Maass form J_-u is given by*

$$J_-\psi_j(\zeta) = \zeta^{-2\beta}\chi_\alpha(\eta_{\sigma\circ\delta(j)})\psi_{\lambda_{2,-}\circ\sigma\circ\delta(j)}(\frac{1}{\zeta}) \tag{4.0.16}$$

where the permutations $\lambda_{2,-}$, σ, δ respectively the $\eta_j \in \Gamma_0(4)$ are determined through the coset representatives R_j by

$$j_{2,+}R_j = \theta_j R_{\lambda_{2,-}(j)},\ j_{0,-}R_j j_{0,-} = \gamma_j R_{\sigma(j)},\ R_j S = \eta_j R_{\delta(j)}$$

with θ_j, γ_j, $\eta_j \in \Gamma_0(4)$ for $1 \le j \le 6$.

Proof. Set $j_\pm := j_{2,\pm}$ and $J_\pm u(z) := u(j_\pm z)$. Then J_-u is a Maass form for $\Gamma_0(4)$ and character χ_α whereas J_+u is a Maass form for $\Gamma_0(4)$ and character $\chi_{-\alpha}$. The vector valued Maass form $J_+\underline{u} = (J_+u_j)_{1\le j\le 6}$ is given by $J_+u_j(z) = u(j_+R_jz)$. Therby we have choosen the representatives R_j of the cosets in $SL(2,\mathbb{Z}) = \bigcup_{1\le j\le 6}\Gamma_0(4)R_j$ as follows:

$$R_1 = id_2, R_j = ST^{j-2},\ 2 \le j \le 5, R_6 = ST^2S.$$

But $j_+R_j = \theta_jR_{\lambda_{2,-}(j)}$ for some $\theta_j \in \Gamma_0(4)$ and some permutation $\lambda_{2,-}$ of the set $\{1,2,\dots,6\}$ and hence $J_+u_j(z) = \chi_\alpha(\theta_j)u(R_{\lambda_{2,-}(j)}z)$. It turns out that $\theta_j = B^{-1}$ for $1 \le j \le 3$ and $\theta_j = id_2$ for $4 \le j \le 6$. Hence $\chi_\alpha(\theta_j) = 1$ and

$$J_+u_j(z) = u_{\lambda_{2,-}(j)}(z),\ 1 \le j \le 6, \tag{4.0.17}$$

with $\lambda_{2,-}$ the permutation

$$\lambda_{2,-} = \frac{1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6}{6\ \ 4\ \ 5\ \ 2\ \ 3\ \ 1}. \tag{4.0.18}$$

Consider next $J_+u_j(-z^*) = J_+u_j(j_{0,-}z)$. Then

$$J_+u_j(j_{0,-}z) = u(j_+R_jj_{0,-}z) = u(j_+j_{0,-}j_{0,-}R_jj_{0,-}z).$$

If $j_{0,-}R_j j_{0,-} = \gamma_j R_{\sigma(j)}$ then $J_+u_j(j_{0,-}z) = u(j_-\gamma_j j_- j_- R_{\sigma(j)}z)$. But it turns out that $j_-\gamma_j j_- = id_2$ for $j = 1, 2, 6$ respectively $j_-\gamma_j j_- = B$ for $j = 3, 4, 5$, hence $\chi_\alpha(j_-\gamma_j j_- = 1$ and therefore

(4.0.19) $$J_+u_j(-z^*) = J_+u_j(j_{0,-}z) = J_-u_{\sigma(j)}(z).$$

Since furthermore $J_+u_j(Sz) = u(j_+R_jSz) = u(j_+\eta_j R_{\delta(j)}z)$ one finds

(4.0.20) $$J_+u_j(Sz) = \chi_{-\alpha}(\eta_j)u_{\lambda_{2,-}\circ\delta(j)}(z)$$

where δ is the following permutation of the set $\{1, 2, \dots, 6\}$

(4.0.21) $$\delta = \frac{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6}{2 \quad 1 \quad 5 \quad 6 \quad 3 \quad 4}.$$

and $\eta_j = id_2$ for $j = 1, 2, 4, 6$ respectively $\eta_3 = \eta_5^{-1} = T^{-1}B^{-1}$. For $J_+u_j(-Sz^*)$ one gets with (4.0.19) $J_+u_j(-Sz^*) = J_-u_{\sigma(j)}(Sz) = u(j_{2,-}R_{\sigma(j)}Sz)$ and hence
$J_+u_j(-Sz^*) = u(j_{2,-}\eta_{\sigma(j)}R_{\delta\circ\sigma(j)}z) = \chi_\alpha(\eta_{\sigma(j)})J_-u_{\delta\circ\sigma(j)}(z)$. Using the explicit form of the η_j one shows $\chi_\alpha(\eta_{\sigma(j)}) = \chi_{-\alpha}(\eta_j)$ and therefore

(4.0.22) $$J_+u_j(-Sz^*) = \chi_{-\alpha}(\eta_j)J_-u_{\delta\circ\sigma(j)}(z).$$

Define next $v_{\pm,j} = v_{\pm,j}(z)$ as

(4.0.23) $$v_{\pm,j}(z) := J_+u_j(z) \pm J_+u_j(-z^*).$$

Then $v_{\pm,j}(-z^*) = \pm v_{\pm,j}(z)$ and by (4.0.20) respectively (4.0.22)

(4.0.24) $$v_{\pm,j}(Sz) = \chi_{-\alpha}(\eta_j)v_{\pm,\delta(j)}(z)$$

If therefore $\psi'_{\pm,j}(\zeta) := \int_0^{i\infty} \eta(v_{\pm,j}, R_\zeta^\beta)(z)$ one gets from relation (4.0.24)

(4.0.25) $$\psi'_{\pm,j}(\zeta) = \pm\zeta^{-2\beta}\chi_{-\alpha}(\eta_j)\psi'_{\pm,\delta(j)}(\frac{1}{\zeta})$$

and using the identity (4.0.23)

(4.0.26)
$$\psi_{\lambda_{2,-}(j)}(\zeta) \pm J_-\psi_{\sigma(j)}(\zeta) = \pm\zeta^{-2\beta}\chi_{-\alpha}(\eta_j)\left(\psi_{\lambda_{2,-}\circ\delta(j)}(\frac{1}{\zeta}) \pm J_-\psi_{\sigma\circ\delta(j)}(\frac{1}{\zeta})\right).$$

Adding these two equations leads finally to

(4.0.27) $$J_-\psi_j(\zeta) = \zeta^{-2\beta}\chi_\alpha(\eta_{\sigma\circ\delta(j)})\psi_{\lambda_{2,-}\circ\sigma\circ\delta(j)}(\frac{1}{\zeta}).$$

□

Inserting the explicit form of the permutations

(4.0.28) $$\sigma \circ \delta = \frac{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6}{2 \quad 1 \quad 3 \quad 6 \quad 5 \quad 4}$$

respectively

(4.0.29) $$\lambda_{2,-} \circ \sigma \circ \delta = \frac{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6}{4 \quad 6 \quad 5 \quad 1 \quad 3 \quad 2}$$

and the character values

$$\chi_\alpha(\eta_1) = \chi_\alpha(\eta_2) = \chi_\alpha(\eta_4) = \chi_\alpha(\eta_6) = 1$$

respectively

$$\chi_\alpha(\eta_3) = \chi_\alpha(\eta_5)^{-1} = e^{-2\pi i\alpha}$$

one finds

$$\begin{aligned} J_-\psi_1(\zeta) &= \zeta^{-2\beta}\psi_4(\frac{1}{\zeta}) \\ J_-\psi_2(\zeta) &= \zeta^{-2\beta}\psi_6(\frac{1}{\zeta}) \\ J_-\psi_3(\zeta) &= \zeta^{-2\beta}e^{-2\pi i\alpha}\psi_4(\frac{1}{\zeta}) \\ J_-\psi_4(\zeta) &= \zeta^{-2\beta}\psi_1(\frac{1}{\zeta}) \\ J_-\psi_5(\zeta) &= \zeta^{-2\beta}e^{2\pi i\alpha}\psi_3(\frac{1}{\zeta}) \\ J_-\psi_6(\zeta) &= \zeta^{-2\beta}\psi_2(\frac{1}{\zeta}) \end{aligned} \tag{4.0.30}$$

Define the matrix $Q_{2,-}$ through the equation $J_{2,-}\underline{\psi}(\zeta) = \zeta^{-2\beta}Q_{2,-}\underline{\psi}(\frac{1}{\zeta})$. Then one gets

Proposition 4.0.3. *The permutation matrix* $P_{2,-} := \rho_{\chi_\alpha}(S)Q_{2,-}$ *defines a symmetry* $\tilde{P}_{2,-} = \begin{pmatrix} 0 & P_{2,-} \\ P_{2,-} & 0 \end{pmatrix}$ *of the transfer operator*

$$\mathbf{L}_{\beta,\chi_\alpha} = \begin{pmatrix} 0 & \mathcal{L}^+_{\beta,\chi_\alpha} \\ \mathcal{L}^+_{\beta,\chi_\alpha} & 0 \end{pmatrix}$$

for $\Gamma_0(4)$ *and character* χ_α *with* $P_{2,-}^2 = id_6$ *and* $P_{2,-}\rho_{\chi_\alpha}(S) = \rho_{\chi_\alpha}(S)P_{2,-}$ *respectively* $P_{2,-}\rho_{\chi_\alpha}(T) = \rho_{\chi_\alpha}(T^{-1})P_{2,-}$ *and therefore* $P_{2,-}\mathcal{L}^+_{\beta,\chi_\alpha} = \mathcal{L}^-_{\beta,\chi_\alpha}P_{2,-}$. *The permutation matrix* $P_{2,-}$ *corresponds to the permutation* $\lambda_{2,-}\circ\sigma$ *and hence is determined by the coset representatives* $J_{2,-}R_j j_{0,-}$.

Proof. For our choice of coset representatives R_j as given in (2.1.4) one finds for $\rho_{\chi_\alpha}(S)$

$$\rho_{\chi_\alpha}(S) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-2\pi i\alpha} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & e^{2\pi i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{4.0.31}$$

and hence the matrix $Q_{2,-}$ is given by

$$Q_{2,-} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & e^{-2\pi i\alpha} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{2\pi i\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.0.32)$$

For $\rho_{\chi_\alpha}(T)$ one finds

$$\rho_{\chi_\alpha}(T) = \begin{pmatrix} e^{2\pi i\alpha} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2\pi i\alpha} \end{pmatrix}. \quad (4.0.33)$$

A simple calculation then confirms that $P_{2,-}\rho_{\chi_\alpha}(S) = \rho_{\chi_\alpha}(S)P_{2,-}$ respectively $P_{2,-}\rho_{\chi_\alpha}(T) = \rho_{\chi_\alpha}(T^{-1})P_{2,-}$ with

$$P_{2,-} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.0.34)$$

and hence defines a symmetry of the transfer operator $\mathbf{L}_{\beta,\chi_\alpha}$. The matrix $P_{2,-}$ coincides with the permutation matrix P_2 corresponding to the permutation σ_2 in (2.1.6). □

Remark 4.0.4. For the trivial character χ_0 also the map $j_{0,-}z = -z^*$ defines an automorphism of the Maass forms for the group $\Gamma_0(4)$, indeed this is an automorphism for all Hecke congruence subgroups $\Gamma_0(N)$. In this case the permutation $\lambda_{0,-}$ is the trivial permutation and the matrix $Q_{0,-}$ is determined by the permutation $\sigma \circ \delta$. For $\Gamma_0(4)$ this is given by (4.0.28). Using (4.0.31) with $\alpha = 0$ one obtains for $P_{0,-} = \rho_{\chi_0}(S)Q_{0,-}$ just the permutation σ_1 as given in (2.1.5). The symmetry $\tilde{P}_1$ for $\Gamma_0(4)$ and trivial character χ_0 hence corresponds to the automorphism $z \to -z^*$ of the Maass forms for this group.

We have seen that for every eigenfunction $\underline{f} = \underline{f}(\zeta)$ of the operator $P_2\mathcal{L}^+_{\beta,\chi_\alpha}$ with eigenvalue $\lambda = \pm 1$ the function $\underline{\Psi} = \underline{\Psi}(\zeta) = P_2\rho_{\chi_\alpha}(T^{-1}S)P_2\underline{f}(\zeta - 1)$ fulfills the functional equation

$$\underline{\Psi}(\zeta) = \lambda\zeta^{-2\beta}\rho_{\chi_\alpha}(S)P_2\underline{\Psi}(\frac{1}{\zeta}) = \lambda J_-\underline{\Psi}(\zeta) \quad (4.0.35)$$

and hence is an eigenfunction of the involution J_- corresponding to the automorphism $j_- = j_{2,-}$ of the Maass forms for $\Gamma_0(4)$ and character χ_α. Hence this shows

Proposition 4.0.5. *The eigenfunctions $\underline{f} = \underline{f}(\zeta)$ of the operator $P_2\mathcal{L}^+_{\beta,\chi_\alpha}$ with eigenvalue $\lambda = \pm 1$ correspond to Maass forms which are even respectively odd under the involution $J_- = J_{2,-}$.*

Phillips and Sarnak have shown in [18] for a conjugate character $\hat{\chi}_\alpha$ that the Maass cusp forms odd under the corresponding conjugate involution $\hat{J}$ stay cusp forms under the deformation of this character. Hence we get as a corollary of their result

Corollary 4.0.6. *The zero's of the Selberg zeta function for the group $\Gamma_0(4)$ and character χ_α corresponding to eigenfunctions of the operator $P_2\mathcal{L}^+_{\beta,\chi_\alpha}$ with eigenvalue $\lambda = -1$ which for $\alpha = 0$ are on the critical line $\Re\beta = \frac{1}{2}$ stay for all α on this line.*

Remark 4.0.7. The operator $P_2\mathcal{L}^+_{\beta,\chi_\alpha}$ can be used to calculate numerically the Selberg zeta function for small values of $\Im\beta$ and arbitrary $0 \leq \alpha \leq \frac{1}{2}$. These numerical calculations confirm the above Corollary and let us expect that all the zero's of the Selberg function corresponding to the eigenvalue $\lambda = 1$ of the operator $P_2\mathcal{L}^+_{\beta,\chi_\alpha}$ for $\alpha = 0$ leave the critical line when α becomes positive. A detailed discussion of the numerical treatment of the behaviour of the zero's of Selberg's function under the character deformation will appear elsewhere [1].

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