ELLIPTIC CURVES IN MODULI SPACE OF STABLE BUNDLES

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Dedicated to the memory of Eckart Viehweg

Abstract. Let M be the moduli space of rank 2 stable bundles with fixed determinant of degree 1 on a smooth projective curve C of genus $g \geq 2$. When C is generic, we show that any elliptic curve on M has degree (respect to anti-canonical divisor $-K_M$) at least 6, and we give a complete classification for elliptic curves of degree 6. Moreover, if $g > 4$, we show that any elliptic curve passing through the generic point of M has degree at least 12. We also formulate a conjecture for higher rank.

1. Introduction

Let C be a smooth projective curve of genus $g \geq 2$ and $\mathcal{L}$ be a line bundle of degree d on C. Let $M := \mathcal{SU}_C(r, \mathcal{L})^s$ be the moduli space of stable vector bundles on C of rank r and with fixed determinant $\mathcal{L}$, which is a smooth qusi-projective Fano variety with $\mathrm{Pic}(M) = \mathbb{Z} \cdot \Theta$ and $-K_M = 2(r,d)\Theta$, where Θ is an ample divisor. Let B be a smooth projective curve of genus b. The degree of a curve $\phi : B \to M$ is defined to be $\deg\phi^*(-K_M)$. It seems quite natural to ask what is the lower bound of degree and to classify the curves of lowest degree.

When $B = \mathbb{P}^1$, we have determined all $\phi : \mathbb{P}^1 \to M$ with lowest degree in [6] and all $\phi : \mathbb{P}^1 \to M$ passing through the generic point of M with lowest degree in [9]. In fact, one can construct $\phi : \mathbb{P} \to M$ for various projective spaces $\mathbb{P}$ such that $\phi^*(-K_M) = \mathcal{O}_{\mathbb{P}}(2(r,d))$, and $\phi : \mathbb{P}^{r-1} \to M$ passing through the generic point of M such that $\phi^*(-K_M) = \mathcal{O}_{\mathbb{P}^{r-1}}(2r)$. Then it was proved in [6] and [9] that the images of lines in these projective spaces exhaust all minimal rational curves on M (resp. minimal rational curves passing through generic point of M). Some applications of the results were also pointed out in [6] and [9]. Thus it is natural to ask what are the situation when $b > 0$. This note is a start to study the case of $b = 1$. It may happen that the normalization of $\phi(B)$ is $\mathbb{P}^1$. To avoid this case, we call $\phi : B \to M$

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an essential elliptic curve of M if the normalization of $\phi(B)$ is an elliptic curve.

It is easy to construct essential elliptic curves of degree $6(r,d)$ on M, and essential elliptic curves of degree $6r$ that pass through the generic point of M. For example, for smooth elliptic curves $B\subset\mathbb{P}$ of degree 3, the morphism $\phi:\mathbb{P}\to M$ defines essential elliptic curves $\phi|_B:B\to M$ of degree $6(r,d)$ (See Example 3.6), which are called **elliptic curves of split type**. For smooth elliptic curves $B\subset\mathbb{P}^{r-1}$ of degree 3, the morphism $\phi:\mathbb{P}^{r-1}\to M$ defines essential elliptic curves $\phi|_B:B\to M$ of degree $6r$ passing through the generic point of M (See Example 3.5), which are called **elliptic curves of Hecke type**. Are they minimal elliptic curves of M (resp. minimal elliptic curves passing through generic point of M)? Do they exhaust all minimal essential elliptic curves on M (See Conjecture 4.8 for detail)?

In this note, we consider the case that $r=2$ and $d=1$, then M is a smooth projective fano manifold of dimension $3g-3$. When C is generic, we show that any essential elliptic curve $\phi:B\to M$ has degree at least 6, and it must be an **elliptic curve of split type** if it has degree 6 (See Theorem 4.6). When $g>4$ and C is generic, we show that any essential elliptic curve $\phi:B\to M$ passing through the generic point of M have degree at least 12 (See Theorem 4.7). When C is generic, there is no nontrivial morphism from C to an elliptic curve, which implies that $\mathrm{Pic}(C\times B)=\mathrm{Pic}(C)\times\mathrm{Pic}(B)$. It is the condition that we need through the whole paper.

We give a brief description of the article. In Section 2, we show a formula of degree for general case. In Section 3, we show how the general formula implies the known case $B=\mathbb{P}^1$ and construct the examples of essential elliptic curves of degree $6(r,d)$ and $6r$. In Section 4, we prove the main theorems (Theorem 4.6 and Theorem 4.7), which is the special case $r=2$, $d=1$ of Conjecture 4.8. Although I believe the conjecture, I leave the case of $r>2$ to other occasion.

2. The degree formula of curves in moduli spaces

Let C be a smooth projective curve of genus $g\geq 2$ and $\mathcal{L}$ a line bundle on C of degree d. Let $M=\mathcal{SU}_C(r,\mathcal{L})^s$ be the moduli spaces of stable bundles on C of rank r, with fixed determinant $\mathcal{L}$. It is well-known that $Pic(M)=\mathbb{Z}\cdot\Theta$, where Θ is an ample divisor.

Lemma 2.1. *For any smooth projective curve B of genus b, if*

$$\phi:B\to M$$

is defined by a vector bundle E on $C \times B$, then

$$\deg\phi^*(-K_M) = c_2(\mathcal{E}nd^0(E)) = 2rc_2(E) - (r-1)c_1(E)^2 := \Delta(E)$$

Proof. In general, there is no universal bundle on $C \times M$, but there exist vector bundle $\mathcal{E}nd^0$ and projective bundle $\mathcal{P}$ on $C \times M$ such that $\mathcal{E}nd^0|_{C\times\{[V]\}} = \mathcal{E}nd^0(V)$ and $\mathcal{P}|_{C\times\{[V]\}} = \mathbb{P}(V)$ for any $[V] \in M$. Let $\pi : C \times M \to M$ be the projection, then $T_M = R^1\pi_*(\mathcal{E}nd^0)$, which commutes with base changes since $\pi_*(\mathcal{E}nd^0) = 0$.

For any curve $\phi : B \to M$, let $X := C \times B$, $\mathbb{E} = (id \times \phi)^*\mathcal{E}nd^0$ and $\pi : X = C \times B \to B$ still denote the projection. Then $\phi^*T_M = R^1\pi_*\mathbb{E}$. By Riemann-Roch theorem, we have

$$\deg\phi^*(-K_M) = \chi(R^1\pi_*\mathbb{E}) + (r^2-1)(g-1)(b-1).$$

By using Leray spectral sequence and $\chi(\mathbb{E}) = \deg(ch(\mathbb{E}) \cdot td(T_X))_2$, we have $\chi(R^1\pi_*\mathbb{E}) = -\chi(\mathbb{E}) = c_2(\mathbb{E}) - (r^2-1)(g-1)(b-1)$, hence

$$\deg\phi^*(-K_M) = c_2(\mathbb{E}).$$

If $\phi : B \to M$ is defined by a vector bundle E on $X = C \times B$, then $\mathbb{E} = \mathcal{E}nd^0(E)$ (cf. the proof of lemma 2.1 in [9]). Thus

$$\deg\phi^*(-K_M) = c_2(\mathcal{E}nd^0(E)) = 2rc_2(E) - (r-1)c_1(E)^2.$$

□

Let $f : X \to C$ be the projection. Then, for any vector bundle E on X, there is a relative Harder-Narasimhan filtration (cf Theorem 2.3.2, page 45 in [5])

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that $F_i = E_i/E_{i-1}$ $(i = 1, \ldots n)$ are flat over C and its restriction to general fiber $X_p = f^{-1}(p)$ is the Harder-Narasimhan filtration of $E|_{X_p}$. Thus F_i are semi-stable of slop μ_i at generic fiber of $f : X \to B$ with $\mu_1 > \mu_2 > \cdots > \mu_n$. Then we have the following theorem

Theorem 2.2. *For any vector bundle E of rank r on X, let*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

be the relative Harder-Narasimhan filtration over C with $F_i = E_i/E_{i-1}$ and $\mu_i = \mu(F_i|_{f^{-1}(x)})$ for generic $x \in C$. Let $\mu(E)$ and $\mu(E_i)$ denote the slop of $E|_{\pi^{-1}(b)}$ and $E_i|_{\pi^{-1}(b)}$ for generic $b \in B$. Then, if

$$\mathrm{Pic}(C \times B) = \mathrm{Pic}(C) \times \mathrm{Pic}(B),$$

we have the following formula

$$\Delta(E) = 2r \left(\begin{array}{c} \sum_{i=1}^{n} \left(c_2(F_i) - \frac{\mathrm{rk}(F_i) - 1}{2\,\mathrm{rk}(F_i)} c_1(F_i)^2 \right) \\ + \sum_{i=1}^{n-1} (\mu(E) - \mu(E_i))\mathrm{rk}(E_i)(\mu_i - \mu_{i+1}) \end{array} \right). \tag{2.1}$$

Proof. It is easy to see that

$$\begin{aligned} 2c_2(E) &= 2\sum_{i=1}^{n} c_2(F_i) + 2\sum_{i=1}^{n} c_1(E_{i-1})c_1(F_i) \\ &= 2\sum_{i=1}^{n} c_2(F_i) + c_1(E)^2 - \sum_{i=1}^{n} c_1(F_i)^2. \end{aligned}$$

Thus

$$\Delta(E) = 2r\sum_{i=1}^{n} c_2(F_i) + c_1(E)^2 - r\sum_{i=1}^{n} c_1(F_i)^2.$$

Let r_i be the rank of F_i and d_i be the degree of F_i on the generic fiber of $\pi : C \times B \to B$. Then we can write

$$c_1(F_i) = f^*\mathcal{O}_C(d_i) + \pi^*\mathcal{O}_B(r_i\mu_i)$$

where $\mathcal{O}_C(d_i)$ (resp. $\mathcal{O}_B(r_i\mu_i)$) denotes a divisor of degree d_i (resp. degree $r_i\mu_i$) of C (resp. B). Note that

$$c_1(F_i)^2 = 2d_ir_i\mu_i, \qquad c_1(E)^2 = 2d\sum_{i=1}^{n} r_i\mu_i$$

we have

$$\begin{aligned} \Delta(E) &= 2r\left(\sum_{i=1}^{n} c_2(F_i) + \mu(E)\sum_{i=1}^{n} r_i\mu_i - \sum_{i=1}^{n} d_ir_i\mu_i\right) \\ &= 2r\left(\sum_{i=1}^{n}(c_2(F_i) - (r_i - 1)d_i\mu_i) + \mu(E)\sum_{i=1}^{n} r_i\mu_i - \sum_{i=1}^{n} d_i\mu_i\right). \end{aligned}$$

Let $\deg(E_i)$ denote the degree of E_i on the generic fiber of

$$\pi : C \times B \to B.$$

Using $d_i = \deg(E_i) - \deg(E_{i-1})$ and $r_i = \mathrm{rk}(E_i) - \mathrm{rk}(E_{i-1})$, we have

$$\mu(E)\sum_{i=1}^{n} r_i\mu_i - \sum_{i=1}^{n} d_i\mu_i = \sum_{i=1}^{n-1}(\mu(E) - \mu(E_i))\mathrm{rk}(E_i)(\mu_i - \mu_{i+1}).$$

Since $d_i\mu_i = c_1(F_i)^2/2r_i$, we get the formula

$$\Delta(E) = 2r\left(\begin{array}{l}\displaystyle\sum_{i=1}^{n}\left(c_2(F_i) - \frac{r_i - 1}{2r_i}c_1(F_i)^2\right) \\ \displaystyle+\sum_{i=1}^{n-1}(\mu(E)-\mu(E_i))\mathrm{rk}(E_i)(\mu_i - \mu_{i+1})\end{array}\right).$$

□

Remark 2.3. I do not know if the formula holds without the assumption that $\mathrm{Pic}(C \times B) = \mathrm{Pic}(C) \times \mathrm{Pic}(B)$. On the other hand, the assumption holds when B is an elliptic curve and C is generic.

Theorem 2.4. *For any torsion free sheaf $\mathcal{F}$ on $X = C \times B$, if its restriction to a fiber of $f : X = C \times B \to C$ is semi-stable, then*

$$\Delta(\mathcal{F}) = 2\,\mathrm{rk}(\mathcal{F})\,c_2(\mathcal{F}) - (\mathrm{rk}(\mathcal{F}) - 1)c_1(\mathcal{F})^2 \geq 0.$$

If the determinants $\{\det(\mathcal{F}^{**})_x\}_{x\in C}$ are isomorphic each other, then $\Delta(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is locally free and satisfies

- *All the bundles $\{\mathcal{F}_x := \mathcal{F}|_{\{x\}\times B}\}_{x\in C}$ are semi-stable and s-equivalent each other.*
- *All the bundles $\{\mathcal{F}_y := \mathcal{F}|_{C\times\{y\}}\}_{y\in B}$ are isomorphic each other.*

Proof. Since $\Delta(\mathcal{F}) \geq \Delta(\mathcal{F}^{**})$, we can assume that $\mathcal{F}$ is a vector bundle. There is a $x \in C$ such that $\mathcal{F}_x = \mathcal{F}|_{\{x\}\times B}$ is semi-stable, so is $\mathcal{E}nd^0(\mathcal{F})_x = \mathcal{E}nd^0(\mathcal{F}_x)$. Thus, by a theorem of Faltings (cf. Theorem I.2. of [1]), there is a vector bundle V on B such that

$$\mathrm{H}^0(\mathcal{E}nd^0(\mathcal{F})_x \otimes V) = \mathrm{H}^1(\mathcal{E}nd^0(\mathcal{F})_x \otimes V) = 0,$$

which defines a global section $\vartheta(V)$ of the line bundle

$$\Theta(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V) = (\det f_!(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V))^{-1}$$

such that $\vartheta(V)(x) \neq 0$. By Grothendieck-Riemann-Roch theorem,

$$\begin{aligned} c_1(\det f_!(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)) &= f_*(\mathrm{ch}(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)\mathrm{td}(\pi^*T_B))_2 \\ &= -c_2(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V) \end{aligned}$$

which means that the line bundle $\Theta(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)$ has degree

$$c_2(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V) = \mathrm{rk}(V)\cdot c_2(\mathcal{E}nd^0(\mathcal{F})) = \mathrm{rk}(V)\cdot\Delta(\mathcal{F})$$

with a nonzero global section $\vartheta(V)$. Thus $\Delta(\mathcal{F}) \geq 0$.

If $\Delta(\mathcal{F}) = 0$, then $\mathcal{F} = \mathcal{F}^{**}$ must be locally free and $\vartheta(V)(x) \neq 0$ for any $x \in C$, which means that for any $x \in C$, we have

$$\mathrm{H}^0(\mathcal{E}nd^0(\mathcal{F})_x \otimes V) = \mathrm{H}^1(\mathcal{E}nd^0(\mathcal{F})_x \otimes V) = 0.$$

Then, by the theorem of Faltings, the bundles

$$\{\,\mathcal{E}nd^0(\mathcal{F})_x\,\}_{x\in C}$$

are all semi-stable. Thus, for any $x \in C$, the bundle $\mathcal{F}_x := \mathcal{F}|_{\{x\}\times B}$ is semi-stable. The bundle $\mathcal{F}$ defines a morphism $\phi_{\mathcal{F}} : C \to \mathcal{U}_B$ from C to the moduli space $\mathcal{U}_B$ of semi-stable bundles on B, the line bundle $\Theta(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)$ clearly descends to a line bundle on $\mathcal{U}_B$. If the determinants $\det(\mathcal{F}_x)$ $(x \in C)$ are fixed, then

$$\deg(\Theta(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)) = 0$$

means that all $\{\mathcal{F}_x\}_{x\in C}$ are s-equivalence.

By using a technique of [4] (see Step 5 in the proof of Theorem 4.2 in [4], see also the proof of Theorem I.4 in [1]), we will show

$$\mathcal{F}|_{C\times\{y_1\}} \cong \mathcal{F}|_{C\times\{y_2\}}, \quad \forall\; y_1,\, y_2 \in B.$$

Choose a nontrivial extension $0 \to V \to V' \xrightarrow{q_1} \mathcal{O}_{y_1} \to 0$ on B, let $\mathfrak{Q}$ be the Quot-scheme of rank 0 and degree 1 quotients of V', and

$$0 \to \mathcal{K} \to p_B^* V' \to \mathfrak{T} \to 0$$

be the tautological exact sequence on $B \times \mathfrak{Q}$. Fix a point $x_1 \in C$, then the set $q \in \mathfrak{Q}$ such that $\mathrm{H}^0(\mathcal{F}_{x_1} \otimes \mathcal{K}_q) = \mathrm{H}^1(\mathcal{F}_{x_1} \otimes \mathcal{K}_q) = 0$ is an open set $U \subset \mathfrak{Q}$ and $U \neq \emptyset$ since $q_1 = (0 \to V \to V' \xrightarrow{q_1} \mathcal{O}_{y_1} \to 0) \in U$.

Let $\Gamma \subset B \times \mathbb{P}(V')$ be the graph of $\mathbb{P}(V') \xrightarrow{p} B$, then

$$p_B^*V' \to p_B^*V'|_{\Gamma} = p^*V' \to \mathcal{O}(1) \to 0$$

induces a quotient $p_B^*V' \to {}_{\Gamma}\mathcal{O}(1) \to 0$ on $B \times \mathbb{P}(V')$, which defines a morphism $\mathbb{P}(V') \to \mathfrak{Q}$. It is easy to see that $\mathbb{P}(V') \to \mathfrak{Q}$ is surjective (in fact, it is a isomorphism). Thus there is an open $B_1 \subset B$ with $y_1 \in B_1$ such that for any $y \in B_1$ there exists an exact sequence

$$0 \to \mathcal{K}_q \to V' \xrightarrow{q} \mathcal{O}_y \to 0 \tag{2.2}$$

such that $\mathrm{H}^0(\mathcal{F}_{x_1} \otimes \mathcal{K}_q) = \mathrm{H}^1(\mathcal{F}_{x_1} \otimes \mathcal{K}_q) = 0$, which implies

$$\mathrm{H}^0(\mathcal{F}_x \otimes \mathcal{K}_q) = \mathrm{H}^1(\mathcal{F}_x \otimes \mathcal{K}_q) = 0 \qquad \forall\; x \in C$$

since $\mathcal{F}_x$ is s-equivalent to $\mathcal{F}_{x_1}$ for any $x \in C$. Pull back the exact sequence (2.2) by $\pi : C \times B \to B$ and tensor with $\mathcal{F}$, we have the exact sequence

$$0 \to \mathcal{F} \otimes \pi^*\mathcal{K}_q \to \mathcal{F} \otimes \pi^*V' \to \mathcal{F}_y \to 0. \tag{2.3}$$

Take direct images of (2.2) under $f : C \times B \to C$, we have

$$\mathcal{F}_y \cong f_*(\mathcal{F} \otimes \pi^*V')\,, \quad \forall\; y \in B_1$$

which implies that all $\{\mathcal{F}_y\}_{y\in B}$ are isomorphic each other. $\square$

We will need the following lemma in the later computation, whose proof are straightforward computations (see [2] for the case of rank 1).

Lemma 2.5. *Let X be a smooth projective surface and $j : D \hookrightarrow X$ be an effective divisor. Then, for any vector bundle V on D, we have*

$$c_1(j_*V) = \mathrm{rk}(V) \cdot D$$
$$c_2(j_*V) = \frac{\mathrm{rk}(V)(\mathrm{rk}(V)+1)}{2} D^2 - j_*c_1(V).$$

Recall that $X_t = f^{-1}(t)$ denotes the fiber of $f : X \to C$ and for any vector bundle $\mathcal{F}$ on X, $\mathcal{F}_t$ denote the restrictions of $\mathcal{F}$ to X_t.

Lemma 2.6. *Let $\mathcal{F}_t \to W \to 0$ be a locally free quotient and*

$$0 \to \mathcal{F}' \to \mathcal{F} \to {}_{X_t}W \to 0$$

be the elementary transformation of $\mathcal{F}$ along W at $X_t \subset X$. Then

$$\Delta(\mathcal{F}) = \Delta(\mathcal{F}') + 2r(\mu(\mathcal{F}_t) - \mu(W))\mathrm{rk}(W).$$

3. Mminimal rational curves and examples of elliptic curves on moduli spaces

When $B = \mathbb{P}^1$, the condition $\mathrm{Pic}(C \times B) = \mathrm{Pic}(C) \times \mathrm{Pic}(B)$ always hold and any morphism $B \to M$ is defined by a vector bundle on $C \times B$ (cf. Lemma 2.1 of [9]).

Recall that given two nonnegative integers k, ℓ, a vector bundle W of rank r and degree d on C is (k, ℓ)-stable, if, for each proper subbundle W' of W, we have

$$\frac{\deg(W') + k}{\mathrm{rk}(W')} < \frac{\deg(W) + k - \ell}{r}.$$

The usual stability is equivalent to $(0, 0)$-stability. The (k, ℓ)-stability is an open condition. The proofs of following lemmas are easy and elementary (cf. [7]).

Lemma 3.1. *If $g \geq 3$, M contains $(0, 1)$-stable and $(0, 1)$-stable bundles. M contains a $(1, 1)$-stable bundle W except $g = 3$, d, r both even.*

Lemma 3.2. *Let $0 \to V \to W \to \mathcal{O}_p \to 0$ be an exact sequence, where $\mathcal{O}_p$ is the 1-dimensional skyscraper sheaf at $p \in C$. If W is (k, ℓ)-stable, then V is $(k, \ell - 1)$-stable.*

A curve $B \to M$ defined by E on $C \times B$ passes through the generic point of M implies that $E_y := E|_{C\times\{y\}}$ is $(1,1)$-stable for generic $y \in B$. Thus in the formula (2.1) of Theorem 2.2 we have

$$(\mu(E) - \mu(E_i))\mathrm{rk}(E_i) > 1. \tag{3.1}$$

On the other hand, any semi-stable bundle on $B = \mathbb{P}^1$ must have integer slop. By the formula (2.1) in Theorem 2.2, we have

$$\Delta(E) > 2r$$

if E is not semi-stable on the generic fiber of $f : X = C \times \mathbb{P}^1 \to C$.

When E is semi-stable on the generic fiber of $f : X \to C$, by tensor E with a line bundle, we can assume that E is trivial on the generic fiber of $f : X \to C$. Thus $\Delta(E) = 2rc_2(E) \geq 2r$ and there must be a fiber $X_t = f^{-1}(t)$ such that $E_t = E|_{X_t}$ is not semi-stable by Theorem 2.4. If $\Delta(E) = 2r$, by Lemma 2.6, we must have $\mathrm{rk}(W) = 1$, $\mu(W) = -1$ and $\Delta(\mathcal{F}') = 0$ in Lemma 2.6. Thus $\Delta(E) = 2r$ if and only if E satisfies

$$0 \to f^*V \to E \to {}_{X_t}\mathcal{O}_{\mathbb{P}^1}(-1) \to 0$$

which defines a so called Hecke curve. Therefore we get the main theorem in [9].

Theorem 3.3. *If $g \geq 3$, then any rational curve of M passing through the generic point of M has at least degree $2r$ with respect to $-K_M$. It has degree $2r$ if and only if it is a Hecke curve except $g = 3$, $r = 2$ and $(2,d) = 2$.*

At the end of this section, we give some examples of elliptic curves on M. Let us recall the construction of Hecke curves. Let $\mathcal{U}_C(r, d-1)$ be the moduli space of stable bundles of rank r and degree $d-1$. Let

$$\mathfrak{O} \subset \mathcal{U}_C(r, d-1)$$

be the open set of $(1,0)$-stable bundles. Let $C \times \mathfrak{O} \xrightarrow{\psi} J^d(C)$ be defined as $\psi(x, V) = \mathcal{O}_C(x) \otimes \det(V)$ and

$$\mathscr{R}_C := \psi^{-1}(\mathcal{L}) \subset C \times \mathfrak{O},$$

which consists of the points (x, V) such that V are $(1,0)$-stable bundles on C with $\det(V) = \mathcal{L}(-x)$. There exists a projective bundle

$$p : \mathscr{P} \to \mathscr{R}_C$$

such that for any $(x, V) \in \mathscr{R}_C$ we have $p^{-1}(x, V) = \mathbb{P}(V_x^\vee)$. Let

$$V_x^\vee \otimes \mathcal{O}_{\mathbb{P}(V_x^\vee)} \to \mathcal{O}_{\mathbb{P}(V_x^\vee)}(1) \to 0$$

be the universal quotient, $f : C \times \mathbb{P}(V_x^\vee) \to C$ be the projection, and

$$0 \to \mathscr{E}^\vee \to f^*V^\vee \to {}_{\{x\}\times\mathbb{P}(V_x^\vee)}\mathcal{O}_{\mathbb{P}(V_x^\vee)}(1) \to 0$$

where $\mathscr{E}^\vee$ is defined to the kernel of the surjection. Take dual, we have

$$0 \to f^*V \to \mathscr{E} \to {}_{\{x\}\times\mathbb{P}(V_x^\vee)}\mathcal{O}_{\mathbb{P}(V_x^\vee)}(-1) \to 0, \tag{3.2}$$

which, at any point $\xi = (V_x^\vee \to \Lambda \to 0) \in \mathbb{P}(V_x^\vee)$, gives exact sequence

$$0 \to V \xrightarrow{\iota} \mathscr{E}_\xi \to \mathcal{O}_x \to 0$$

on C such that $\ker(\iota_x) = \Lambda^\vee \subset V_x$. V being $(1,0)$-stable implies stability of $\mathscr{E}_\xi$. Thus (3.2) defines

$$\Psi_{(x,V)} : \ \mathbb{P}(V_x^\vee) = p^{-1}(x,V) \to M. \tag{3.3}$$

Definition 3.4. The images (under $\{\Psi_{(x,V)}\}_{(x,V)\in\mathscr{R}_C}$) of lines in the fibres of $p : \mathscr{P} \to \mathscr{R}_C$ are the so called **Hecke curves** in M. The images (under $\{\Psi_{(x,V)}\}_{(x,V)\in\mathscr{R}_C}$) of elliptic curves in the fibres of

$$p : \mathscr{P} \to \mathscr{R}_C$$

are called **elliptic curves of Hecke type**.

It is known (cf. [7, Lemma 5.9]) that the morphisms in (3.3) are closed immersions. By a straightforward computation, we have

$$\Psi^*_{(x,V)}(-K_M) = \mathcal{O}_{\mathbb{P}(V_x^\vee)}(2r). \tag{3.4}$$

For any point $[W] \in M$ and $(W_x \to \mathbb{C} \to 0) \in \mathbb{P}(W_x)$, where W is $(1,1)$-stable, we define a $(1,0)$-stable bundle V by

$$0 \to V \xrightarrow{\alpha} W \to {}_x\mathbb{C} \to 0.$$

Then the images of $p^{-1}(x,V) = \mathbb{P}(V_x^\vee)$ are projective spaces that pass through $[W] \in M$, and the images of lines $\ell \subset \mathbb{P}(V_x^\vee)$ that pass through $[\ker(\alpha_x)] \in \mathbb{P}(V_x^\vee)$ are Hecke curves passing through $[W] \in M$.

Example 3.5. When $g \geq 4$ and $r > 2$, for generic $[W] \in M$, the images of smooth elliptic curves $B \subset \mathbb{P}(V_x^\vee)$ with degree 3 and $[\ker(\alpha_x)] \in B$ are smooth elliptic curves on M that pass through $[W] \in M$, which have degree $6r$ by (3.4).

If we do not require the curve $\phi : B \to M$ passing through generic point of M, we may construct rational curves and elliptic curves with smaller degree. Let us recall the Construction 2.3 from [6].

For any given r and d, let r_1, r_2 be positive integers and d_1, d_2 be integers that satisfy the equalities $r_1 + r_2 = r$, $d_1 + d_2 = d$ and

$$r_1\frac{d}{(r,d)} - d_1\frac{r}{(r,d)} = 1, \quad d_2\frac{r}{(r,d)} - r_2\frac{d}{(r,d)} = 1.$$

Let $\mathcal{U}_C(r_1,d_1)$ (resp. $\mathcal{U}_C(r_2,d_2)$) be the moduli space of stable vector bundles with rank r_1 (resp. r_2) and degree d_1 (resp. d_2). Then, since

$(r_1, d_1) = 1$ and $(r_2, d_2) = 1$, there are universal vector bundles $\mathcal{V}_1$, $\mathcal{V}_2$ on $C \times \mathcal{U}_C(r_1, d_1)$ and $C \times \mathcal{U}_C(r_2, d_2)$ respectively. Consider

$$\mathcal{U}_C(r_1, d_1) \times \mathcal{U}_C(r_2, d_2) \xrightarrow{\det(\bullet)\times\det(\bullet)} J_C^{d_1} \times J_C^{d_2} \xrightarrow{(\bullet)\otimes(\bullet)} J_C^d,$$

let $\mathcal{R}(r_1, d_1)$ be its fiber at $[\mathcal{L}] \in J_C^d$. The pullback of $\mathcal{V}_1$, $\mathcal{V}_2$ by the projection $C \times \mathcal{R}(r_1, d_1) \to C \times \mathcal{U}_C(r_i, d_i)$ $(i = 1, 2)$ is still denoted by $\mathcal{V}_1$, $\mathcal{V}_2$ respectively. Let $p : C \times \mathcal{R}(r_1, d_1) \to \mathcal{R}(r_1, d_1)$ and

$$\mathcal{G} = R^1 p_*(\mathcal{V}_2^\vee \otimes \mathcal{V}_1),$$

which is locally free of rank $r_1 r_2 (g-1) + (r, d)$. Let

$$q : P(r_1, d_1) = \mathbb{P}(\mathcal{G}) \to \mathcal{R}(r_1, d_1)$$

be the projective bundle parametrzing 1-dimensional subspaces of $\mathcal{G}_t$ $(t \in \mathcal{R}(r_1, d_1))$ and $f : C \times P(r_1, d_1) \to C$, $\pi : C \times P(r_1, d_1) \to P(r_1, d_1)$ be the projections. Then there is a universal extension

$$0 \to (id \times q)^* \mathcal{V}_1 \otimes \pi^* \mathcal{O}_{P(r_1,d_1)}(1) \to \mathcal{E} \to (id \times q)^* \mathcal{V}_2 \to 0 \tag{3.5}$$

on $C \times P(r_1, d_1)$ such that for any $x = ([V_1], [V_2], [e]) \in P(r_1, d_1)$, where $[V_i] \in \mathcal{U}_C(r_i, d_i)$ with $\det(V_1) \otimes \det(V_2) = \mathcal{L}$ and $[e] \subset \mathrm{H}^1(C, V_2^\vee \otimes V_1)$ being a line through the origin, the bundle $\mathcal{E}|_{C \times \{x\}}$ is the isomorphic class of vector bundles E given by extensions

$$0 \to V_1 \to V \to V_2 \to 0$$

that defined by vectors on the line $[e] \subset \mathrm{H}^1(C, V_2^\vee \otimes V_1)$. Then V must be stable by [6, Lemma 2.2], and the sequence (3.5) defines

$$\Phi : P(r_1, d_1) \to \mathcal{SU}_C(r, \mathcal{L})^s = M.$$

On each fiber $q^{-1}(\xi) = \mathbb{P}(\mathrm{H}^1(V_2^\vee \otimes V_1))$ at $\xi = (V_1, V_2)$, the morphisms

$$\Phi_\xi := \Phi|_{q^{-1}(\xi)} : q^{-1}(\xi) = \mathbb{P}(\mathrm{H}^1(V_2^\vee \otimes V_1)) \to M \tag{3.6}$$

is birational and $\Phi_\xi^*(-K_M) = \mathcal{O}_{\mathbb{P}(\mathrm{H}^1(V_2^\vee \otimes V_1))}(2(r, d))$ by [6, Lemma 2.4].

Example 3.6. The images of lines $\ell \subset \mathbb{P}(\mathrm{H}^1(V_2^\vee \otimes V_1))$ are rational curves of degree $2(r, d)$ on M, which is clearly the minimal degree since $-K_M = 2(r, d)\Theta$. For smooth elliptic curves $B \subset \mathbb{P}(\mathrm{H}^1(V_2^\vee \otimes V_1))$ of degree 3, the images of $\Phi_\xi : B \to M$ are of degree $6(r, d)$. For any smooth elliptic curve $B \subset q^{-1}(\xi)$ $(\forall\ \xi \in \mathcal{R}(r_1, d_1))$, the images of $\Phi_\xi : B \to M$ are called **elliptic curves of split type**.

4. Minimal elliptic curves on moduli spaces

In this section, we consider the moduli space M of rank 2 stable bundles on C with a fixed determinant $\mathcal{L}$ of degree 1. We also assume that the curve C is generic in the sense that C admits no surjective morphism to an elliptic curve. With this assumption, we know that $\mathrm{Pic}(C \times B) = \mathrm{Pic}(C) \times \mathrm{Pic}(B)$ for any elliptic curve B.

For a morphism $\phi : B \to M$, it may happen that the normalization of $\phi(B)$ is a rational curve. To avoid this case, we make the following definition

Definition 4.1. $\phi : B \to M$ is called an essential elliptic curve of M if the normalization of $\phi(B)$ is an elliptic curve.

For any morphism $\phi : B \to M$, let E be the vector bundle on $X = C \times B$ that defines ϕ. It will be free to tensor E with a pull-back of line bundles on B. In this section, B will always denote an elliptic curve.

Proposition 4.2. *Let $\phi : B \to M$ be an essential elliptic curve of M defined by a vector bundle E. If E is not semi-stable on the generic fiber of $f : X \to C$, then*

$$\Delta(E) \geq 6.$$

If $g = g(C) \geq 4$ and the curve $\phi : B \to M$ passes through the generic point of M, then

$$\Delta(E) > 12.$$

Proof. Let $0 \to E_1 \to E \to F_2 \to 0$ be the relative Harder-Narasimhan filtration over C. Then we have exact sequence

$$0 \to E_1|_{X_t} \to E|_{X_t} \to F_2|_{X_t} \to 0$$

on each fiber $X_t = \{t\} \times B$ of $f : X \to C$ since E_1, F_2 are flat over C. Thus E_1 is locally free (cf. Lemma 1.27 of [8]) and

$$\Delta(E) = 4c_2(F_2) + 4(\mu(E) - \mu(E_1))(\mu_1 - \mu_2) \tag{4.1}$$

where $\mu_1 = \deg(E_1|_{X_t})$, $\mu_2 = \deg(F_2|_{X_t})$ for $t \in C$ (cf. Theorem 2.2).

That $0 \to E_1 \to E \to F_2 \to 0$ is the relative Harder-Narasimhan filtration over C means for almost $t \in C$ the exact sequences

$$0 \to E_1|_{X_t} \to E|_{X_t} \to F_2|_{X_t} \to 0$$

are the Harder-Narasimhan filtration of $E|_{X_t}$, which in particular means that F_2 is locally free over $f^{-1}(C \setminus T)$ where $T \subset C$ is a finite set. Thus

$$0 \to E_1|_{C \times \{y\}} \to E|_{C \times \{y\}} \to F_2|_{C \times \{y\}} \to 0\ , \quad \forall\, y \in B \tag{4.2}$$

are exact sequences, which imply that F_2 is also B-flat.

If $c_2(F_2)=0$, then F_2 is a line bundle and there are line bundles V_1, V_2 on C such that

$$E_1=f^*V_1\otimes\pi^*\mathcal{O}(\mu_1),\quad F_2=f^*V_2\otimes\pi^*\mathcal{O}(\mu_2)$$

where $\mathcal{O}(\mu_i)$ denote line bundles on B of degree μ_i. Replace E by $E\otimes\pi^*\mathcal{O}(-\mu_2)$, we can assume that E satisfies

$$0\to f^*V_1\otimes\pi^*\mathcal{O}(\mu_1-\mu_2)\to E\to f^*V_2\to 0. \tag{4.3}$$

Let $d_i=\deg(V_i)$ and $J=\{(L_1,L_2)\in J_C^{d_1}\times J_C^{d_2}\,|\,L_1\otimes L_2=\mathcal{L}\}$. Then there is a projective bundle $q:P\to J$ and an universal extension

$$0\to(id\times q)^*\mathcal{V}_1\otimes\pi^*\mathcal{O}_P(1)\to\mathcal{E}\to(id\times q)^*\mathcal{V}_2\to 0 \tag{4.4}$$

on $C\times P$ such that for any $x=([V_1],[V_2],[e])\in P$, where $[V_i]\in J_C^{d_i}$ with $V_1)\otimes V_2=\mathcal{L}$ and $[e]\subset \mathrm{H}^1(C,V_2^{-1}\otimes V_1)$ being a line through the origin, the bundle $\mathcal{E}|_{C\times\{x\}}$ is the isomorphic class of vector bundles V given by extensions $0\to V_1\to V\to V_2\to 0$ that defined by vectors on the line $[e]\subset\mathrm{H}^1(C,V_2^{-1}\otimes V_1)$, where $\mathcal{V}_i$ denote the pullback (under $C\times J\to C\times J_C^{d_i}$) of universal line bundles, and $\pi:C\times P\to P$ denote the projection. Thus the exact sequence (4.3) induces a morphism

$$\psi:B\to\mathbb{P}^{d_2-d_1+g-2}=q^{-1}(V_1,V_2)\subset P \tag{4.5}$$

such that $\mathcal{O}(\mu_1-\mu_2)=\psi^*\mathcal{O}_P(1)$ and $\phi:B\to M$ factors through $\psi:B\to\psi(B)\subset\mathbb{P}^{d_2-d_1+g-2}$, which implies that the normalization of $\psi(B)$ is an elliptic curve. Hence $\mu_1-\mu_2\ge 3$ and $\Delta(E)\ge 6$ by (4.1). If $\phi:B\to M$ passes through the generic point, then $\mu(E)-\mu(E_1)>1$ and $\Delta(E)>12$.

If $c_2(F_2)\neq 0$, F_2 is not locally free, which implies that there is a $y_0\in B$ such that $F_2|_{C\times\{y_0\}}$ has torsion $\tau(F_2|_{C\times\{y_0\}})\neq 0$ since F_2 is B-flat (cf. Lemma 1.27 of [8]). Let

$$0\to\tau(F_2|_{C\times\{y_0\}})\to F_2|_{C\times\{y_0\}}\to F_2^0\to 0. \tag{4.6}$$

Then F_2^0 being a quotient line bundle of $E|_{C\times\{y_0\}}$ implies

$$\deg(F_2^0)>\mu(E|_{C\times\{y_0\}})=\frac{1}{2}$$

since $E|_{C\times\{y_0\}}$ is stable. By sequences (4.2) and (4.6), we have

$$\mu(E_1)=\deg(E_1|_{C\times\{y_0\}})=1-\deg(F_2^0)-\dim\tau(F_2|_{C\times\{y_0\}})\le -1$$

which, by the formula (4.1), implies that

$$\Delta(E)\ge 4c_2(F_2)+4(\frac{1}{2}+1)(\mu_1-\mu_2)\ge 10.$$

When $\phi:B\to M$ passes through a generic point, in order to show $\Delta(E)>12$, we note that $c_2(F_2)\neq 0$ and F_2 being C-flat also imply that there exists a $t_0 \in C$ such that $F_2|_{X_{t_0}}$ has torsion $\tau(F_2|_{X_{t_0}}) \neq 0$. Let $0 \to \tau(F_2|_{X_{t_0}}) \to F_2|_{X_{t_0}} \to \mathcal{Q} \to 0$ and $E' = \ker(E \to {}_{X_{t_0}}\mathcal{Q})$, then

$$0 \to E' \to E \to {}_{X_{t_0}}\mathcal{Q} \to 0$$

which, for any $y \in B$, induces exact sequence

$$0 \to E'|_{C\times\{y\}} \to E|_{C\times\{y\}} \to {}_{(t_0,y)}\mathcal{Q} \to 0. \tag{4.7}$$

Thus all $E'_y := E'|_{C\times\{y\}}$ are semi-stable of degree 0. If $\phi : B \to M$ passes through a generic point, then there is a $y_0 \in B$ such that E_{y_0} is $(1,1)$-stable on $X_{y_0} = C \times \{y_0\}$, thus E'_{y_0} is stable by (4.7) and Lemma 3.2. This implies that $\Delta(E') > 0$. Otherwise $\{E'_y\}_{y\in B}$ are s-equivalent by applying Theorem 2.4 to $\pi : X \to B$, which implies $E' = f^*V \otimes \pi^*L$ for a stable bundle V on C and a line bundle L on B. Then $E_t = E'_t = L \oplus L$ for any $t \neq t_0$, which is a contradiction since E is not semi-stable on the generic fiber of $f : X \to C$.

To compute $\Delta(E')$, consider the Harder-Narasimhan filtration

$$0 \to E'_1 \to E' \to F'_2 \to 0$$

over C, let $\mu'_1 = \deg(E'_1|_{X_t})$, $\mu'_2 = \deg(F'_2|_{X_t})$ for $t \in C$, then

$$\Delta(E') = 4c_2(F'_2) + 4(\mu(E') - \mu(E'_1))(\mu'_1 - \mu'_2) \geq 8.$$

To see it, we can assume $c_2(F'_2) = 0$, then there are line bundles V'_i on C and line bundles $\mathcal{O}(\mu'_i)$ on B of degree μ'_i such that

$$0 \to f^*V'_1 \otimes \pi^*\mathcal{O}(\mu'_1 - \mu'_2) \to E' \otimes \pi^*\mathcal{O}(-\mu'_2) \to f^*V'_2 \to 0$$

which defines a morphism $\psi : B \to \mathbb{P}$ to a projective space such that $\mathcal{O}(\mu'_1 - \mu'_2) = \psi^*\mathcal{O}_{\mathbb{P}}(1)$. Thus $\mu'_1 - \mu'_2 \geq 2$ and $\Delta(E') \geq 8$. Then

$$\Delta(E) = \Delta(E') + 4(\mu(E|_{X_{t_0}}) - \mu(\mathcal{Q})) \geq \Delta(E') + 6 \geq 14.$$

□

Now we consider the case that E is semi-stable on the generic fiber of $f : X \to C$. We can assume $0 \leq \deg(E|_{X_t}) \leq 1$ on $X_t = f^{-1}(t)$.

Proposition 4.3. *When E is semi-stable of degree 1 on the generic fiber of $f : X \to C$, we have $\Delta(E) \geq 10$. If $g > 4$ and $\phi : B \to M$ passes through the generic point, then $\Delta(E) \geq 14$.*

Proof. It is easy to see that there is a unique stable rank 2 vector bundle with a fixed determinant of degree 1 on an elliptic curve. Thus $\Delta(E) > 0$ if and only if there exists $t_1 \in C$ such that $E_{t_1} = E|_{X_{t_1}}$ is not semi-stable.

Let $E_{t_1} \to \mathcal{O}(\mu_1) \to 0$ be the quotient of minimal degree and

$$0 \to E^{(1)} \to E \to {}_{X_{t_1}}\mathcal{O}(\mu_1) \to 0$$

be the elementary transformation of E along $\mathcal{O}(\mu_1)$ at X_{t_1}. If $E^{(i)}$ is defined and $\Delta(E^{(i)}) > 0$, let $t_{i+1} \in C$ such that $E^{(i)}_{t_{i+1}} = E^{(i)}|_{X_{t_{i+1}}}$ is not semi-stable and $E^{(i)}_{t_{i+1}} \to \mathcal{O}(\mu_{i+1}) \to 0$ be the quotient of minimal degree, then we define $E^{(i+1)}$ to be the elementary transformation of $E^{(i)}$ along $\mathcal{O}(\mu_{i+1})$ at $X_{t_{i+1}}$, namely $E^{(i+1)}$ satisfies the exact sequence

$$0 \to E^{(i+1)} \to E^{(i)} \to {}_{X_{t_{i+1}}}\mathcal{O}(\mu_{i+1}) \to 0. \tag{4.8}$$

Let s be the minimal integer such that $\Delta(E^{(s)}) = 0$. Then

$$\Delta(E) = 2 \cdot s - 4 \sum_{i=1}^{s} \mu_i \tag{4.9}$$

where $\mu_i \leq 0$ $(i = 1\,, 2\,, ...\,, s)$. Take direct image of (4.8), we have

$$0 \to f_*E^{(s)} \to f_*E^{(s-1)} \to {}_{t_s}\mathrm{H}^0(\mathcal{O}(\mu_s)) \to 0 \tag{4.10}$$

(since $R^1 f_*E^{(s)} = 0$) and $\deg(f_*E^{(i+1)}) \leq \deg(f_*E^{(i)})$, which imply

$$\deg(f_*E^{(s)}) \leq \deg(f_*E) - \dim \mathrm{H}^0(\mathcal{O}(\mu_s)). \tag{4.11}$$

Restrict (4.8) to a fiber $X_y = \pi^{-1}(y)$, we have exact sequence

$$0 \to E^{(i+1)}_y \to E^{(i)}_y \to {}_{(t_{i+1},y)}\mathbb{C} \to 0,$$

which implies that

$$\deg(E^{(s)}_y) = \deg(E_y) - s = 1 - s. \tag{4.12}$$

On the other hand, by Theorem 2.4, $\Delta(E^{(s)}) = 0$ implies that there exist a stable rank 2 vector bundle V of degree 1 on B and a line bundle L on C such that $E^{(s)} = \pi^*V \otimes f^*L$. It is easy to see

$$\deg(E^{(s)}_y) = 2\deg(L) = 2\deg(f_*E^{(s)}).$$

Thus, combine (4.11) and (4.12), we have the inequality

$$s \geq 1 - 2\deg(f_*E) + 2\dim \mathrm{H}^0(\mathcal{O}(\mu_s)). \tag{4.13}$$

We claim that $\deg(f_*E) \leq -1$. To show it, consider

$$0 \to \mathcal{F}' := f^*(f_*E) \to E \to \mathcal{F} \to 0 \tag{4.14}$$

where $\mathcal{F}$ is locally free on $f^{-1}(C \setminus T)$ and $T \subset C$ is a finite set such that E_t $(t \in T)$ is not semi-stable. Thus, for any $y \in B$, the sequence

$$0 \to \mathcal{F}'_y \to E_y \to \mathcal{F}_y \to 0 \tag{4.15}$$

is still exact, which implies that $\mathcal{F}$ is B-flat (cf. Lemma 2.1.4 of [5]). The sequence (4.15) already implies $\deg(f_*E) = \deg(\mathcal{F}'_y) \leq 0$ since E_y is stable of degree 1. Thus $\mathcal{F}$ can not be locally free since

$$4 \cdot c_2(\mathcal{F}) = \Delta(E) - 4 \cdot \deg(f_*E) + 2 > 0.$$

Then there is at least a $y_0 \in B$ such that $\mathcal{F}_{y_0}$ has torsion, otherwise $\mathcal{F}$ is locally free (cf. Lemma 1.27 of [8]). The stability of E_{y_0} implies that $\mathcal{F}_{y_0}/torsion$ has degree at least 1. Thus $\deg(\mathcal{F}_{y_0}) \geq 2$ and

$$\deg(f_*E) = \deg(\mathcal{F}'_{y_0}) \leq -1,$$

which means $s \geq 3 + 2\dim \mathrm{H}^0(\mathcal{O}(\mu_s))$. Therefore, if $\mu_s < 0$, we have $\Delta(E) \geq 2 \cdot s + 4 \geq 10$ by (4.9). If $\mu_s = 0$, by tensoring E with $\pi^*\mathcal{O}(\mu_s)^{-1}$, we may assume $\dim \mathrm{H}^0(\mathcal{O}(\mu_s)) = 1$, then $s \geq 5$ and

$$\Delta(E) \geq 10.$$

If $\phi : B \to M$ passes through the generic point of M, we claim that $\deg(f_*E) \leq -2$, which implies $\Delta(E) \geq 14$. To prove the claim, assume $\deg(f_*E) = -1$, we will show that $\phi(B)$ lies in a given divisor. Note that $\mathcal{F}_y$ must be locally free of degree 2 for generic $y \in B$ (if $\mathcal{F}_y$ has nontrivial torsion, then E_y has a quotient line bundle of degree at most 1, which is impossible since E_y is $(1,1)$-stable for generic $y \in B$). Thus E_y satisfies $0 \to \xi \to E_y \to \xi^{-1} \otimes \mathcal{L} \to 0$ where ξ is a line bundle of degree -1 on C. The locus of such bundles has dimension at most $g + h^1(\xi^2 \otimes \mathcal{L}^{-1}) - 1 = 2g + 1 < \dim(M)$ when $g > 4$. We are done. $\square$

Now we consider the case that E is semi-stable of degree 0 on the generic fiber of $f : X \to C$. If E is semi-stable on every fiber of $f : X \to C$, then E induces a non-trivial morphism

$$\varphi_E : C \to \mathbb{P}^1$$

(cf. [3]) such that $\varphi_E^*\mathcal{O}_{\mathbb{P}^1}(1) = \Theta(E) = (\det f_!E)^{-1}$, which has degree $c_2(E)$ by Grothendieck-Riemann-Roch theorem. Thus

$$\Delta(E) = 4 \cdot c_2(E) = 4 \cdot \deg(\varphi_E) \geq 8. \tag{4.16}$$

If there is a $t_0 \in C$ such that $E_{t_0} = E|_{X_{t_0}}$ is not semi-stable on $X_{t_0} = f^{-1}(t_0)$, let $E_{t_0} \to \mathcal{O}(\mu) \to 0$ be the quotient line bundle of minimal degree μ and $E' = \mathrm{kernel}\,(E \to {}_{X_{t_0}}\mathcal{O}(\mu) \to 0\,)$, then we have

Lemma 4.4. *If $\Delta(E') = 0$, then there is a semi-stable vector bundle V on C and a line bundle L of degree 0 on B such that*

$$E' = f^*V \otimes \pi^*L.$$

Proof. By the definition, $\{E'_t = E'|_{\{t\}\times B}\}_{t\in C}$ and $\{E'_y = E'|_{C\times\{y\}}\}_{y\in B}$ are families of semi-stable bundles of degree 0. Apply Theorem 2.4 to $f : X \to C$ (resp. $\pi : X \to B$), then $\Delta(E') = 0$ implies that $\{E'_t\}_{t\in C}$ (resp. $\{E'_y\}_{y\in B}$) are isomorphic each other. By tensor E (thus E') with π^*L^{-1} (where L is a line bundle of degree 0 on B), we can assume that $\mathrm{H}^0(E'_t) \neq 0$ ($\forall\ t \in C$), which have dimension at most 2 since E'_t is semi-stable of degree 0. If $\mathrm{H}^0(E'_t)$ has dimension 2, then $E' = f^*(f_*E')$ and we are done.

If $\mathrm{H}^0(E'_t)$ has dimension 1, we will show a contradiction. In fact, by the definition of E', we have an exact sequence

$$0 \to E' \to E \to {}_{X_{t_0}}\mathcal{O}(\mu) \to 0 \tag{4.17}$$

where $\mathcal{O}(\mu)$ is a line bundle on $\{t_0\} \times B \cong B$ of degree $\mu < 0$. Then

$$V_1 := f_*E = f_*E'$$

is a line bundle on C. Since $\{E'_t\}_{t\in C}$ are isomorphic each other and $\mathrm{H}^0(E'_t)$ has dimension 1, we have the exact sequence

$$0 \to f^*V_1 \to E' \to f^*V_2 \otimes \pi^*L_0 \to 0 \tag{4.18}$$

for a line bundle V_2 on C and a degree 0 line bundle L_0 on B. If $L_0 \neq \mathcal{O}_B$, then $R^if_*(f^*(V_2^{-1} \otimes V_1) \otimes L_0) = V_2^{-1} \otimes V_1 \otimes \mathrm{H}^i(L_0) = 0$ ($i = 0, 1$), which implies $\mathrm{H}^1(X, f^*(V_2^{-1} \otimes V_1) \otimes L_0) = 0$ and (4.18) is splitting. This is impossible since E'_y is semi-stable of degree 0 and we can show that $\deg(V_1) = \deg(f_*E) \leq -1$ in the following.

To prove that $\deg(f_*E) \leq -1$, we consider the exact sequence

$$0 \to f^*f_*E \to E \to \mathcal{F} \to 0 \tag{4.19}$$

where $\mathcal{F}|_{f^{-1}(C\setminus\{t_0\})}$ is locally free of rank 1 by (4.18). But $\mathcal{F}$ is not locally free (otherwise $c_2(E) = (c_1(E) - c_1(f^*f_*E)) \cdot c_1(f^*f_*E) = 0$) and for any $y \in B$ the restriction of (4.19) to $X_y = \pi^{-1}(y)$

$$0 \to f_*E \to E_y \to \mathcal{F}_y \to 0 \tag{4.20}$$

is exact, which means that $\mathcal{F}$ is B-flat (cf. Lemma 2.1.4 of [5]). Thus, by Lemma 1.27 of [8], there is a $y_0 \in B$ such that $\mathcal{F}_{y_0}$ has torsion $\tau \neq 0$ since $\mathcal{F}$ is not locally free. Then, since E_{y_0} is stable of degree 1,

$$\deg(\mathcal{F}_{y_0}) \geq 1 + \deg(\mathcal{F}_{y_0}/\tau) > 1 + \mu(E_{y_0}) = \frac{3}{2}$$

which implies $\deg(f_*E) \leq -1$ by (4.20).

We have shown that L_0 has to be $\mathcal{O}_B$ and (4.18) has to be

$$0 \to f^*V_1 \to E' \to f^*V_2 \to 0 \tag{4.21}$$

which is determined by a class of $\mathrm{H}^1(X, f^*(V_1 \otimes V_2^{-1}))$. However, note $R^1f_*(f^*(V_1 \otimes V_2^{-1})) = V_1 \otimes V_2^{-1} \otimes \mathrm{H}^1(\mathcal{O}_B) = V_1 \otimes V_2^{-1}$ and

$$\mathrm{H}^0(C, V_1 \otimes V_2^{-1}) = 0,$$

by using Leray spectral sequence, we have

$$\mathrm{H}^1(C, V_1 \otimes V_2^{-1}) \cong \mathrm{H}^1(X, f^*(V_1 \otimes V_2^{-1})).$$

Hence there exists an extension $0 \to V_1 \to V \to V_2 \to 0$ on C such that $E' \cong f^*V$, which contradicts the assumption

$$\dim(\mathrm{H}^0(\{t\} \times B, E'_t)) = 1.$$

□

Proposition 4.5. *When E is semi-stable of degree 0 on the generic fiber of $f : X \to C$, we have $\Delta(E) \geq 8$. If C is not hyper-elliptic and $\phi : B \to M$ passes through a $(1,1)$-stable bundle, assume that E defines an essential elliptic curve, then $\Delta(E) \geq 12$.*

Proof. If E is semi-stable on each fiber $X_t = f^{-1}(t)$, then E induces a non-trivial morphism $\varphi_E : C \to \mathbb{P}^1$. By (4.16), $\Delta(E) \geq 8$.

If there is a $t_0 \in C$ such that E_{t_0} is not semi-stable, then we have

$$0 \to E' \to E \to {}_{X_{t_0}}\mathcal{O}(\mu) \to 0$$

where $\mathcal{O}(\mu)$ is a line bundle of degree μ on B. If $\Delta(E') \neq 0$, then $\Delta(E') > 0$ by Theorem 2.4. On the other hand, $c_1(E')^2 = 0$ since E' has degree 0 on the generic fiber of $X \to C$ and $\mathrm{Pic}(C \times B) = \mathrm{Pic}(C) \times \mathrm{Pic}(B)$. Thus $\Delta(E') = 4 \cdot c_2(E') \geq 4$, and by Lemma 2.6

$$\Delta(E) = \Delta(E') - 4\mu \geq 8.$$

If $\Delta(E') = 0$, by Lemma 4.4, we can assume that $E' = f^*V$, then the sequence (4.17) induces a nontrivial morphism $\varphi : B \to \mathbb{P}(V_{t_0}^\vee)$ such that $\mathcal{O}(-\mu) = \varphi^*\mathcal{O}_{\mathbb{P}(V_{t_0}^\vee)}(1)$. Thus $\Delta(E) = -4\mu \geq 8$.

Now we assume that C is not hyper-elliptic and $\phi : B \to M$ passes through a $(1,1)$-stable bundle. If E is semi-stable on each fiber X_t, then $\Delta(E) = 4 \cdot \deg(\varphi_E) \geq 12$ by (4.16) since C is not hyper-elliptic.

If there is $t_0 \in C$ such that E_{t_0} is not semi-stable, we claim $\Delta(E') > 0$ since $\phi : B \to M$ passes through a $(1,1)$-stable bundle. Otherwise, $E' = f^*V$ where V is a $(1,0)$-stable by Lemma 3.2, then sequence (4.17) implies that $\phi : B \to M$ factors through a Hecke curve, which implies that $\phi : B \to M$ is not an essential elliptic curve. If E' is semi-stable on each fiber X_t, then E' defines a nontrivial morphism $\varphi_{E'} : C \to \mathbb{P}^1$ such that $\varphi^*\mathcal{O}_{\mathbb{P}^1}(1) = \Theta(E') = (\det f_!E')^{-1} = c_2(E')$. Thus $\Delta(E') = 4 \cdot \deg(\varphi_{E'}) \geq 12$ and $\Delta(E) = \Delta(E') - 4\mu \geq 16$.

If there is $t'_0 \in C$ such that E'_{t_0} is not semi-stable, then we have

$$0 \to \mathcal{F} \to E' \to {}_{X_{t'_0}}\mathcal{O}(\mu') \to 0 \tag{4.22}$$

where $\mathcal{F}_y = \mathcal{F}|_{C \times \{y\}}$ is stable of degre -1 for generic $y \in B$ since E'_y is stable of degree 0. If $\Delta(\mathcal{F}) \neq 0$, it is clear that $\Delta(\mathcal{F}) = 4 \cdot c_2(\mathcal{F}) \geq 4$ and $\Delta(E) = \Delta(\mathcal{F}) - 4\mu' - 4\mu \geq 12$. If $\Delta(\mathcal{F}) = 0$, by Theorem 2.4, there is a stable vector bundle V' on C such that $\mathcal{F}_y \cong V'$ for all $y \in B$.

Then we can choose $\mathcal{F} = f^*V'$, the sequence (4.22) induces a nontrivial morphism $\varphi : B \to \mathbb{P}(V'^{\vee}_{t'_0})$ such that $\mathcal{O}(-\mu') = \varphi^*\mathcal{O}_{\mathbb{P}(V'^{\vee}_{t'_0})}(1)$. Thus $\Delta(E') = -4\mu' \geq 8$ and $\Delta(E) = \Delta(E') - 4\mu \geq 12$.

□

We have seen in Example 3.6 the existence of essential elliptic curves of degree $6(r, d)$ (which is 6 in our case). Then we have shown

Theorem 4.6. *Let $M = \mathcal{SU}_C(2, \mathcal{L})$ be the moduli space of rank two stable bundles on C with a fixed determinant of degree 1. Then, when C is generic, any essential elliptic curve $\phi : B \to M$ has degree*

$$\deg \phi^*(-K_M) \geq 6$$

and $\deg\phi^*(-K_M) = 6$ if and only if $\phi : B \to M$ factors through

$$\phi : B \xrightarrow{\psi} q^{-1}(\xi) = \mathbb{P}(\mathrm{H}^1(V_2^{\vee} \otimes V_1)) \xrightarrow{\Phi_\xi} M$$

for some $\xi = (V_1, V_2)$ such that $\psi^*\mathcal{O}_{\mathbb{P}(\mathrm{H}^1(V_2^{\vee}\otimes V_1))}(1)$ has degree 3.

Proof. By Proposition 4.2, Proposition 4.3 and Proposition 4.5, we have $\Delta(E) \geq 6$. The possible case $\Delta(E) = 6$ occurs only in Proposition 4.2 when $c_2(F_2) = 0$. This implies that E must satisfy

$$0 \to f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1 - \mu_2) \to E \to f^*V_2 \to 0$$

which defines $\psi : B \to \mathbb{P}(\mathrm{H}^1(V_2^{\vee} \otimes V_1))$ such that $\psi^*\mathcal{O}_{\mathbb{P}(\mathrm{H}^1(V_2^{\vee}\otimes V_1))}(1)$ has degree $\mu_1 - \mu_2$. Then $\Delta(E) = 6$ and (4.1) imply $\mu_1 - \mu_2 = 3$. □

Theorem 4.7. *When $g > 4$ and C is generic, any essential elliptic curve $\phi : B \to M = \mathcal{SU}_C(2, \mathcal{L})$ that passes through the generic point must have $\deg\phi^*(-K_M) \geq 12$.*

For $r > 2$, let $M = \mathcal{SU}_C(r, \mathcal{L})$ where $\mathcal{L}$ is a line bundle of degree d. What is the minimal degree of essential elliptic curves on M ? I expect the following conjecture to be true.

Conjecture 4.8. *Let $\phi : B \to M = \mathcal{SU}_C(r, \mathcal{L})^s$ be an essential elliptic curve defined by a vector bundle E on $C \times M$. Then, when C is a generic curve, we have*

$$\deg\phi^*(-K_M) \geq 6(r, d).$$

When $(r, d) \neq r$, then $\deg\phi^*(-K_M) = 6(r, d)$ if and only if it is an elliptic curve of split type in Example 3.6. If $\phi : B \to M$ passes through the generic point and $g > 4$, then $\deg\phi^*(-K_M) \geq 6r$.

References

[1] G. Faltings: Stable G-bundles and projective connections, J. Algebraic Geom. 2 (1993), 507-568

[2] R. Friedman: Algebraic surfaces and holomorphic vector bundles, Universitext, Springer Verlag (1998), 291-345

[3] R. Friedman, J.W. Morgan and E. Witten: Vector bundles over elliptic fibrations, J. Algebraic Geom. 8 (1999), no. 2, 279-401

[4] G. Hein: Generalized Albanese morphisms, Compositio Math. 142 (2006), 719-733

[5] D. Huybrechts and M. Lehn: The geometry of moduli spaces of sheaves, Aspects of Mathematics, Vieweg, Braunschweig, (1997).

[6] N. Mok and X. Sun: Remarks on lines and Minimal rational curves, Science in China, Series A: Mathematics **52** (2005), no. 4, 617-630.

[7] M. S. Narasimhan and S. Ramanan: Geometry of Hecke cycles I, C. P. Ramanujam-a tribute, Springer Verlag (1978), 291-345

[8] C. Simpson: Moduli of representations of the fundamental group of a smooth projective variety I , I.H.E.S. Publications Mathématiques **79** (1994), 47–129.

[9] X. Sun: Minimal rational curves on moduli spaces of stable bundles , Math. Ann. **331** (2005), 925-973.

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