A Functional Approach to FBSDEs and Its Application in Optimal Portfolios

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Abstract

In Liang et al (2009), the current authors demonstrated that BSDEs can be reformulated as functional differential equations, and as an application, they solved BSDEs on general filtered probability spaces. In this paper the authors continue the study of functional differential equations and demonstrate how such approach can be used to solve FBSDEs. By this approach the equations can be solved in one direction altogether rather than in a forward and backward way. The solutions of FBSDEs are then employed to construct the weak solutions to a class of BSDE systems (not necessarily scalar) with quadratic growth, by a nonlinear version of Girsanov's transformation. As the solving procedure is constructive, the authors not only obtain the existence and uniqueness theorem, but also really work out the solutions to such class of BSDE systems with quadratic growth. Finally an optimal portfolio problem in incomplete markets is solved based on the functional differential equation approach and the nonlinear Girsanov's transformation.

Keywords: FBSDE, Quadratic BSDE, weak solution, Girsanov's theorem, functional differential equation, optimal portfolio

Mathematical subject classifications (2000): 60H30, 65C30, 91B28

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1 Introduction

Backward stochastic differential equations (BSDE) provide a new perspective to look at the infinitesimal behavior of Markov processes, and they have been found intrinsically linked to a class of nonlinear partial differential equations (PDEs). In a fundamental paper by Pardoux sand Peng [34], they solved a class of nonlinear BSDEs with Lipschitz drivers for the first time. An intrinsic connection to nonlinear PDEs, which is now well known as nonlinear Feynman-Kac formula, was later established by Peng [37] and Pardoux and Peng [35]. Its applications in finance were discovered by Duffie and Epstein [13] and El Karoui et al [16]. For a more comprehensive review of the BSDE theory, we refer to [14], [15] and [41] and reference therein.

In Liang et al [26], the current authors demonstrate that BSDEs can be reformulated as functional differential equations. As an application we can solve BSDEs on general filtered probability spaces, and in particular without the requirement of martingale representation. In this paper we try to apply such *functional differential equation approach*, or *functional approach* for short, to forward backward stochastic differential equations (FBSDEs), and demonstrate how it can be used to solve a financial optimal portfolio problem in incomplete markets.

Let us recall such idea, which is the first main ingredient of the paper. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a filtered probability space satisfying the usual conditions. Given a semimartingale $(Y_t)_{t\in[0,T]}$ with the following decomposition:

$$Y_t = M_t - V_t, \quad \text{for } t \in [0,T],$$

where M is local martingale and V is a finite variation process, if we further know the terminal data of Y, say $Y_T = \xi$ for some $\mathcal{F}_T$-measurable random variable ξ, then we also have $\xi = M_T - V_T$. Therefore

$$\begin{cases} Y_t = E[\xi + V_T|\mathcal{F}_t] - V_t, \\ M_t = E[\xi + V_T|\mathcal{F}_t], \qquad \text{for } t \in [0,T]. \end{cases} \tag{1.1}$$

Before we apply the above relationship (1.1) to a specific FBSDE setting, let us first look at it from potential theory point of view. For a given domain $D \subset R^d$, a real-valued function f is called superharmonic in D if

$$\int_{\partial B(0,r)} f(x+y)\sigma_r(dy) \leq f(x), \quad \text{for } x \in D \text{ and } r < dist(x, \partial D),$$

where σ_r is the measure on the surface of the ball $\partial B(0,r)$, normalized to have the total mass 1. f is called harmonic if furthermore the equality holds. If f is superharmonic in D, then there exists a unique positive Borel measure on D such that the following Riesz decomposition holds:

$$f(x) = h(x) + G_D\mu(x), \quad \text{for } x \in D,$$

where h is harmonic in D and $G_D\mu$ is the Green's potential of μ on D, i.e. $G_D\mu(x) = \int_D G(x,y)\mu(dy)$ with $G(x,y)$ as the Green's function of the Laplace equation in R^d. Given the boundary data of such superharmonic function f, by the above Riesz decomposition, the Green's potential $G_D\mu$ is often used to study f.

The probabilistic counterpart of the above Riesz decomposition is the Doob-Meyer decomposition. For a supermartingale $(Y_t)_{t\in[0,T]}$ with Càdlàg sample paths, the Doob-Meyer decomposition says that there exists a unique increasing predictably measurable process V starting from $V_0 = 0$ such that M defined by $M_t = Y_t + V_t$ for $t \in [0,T]$ is a martingale. The above relationship (1.1) we just established tells us exactly the same thing as in the potential theory: given the terminal data of a supermartingale (which is a semimartingale) defined on $[0,T]$, we can study such supermartingale by investigating the increasing predictably measurable process V.

Now we apply the above relationship (1.1) to a specific FBSDE, which will be served as an auxiliary equation later. For any given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{Q})$ satisfying the usual conditions, with a d-dimensional Brownian motion $B = (B^1, \cdots, B^d)^*$, (The superscript * denotes the matrix transposition)

$$\begin{cases} dX_t = f(t, Y_t, Z_t)dt + dB_t, \\ X_0 = x \in R^d, \\ dY_t = -h(t, Y_t, Z_t)dt + Z_t dB_t, \\ Y_T = \phi(X_T). \end{cases} \tag{1.2}$$

with the coefficients $f : [0,T] \times R^n \times R^{n\times d} \to R^d$, $h : [0,T] \times R^n \times R^{n\times d} \to R^n$ and $\phi : R^d \to R^n$ satisfying Condition 1 to be introduced later.

To solve FBSDE (1.2), for $\tau \in [0,T]$, we consider the following functional differential equation on $[\tau, T]$:

$$V_t = \int_\tau^t h(s, Y(V,X)_s, Z(V,X)_s)ds \tag{1.3}$$

together with the forward process X:

$$X_t = x + \int_\tau^t f(s, Y(V,X)_s, Z(V,X)_s)ds + \int_\tau^t dB_s, \tag{1.4}$$

where

$$\begin{cases} Y(V,X)_t = E[\phi(X_T) + V_T|\mathcal{F}_t] - V_t, \\ \displaystyle\int_\tau^T Z(V,X)_s dB_s = \phi(X_T) + V_T - E[\phi(X_T) + V_T|\mathcal{F}_\tau]. \end{cases} \tag{1.5}$$

If we can solve (V,X) for functional differential equation (1.3) (1.4) with $\tau = 0$, then by Lemma 8 in Section 2.2, $(Y(V,X), Z(V,X), X)$ will provide us with the solution to (1.2). For the notation's simplicity, let us denote $\Psi = (V,X)^*$ and $F = (h,f)^*$. Then (1.3) (1.4) are simplified to:

$$\Psi_t = \chi_x + \int_\tau^t F(s, Y(\Psi)_s, Z(\Psi)_s)ds + \int_\tau^t \chi_1 dB_s \tag{1.6}$$

where $\chi_x \in R^{n+d}$ with the first n components being 0 and next d components being $x \in R^d$. Note that by functional differential equation (1.6), we can solve all the components (Y,Z,X) in one direction altogether. We consider the solutions to (1.6) in the following space:

- $\mathcal{C}([0,T];R^{n+d})$: the space of continuous and $\mathcal{F}_t$-adapted processes $(\Psi_t)_{t\in[0,T]}$ valued in R^{n+d} such that $\sup_{t\in[0,T]}|\Psi_t| \in \mathcal{L}^2(\Omega,\mathcal{F}_T,\mathbf{Q})$ and endowed with the following norm:

$$||\Psi||_{\mathcal{C}[0,T]} = \sqrt{E \sup_{t\in[0,T]} |\Psi_t|^2}.$$

In Section 2.2 we will mainly solve functional differential equation (1.6) and prove the following theorem:

Theorem 1 *If the coefficients satisfy Condition 1, then there exists a unique solution $\Psi \in \mathcal{C}([0,T];R^{n+d})$ to functional differential equation (1.6).*

The second main ingredient of this paper is a nonlinear version of Girsanov's transformation, which is employed to connect the above FBSDE (1.2) and a class of BSDE systems with quadratic growth. Namely we will use strong solutions of FBSDEs to construct weak solutions to a class of BSDE systems with quadratic growth.

In the PDE theory, if the nonlinear terms in equations have at most quadratic growth with respect to the gradient of solutions, the nature of the equations completely change. In the BSDE theory, there is a class of BSDEs with quadratic growth corresponding to such PDEs, and they are usually called quadratic BSDEs. The study of quadratic BSDEs was initialized by Kobylanski [23] using the idea of the Cole-Hopf transformation adapted from the PDE theory. Her result was substantially developed and generalized by Briand and Hu [5] and [6], where they extended to the equations with the unbounded terminal data and with the convex driver. On the other hand, Quadratic BSDEs have found a lot of applications in finance. For example, they appear naturally when one wants to derive the value function for the maximization of expected utility, use indifference pricing idea to hedge a contingent claim written on nontradeable underlying assets, or consider the risk measure. See for example [4] [17] [21] [32] [33] and [39].

In this paper we mainly consider the following quadratic BSDE system (not necessarily scalar):

$$\begin{cases} dY_t = -h(t,Y_t,Z_t)dt - Z_t f(t,Y_t,Z_t)dt + Z_t dW_t, \\ Y_T = \phi(W_T) \end{cases} \tag{1.7}$$

where $W = (W^1,\cdots,W^d)^*$ is a d-dimensional Brownian mtion starting from $x \in R^d$. The coefficients h, f and ϕ are supposed to satisfy the following condition:

Condition 1 *All the coefficients $h : [0,T]\times R^n \times R^{n\times d} \to R^n$, $f : [0,T]\times R^n \times R^{n\times d} \to R^d$ and $\phi : R^d \to R^n$ are continuous. Moreover h, f and ϕ are Lipschitz continuous, i.e.*

$$\begin{cases} |h(t,y,z) - h(t,\bar{y},\bar{z})| \le C_1(|y-\bar{y}| + |z-\bar{z}|), \\ |f(t,y,z) - f(t,\bar{y},\bar{z})| \le C_1(|y-\bar{y}| + |z-\bar{z}|), \\ |\phi(x) - \phi(\bar{x})| \le C_2|x-\bar{x}|, \end{cases}$$

and ϕ is uniformly bounded,

$$\sup_{x\in R^d} |\phi(x)| \le M,$$

for $t \in [0,T]$, $y, \bar{y} \in R^n$, $z, \bar{z} \in R^{n\times d}$ and $x, \bar{x} \in R^d$.

Because of the terms with the coefficient f, the equations have at most quadratic growth, i.e. there exists a constant C_3 such that for any $y \in R^n$ and $z \in R^{n\times d}$,

$$|zf(t,y,z)| \le C_3|z|(t + |y| + |z|).$$

The quadratic growth term in (1.7) is more special than the usual one considered in the literature. However this special structure is enough to cover the most examples of quadratic BSDEs known in finance, at least with some extra conditions added. We will consider one specific example from optimal portfolio problems in Section 4. Moreover, all the excising results of quadratic BSDEs are only for the case $n = 1$. The current paper seems to be the first attempt to consider the quadratic BSDE systems.

On the other hand, one may wonder why the terminal data has the special form $\phi(W_T)$. This is only for the presentation's simplicity. The whole paper's results can be extended without difficulty to the case $\phi(X_T)$ where X is driven by stochastic differential equations (SDEs):

$$dX_t^i = X_t^i(b_t^i dt + \sigma_t^i dW_t), \quad \text{for } i = 1, \cdots, m.$$

with the coefficients b^i and σ^i satisfying certain regularity conditions.

To solve (1.7), we will pursue another direction different from the existing method for quadratic BSDEs. Namely we don't use the Cole-Hopf transformation at all and don't assume the underlying probability space and Brownian motion as any given; instead we consider weak solutions of quadratic BSDEs.

Before presenting the definition of weak solutions to (1.7), let us mention some already existing work about weak solutions. One of the first attempts to introduce the weak solutions for BSDEs was Buckdahn et al [7], and Buckdahn and Engelbert [8] further proved the uniqueness of their weak solutions. However the driver of their BSDE does not evolve the martingale representation part Z. On the other hand, the notion of weak solutions for FBSDEs was introduced by Antonelli and Ma [2] and further developed by Delarue and Guatteri [12], and by Ma et al [30] and Ma and Zhang [31] who employed the martingale problem approach.

Definition 2 *A weak solution to BSDE (1.7) is a triple $(\Omega, \mathcal{F}, \mathbf{P}^x)$, $\{\mathcal{F}_t\}$ and $(Y, Z^{\mathbf{P}^x}, W)$ such that*

(1) $(\Omega, \mathcal{F}, \mathbf{P}^x)$ is a complete probability space with the filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions;

(2) under such filtered probability space, $Y, Z^{\mathbf{P}^x}$ and W are $\mathcal{F}_t$-adapted, and Y is a special semimartingale, $Z^{\mathbf{P}^x}$ is the density representation of Y under $\mathbf{P}^x$, and W is a Brownian motion starting from $\mathbf{P}^x(W_0 = x) = 1$;

(3) The increments $\{W_u - W_t : t \le u \le T\}$ must be independent of the σ-algebra $\mathcal{F}_t$;

(4) the following integral equation satisfies:

$$Y_t = \phi(W_T) + \int_t^T h(s, Y_s, Z_s^{\mathbf{P}^x})ds + \int_t^T Z_s^{\mathbf{P}^x} f(s, Y_s, Z_s^{\mathbf{P}^x})ds - \int_t^T Z_s^{\mathbf{P}^x} dW_s. \tag{1.8}$$

Remark 3 *In this paper, the density representation* $Z^{\mathbf{P}^x}$ *means it is the density representation for the martingale part of the special semimartingale* Y*, and we use the superscript* $\mathbf{P}^x$ *to emphasize the dependency of the density representation on the probability measure* $\mathbf{P}^x$*.*

Remark 4 *Our definition of weak solutions is more related to Buckdahn et al [7]. The filtration* $\{\mathcal{F}_t\}$ *plays an important role here. If* $\mathcal{F}_t = \mathcal{F}_t^W$*, i.e. the filtration is generated by the Brownian motion* W *augmented by the* $\mathbf{P}^x$*-null sets in* $\mathcal{F}$*, the solution turns to be a strong solution. In Ma and Zhang [31], such solution is also called a semi-strong solution. Actually the smallest filtration for weak solutions is the filtration* $\{\mathcal{F}_t^{W,Y,Z}\}$ *generated by* W, Y, Z *and satisfying the usual conditions.*

Remark 5 *Condition (3) automatically holds given the Brownian motion* W *with the filtration* $\{\mathcal{F}_t\}$*. In fact such condition simply means* $\{\mathcal{F}_t\}$ *consists, additionally to* $\{\mathcal{F}_t^W\}$*, only of independent experiments. In Buckdahn et al [7], such condition is formulated in terms of martingales, i.e. any* $\mathcal{F}_t^W$*-martingale must be an* $\mathcal{F}_t$*-martingale. In Kurtz [24], such kind of condition is called the compatibility constraint. (3) is extremely useful when we want to identify weak solutions are strong solutions.*

For the notation's simplicity, we will suppress the superscript x of $\mathbf{P}^x$ from now on if no confusion may arise. Now we describe our idea formally before presenting the existence and uniqueness theorem of BSDE (1.7). The basic idea is using the strong solution of FBSDE to construct the weak solution to quadratic BSDE. Let us start with a Brownian motion family B on $(\Omega, \mathcal{F}, \mathbf{Q})$ with the filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, and consider FBSDE (1.2).

Suppose FBSDE (1.2) admits a unique solution $(X, Y, Z^{\mathbf{Q}})$. Then we define a new probability measure $\mathbf{P}$ by

$$\frac{d\mathbf{P}}{d\mathbf{Q}} = \mathscr{E}(N)$$

where $\mathscr{E}(N)$ is the Doléans-Dade exponential of N with

$$N = -\int_0^{\cdot} \langle f(s, Y_s, Z_s^{\mathbf{Q}}), dB_s \rangle_d$$

where $\langle \cdot, \cdot \rangle_d$ denotes the inner product in R^d. Under the new probability measure $\mathbf{P}$, by the Girsanov's theorem, B has the following decomposition:

$$\begin{aligned} B &= (B - [B, N]) + [B, N], \\ &= \left(B + \int_0^{\cdot} f(s, Y_s, Z_s^{\mathbf{Q}}) ds \right) - \int_0^{\cdot} f(s, Y_s, Z_s^{\mathbf{Q}}) ds \end{aligned}$$

where $B - [B, N] = B + \int_0^{\cdot} f(s, Y_s, Z_s^{\mathbf{Q}}) ds$ is a martingale under $\mathbf{P}$, and furthermore by the Lévy's characterization, it is in fact a Brownian motion under $\mathbf{P}$. We further define W by $W = x + B - [B, N]$.

Under the probability measure $\mathbf{P}$ and with the Brownian motion W, let's rewrite the backward equation in FBSDE (1.2):

$$dY_t = -h(t, Y_t, Z_t^{\mathbf{Q}}) dt - Z_t^{\mathbf{Q}} f(t, Y_t, Z_t^{\mathbf{Q}}) dt + Z_t^{\mathbf{Q}} dW_t$$

with $Y_T = \phi(W_T)$. If we can prove $Z^{\mathbf{Q}} = Z^{\mathbf{P}}$, then triple $(\Omega, \mathcal{F}, \mathbf{P})$, $\{\mathcal{F}_t\}$ and $(Y, Z^{\mathbf{P}}, W)$ is just one weak solution we want to find.

There are mainly three steps needed to be verified for the above solving procedure. The first step is about the invariant property of the density representation under the change of probability measure, i.e. $Z^{\mathbf{Q}} = Z^{\mathbf{P}}$; The second step is of course the solvability of FBSDE (1.2); the last step is the Doléans-Dade exponential $\mathscr{E}(N)$ must be a uniform-integrable martingale to guarantee $\mathbf{P}$ is a probability measure. As long as the above three steps are verified, we have the following theorem:

Theorem 6 *If the coefficients satisfy Condition 1, then there exists at least one weak solution* $(\Omega, \mathcal{F}, \mathbf{P})$, $\{\mathcal{F}_t\}$ *and* $(Y, Z^{\mathbf{P}}, W)$ *to BSDE (1.7).*

The paper is organized as follows: In section 2 we verify the above three steps and prove Theorem 6, while in Section 3 the uniqueness and the connection between weak solutions and strong solutions are discussed. Finally we apply our method to an optimal portfolio problem in incomplete markets in Section 4.

2 Weak solutions and existence

2.1 Invariant property of density representation

The following lemma is almost trivial but crucial to our results, which states that the density representation of a special semimartingale is invariant under the equivalent change of probability measure. This observation is firstly made in Liang et al [27], and we recall it here for completeness.

Lemma 7 *Let B be a Brownian motion on $(\Omega, \mathcal{F}, \mathbf{Q})$ with the filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Let $Z^{\mathbf{Q}}$ be the density representation of a special semimartingale Y under $\mathbf{Q}$. If define an equivalent probability measure $\mathbf{P}$ by $\frac{d\mathbf{P}}{d\mathbf{Q}} = \mathscr{E}(N)$ for some uniform integrable martingale $\mathscr{E}(N)$, then $Z_t^{\mathbf{P}} = Z_t^{\mathbf{Q}}$ for a.e. $t \in [0, T]$, a.s.*

Proof. Under the probability measure $\mathbf{Q}$, Y has the canonical decomposition $Y = M - V$ with M being a local martingale and V being a finite variation process, and moreover, M admits the martingale representation:

$$M_t - M_0 = \int_0^t Z_s^{\mathbf{Q}} dB_s, \quad \text{for } t \in [0, T], \ a.s. \tag{2.1}$$

for some predictably measurable process $Z^{\mathbf{Q}}$.

By Girsanov's theorem, Y is still a special semimartingale under $\mathbf{P}$ but with the canonical decomposition $Y = \bar{M} - \bar{V}$, where $\bar{M} = M - [M, N]$ is a martingale, and $\bar{V} = \bar{M} - Y$ is a finite variation process. We also have $\bar{B} = B - [B, N]$ as a Brownian motion under $\mathbf{P}$. Hence under $\mathbf{P}$, (2.1) becomes

$$\bar{M}_t - \bar{M}_0 + [M, N]_t = \int_0^t Z_s^{\mathbf{Q}} d\bar{B}_s + \int_0^t Z_s^{\mathbf{Q}} d[B, N]_s, \quad \text{for } t \in [0, T], \ a.s..$$

By identifying the martingale parts and finite variation parts of the above equality, we must have

$$\bar{M}_t - \bar{M}_0 = \int_0^t Z_s^{\mathbf{Q}} d\bar{B}_s, \quad \text{for } t \in [0,T], \ a.s..$$

But on the other hand under $\mathbf{P}$ we also have

$$\bar{M}_t - \bar{M}_0 = \int_0^t Z_s^{\mathbf{P}} d\bar{B}_s, \quad \text{for } t \in [0,T], \ a.s.$$

for some predictably measurable process $Z^{\mathbf{P}}$, so

$$\int_0^T |Z_s^{\mathbf{P}} - Z_s^{\mathbf{Q}}|^2 ds = 0, \quad a.s.,$$

which proves the claim. ■

Since the density representation usually determines the hedging (or replicating) strategy in finance, a direct consequence of Lemma 7 is that hedging strategy is independent of the choices of the equivalent (martingale) probability measures. Due to Lemma 7, we will not emphasize the dependency of the density representation on the probability measure, and simply write it as Z from now on.

2.2 Functional approach to FBSDEs

The study of FBSDEs was initiated by Antonelli [1], and this subject was further developed in [22] [28] [36] [38] [40] and especially the monograph [29] by Ma and Yong. However most of them either solved the equations locally or assumed some regularity on the coefficients (e.g. smoothness and monotonicity). Recently Delarue [10] solved FBSDEs globally with Lipschitz continuous assumptions on the coefficients by combining the method of contraction mapping and the four-step scheme of FBSDEs.

In this subsection we try to reformulate FBSDE (1.2) as functional differential equation (1.6) and solve such functional differential equation instead. The approach may benefit especially numerical solutions of FBSDEs, because a usual obstacle to numerically solve FBSDEs is one needs to solve (Y, Z) backwards and X forwards at the same time. By introducing a functional differential equation, we can solve all the components (Y, Z, X) in one direction altogether.

We first establish the equivalence between FBSDE (1.2) and functional differential equation (1.6). Besides the space $\mathcal{C}([0,T]; R^n)$ we further introduce the following space:

- $H^2([0,T]; R^n)$: the space of predictably measurable processes endowed with the norm:
$$||Z||_{H^2[0,T]} = \sqrt{E\int_0^T |Z_s|^2 ds}.$$

Lemma 8 *FBSDE (1.2) admits a unique solution* $(Y, Z, X) \in \mathcal{C}([0,T]; R^n) \times H^2([0,T]; R^{n\times d}) \times \mathcal{C}([0,T]; R^d)$ *if and only if functional differential equation (1.6) admits a unique solution* $\Psi \in \mathcal{C}([0,T]; R^{n+d})$*, and therefore by Theorem 1, FBSDE (1.2) admits a unique solution.*

Proof. Suppose (Y, Z, X) is the unique solution to FBSDE (1.2). Since Y is a special (continuous) semimartingale, it admits the following canonical decomposition: $Y_t = M_t - V_t$ for $t \in [0, T]$, where M is a continuous local martingale and V is a continuous finite variation process with $V_0 = 0$. Furthermore by the martingale representation $M_t = M_0 + \int_0^t Z_s dB_s$ for $t \in [0, T]$, we have

$$Y_t = M_0 + \int_0^t Z_s dB_s - V_t, \quad \text{for } t \in [0, T], \tag{2.2}$$

from which we obtain the relationship (1.5). The backward equation in FBSDE (1.2) becomes

$$M_0 + \int_0^t Z_s dB_s - V_t = \phi(X_T) + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s.$$

By taking conditional expectation with respect to $\mathcal{F}_t$ on both sides:

$$\begin{aligned} & M_0 + \int_0^t Z_s dB_s - V_t \\ &= E[\phi(X_T)|\mathcal{F}_t] + E\left[\int_t^T f(s, Y_s, Z_s)ds|\mathcal{F}_t\right] \\ &= E[\phi(X_T)|\mathcal{F}_t] + E\left[\int_0^T f(s, Y_s, Z_s)ds|\mathcal{F}_t\right] - \int_0^t f(s, Y_s, Z_s)ds. \end{aligned}$$

By the uniqueness of the canonical decomposition of Y, and by identifying the martingale part and finite variation part of both sides, V must satisfy (1.3).

On the other hand, if Ψ is the unique solution to functional differential equation (1.6), the relationship (1.5) can be rewritten as

$$\begin{aligned} Y(V, X)_t &= E[\phi(X_T) + V_T|\mathcal{F}_t] - V_t \\ &= E[\phi(X_T) + V_T|\mathcal{F}_t] - \int_0^t f(s, Y(V, X)_s, Z(V, X)_s)ds, \end{aligned}$$

together with

$$\int_t^T Z_s dB_s = \phi(X_T) + V_T - E[\phi(X_T) + V_T|\mathcal{F}_t],$$

from which we deduce $(Y(V, X), Z(V, X), X)$ must satisfy (1.2). ■

The rest of this subsection is devoted to the proof of Theorem 1.

Lemma 9 *If the coefficients satisfy Condition 1, and τ satisfies*

$$\sqrt{T - \tau} \leq \frac{1}{8C_1(1 + C_2)} \wedge 1,$$

then functional differential equation (1.6) admits a unique solution $\Psi \in \mathcal{C}([\tau, T]; R^{n+d})$.

Proof. The mapping defined by (1.6) is denoted by $\mathbb{L}$. We will first show that $\mathbb{L} : \mathcal{C}([\tau, T]; R^{n+d}) \to \mathcal{C}([\tau, T]; R^{n+d})$. In fact for $\Psi \in \mathcal{C}([\tau, T]; R^{n+d})$,

$$\begin{aligned}
&||\mathbb{L}(\Psi)||_{\mathcal{C}[\tau,T]} \\
&\leq |x| + \sqrt{E\left(\int_\tau^T |F(s, Y(\Psi)_s, Z(\Psi)_s)|ds\right)^2} + \sqrt{E \sup_{\tau\leq t\leq T} |\int_\tau^t \chi_1 dB_s|^2} \\
&\leq |x| + \sqrt{T-\tau}\sqrt{E\left(\int_\tau^T |F(s, Y(\Psi)_s, Z(\Psi)_s)|^2 ds\right)} + 2\sqrt{E|\int_\tau^T \chi_1 dB_s|^2} \\
&\leq |x| + (C_1\sqrt{T-\tau} + 2d)\sqrt{\int_\tau^T (s^2 \vee 1) ds} \\
&\quad + C_1\sqrt{T-\tau}\sqrt{E\int_\tau^T |Y(\Psi)_s|^2 ds} + C_1\sqrt{T-\tau}\sqrt{E\int_\tau^T |Z(\Psi)_s|^2 ds}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sqrt{E\int_\tau^T |Y(\Psi)_s|^2 ds} \\
&\leq \sqrt{E\int_\tau^T \{E[\phi(X_T)|\mathcal{F}_s]\}^2 ds} + \sqrt{E\int_\tau^T \{E[V_T|\mathcal{F}_s]\}^2 ds} + \sqrt{E\int_\tau^T |V_s|^2 ds} \\
&\leq \sqrt{\int_\tau^T E|\phi(X_T)|^2 ds} + \sqrt{\int_\tau^T E|V_T|^2 ds} + \sqrt{\int_\tau^T E|V_s|^2 ds} \\
&\leq \sqrt{T-\tau}(2 + C_2)||\Psi||_{C[\tau,T]},
\end{aligned}$$

and by Itô's isometry,

$$\begin{aligned}
\sqrt{E\int_\tau^T |Z(\Psi)_s|^2 ds} &= \sqrt{E\left(\int_\tau^T Z(\Psi)_s dB_s\right)^2} \\
&\leq \sqrt{E|\phi(X_T)|^2} + \sqrt{E|V_T|^2} + \sqrt{E\left\{E[\phi(X_T)|\mathcal{F}_\tau]\right\}^2} \\
&\quad + \sqrt{E\left\{E[V_T|\mathcal{F}_\tau]\right\}^2} \\
&\leq (2 + 2C_2)||\Psi||_{C[\tau,T]}.
\end{aligned}$$

Therefore $||\mathbb{L}(\Psi)||_{\mathcal{C}[\tau,T]} < \infty$. Similarly for $\Psi, \Psi^{'} \in \mathcal{C}([\tau, T]; R^{n+d})$, we have

$$\begin{aligned}
&||\mathcal{L}(\Psi) - \mathcal{L}(\Psi^{'})||_{\mathcal{C}[\tau,T]} \\
&\leq C_1\sqrt{T-\tau}\left(\sqrt{T-\tau}(2 + C_2) + 2 + 2C_2\right)||\Psi - \Psi^{'}||_{\mathcal{C}[\tau,T]} \\
&\leq \frac{1}{2}||\Psi - \Psi^{'}||_{\mathcal{C}[\tau,T]}
\end{aligned}$$

by the condition on τ. Hence $\mathbb{L}$ defined by (1.6) is a contraction mapping on $\mathcal{C}([\tau, T]; R^{n+d})$. ∎

Based on Lemma 9, we next extend to the global solution on $[0, T]$. To do this we pursue the bounded solutions for FBSDE (1.2). First by the Markov property, there exists a Borel-measurable function Φ such that $Y_t = \Phi(t, X_t)$. By checking the proof for Lemma 9, the crucial step to extend to the global solution of (1.6) is that one needs a uniform estimate for the gradient of Φ. We recall a regularity result from Delarue [11].

Lemma 10 *(Delarue [11]) Under Condition 1 on the coefficients, there exists a Borel measurable Φ such that $Y_t = \Phi(t, X_t)$. Moreover there exists a constant C_4 depending on the Lipschitz constants C_1 and C_2, the bound M of the terminal data, the dimension n and d, and the terminal time T such that*

$$|\Phi(t,x)|,\ |\nabla_x \Phi(t,x)| \le C_4, \quad \textit{for } (t,x) \in [0,T] \times R^d.$$

Based on such constant C_4, we make a partition of $[0, T]$ by $\pi : 0 = t_0 \le t_1 \le \cdots \le t_N = T$ with the mesh $|\pi| = \max_{1\le i\le N} |t_i - t_{i-1}|$ such that

$$\sqrt{|\pi|} = \frac{1}{8C_1(1 + C_4)} \wedge 1.$$

We start with $[t_{N-1}, t_N]$ and consider $\Psi(N) = (V(N), X(N)^{t_{N-1},x})^*$ such that

$$\Psi(N)_t = \chi_x + \int_{t_{N-1}}^t F(s, Y(N)_s, Z(N)_s)ds + \int_{t_{N-1}}^t \chi_1 dB_s$$

with

$$\begin{aligned} Y(N)_t &= E[\phi(X(N)_T^{t_{N-1},x}) + V(N)_T|\mathcal{F}_t] - V(N)_t, \\ \int_{t_{N-1}}^T Z(N)_s dB_s &= \phi(X(N)_T^{t_{N-1},x}) + V(N)_T \\ &\quad - E[\phi(X(N)_T^{t_{N-1},x}) + V(N)_T|\mathcal{F}_{t_{N-1}}], \end{aligned}$$

where we used the superscripts (t_{N-1}, x) to emphasize $X(N)^{t_{N-1},x}$ starting from $X(N)_{t_{N-1}}^{t_{N-1},x} = x$. By Lemma 9, there exists a unique solution:

$$\Psi(N) = (V(N), X(N)^{t_{N-1},x})^* \in \mathcal{C}([t_{N-1}, T]; R^{n+d})$$

and we also get $(Y(N), Z(N))$. Moreover, there exists a Borel-measurable function Φ_{N-1} such that $Y(N)_{t_{N-1}} = \Phi_{N-1}(t_{N-1}, x)$ and by Lemma 10,

$$|\nabla_x \Phi_{N-1}(t_{N-1}, x)| \le C_4 \quad \text{for } x \in R^d.$$

In general on $[t_{i-1}, t_i]$ for $1 \le i \le N - 1$, consider $\Psi(i) = (V(i), X(i)^{t_{i-1},x})^*$ such that

$$\Psi(i)_t = \chi_x + \int_{t_{i-1}}^t F(s, Y(i)_s, Z(i)_s)ds + \int_{t_{i-1}}^t \chi_1 dB_s$$

with

$$\begin{aligned} Y(i)_t &= E[\Phi_i(t_i, X(i)_{t_i}^{t_{i-1},x}) + V(i)_{t_i}|\mathcal{F}_t] - V(i)_t, \\ \int_{t_{i-1}}^{t_i} Z(i)_s dB_s &= \Phi_i(t_i, X(i)_{t_i}^{t_{i-1},x}) + V(i)_{t_i} \\ &\quad - E[\Phi_i(t_i, X(i)_{t_i}^{t_{i-1},x}) + V(i)_{t_i}|\mathcal{F}_{t_{i-1}}]. \end{aligned}$$

By Lemma 9 again, there exists a unique solution:

$$\Psi(i) = (V(i), X(i)^{t_{i-1},x})^* \in \mathcal{C}([t_{i-1}, t_i]; R^{n+d})$$

and we get $(Y(i), Z(i))$ as well. Moreover there exists a Borel-measurable function Φ_{i-1} such that $Y(i)_{t_{i-1}} = \Phi_{i-1}(t_{i-1}, x)$ and by Lemma 10,

$$|\nabla_x \Phi_{i-1}(t_{i-1}, x)| \leq C_4 \quad \text{for } x \in R^d.$$

Of course $(V(i), X(i)^{t_{i-1},x})$ for $1 \leq i \leq N$ are not the real solutions to (1.6) on the corresponding time interval $[t_{i-1}, t_i]$, because they start from

$$(V(i)_{t_{i-1}}, X(i)^{t_{i-1},x}_{t_{i-1}}) = (0, x).$$

We need to shift the paths of $(V(i), X(i)^{t_{i-1},x})$ accordingly in order to match the starting points for the solutions to (1.6) on each time interval $[t_{i-1}, t_i]$.

Lemma 11 *If the coefficients satisfy Condition 1, then the global solution $\Psi = (V, X)^*$ to (1.6) is constructed as follows: for $1 \leq i \leq N$,*

$$V_t = V(i)_t + \sum_{j=1}^{i-1} V(j)_{t_j} \quad \textit{for } t_{i-1} \leq t \leq t_i,$$

where we follow the convention $\sum_{j=1}^{0} = 0$ and

$$X_t = \begin{cases} X(1)^{t_0,x}_t & \textit{for } t_0 \leq t \leq t_1; \\ X(2)^{t_1,X_{t_1}}_t & \textit{for } t_1 \leq t \leq t_2; \\ \cdots & \\ X(N)^{t_{N-1},X_{t_{N-1}}}_t & \textit{for } t_{N-1} \leq t \leq t_N \end{cases}$$

with (Y, Z) being constructed as $(Y_t, Z_t) = (Y(i)_t, Z(i)_t)$ for $t_{i-1} \leq t \leq t_i$.

Proof. We only need to show $\Psi = (V, X)^*$ with (Y, Z) satisfying (1.6) for $\tau = 0$. In fact for $t \in [t_{N-1}, t_N]$, by the definition of (V, X),

$$\begin{aligned} V_t - V_{t_{N-1}} &= V(N)_t + \sum_{j=1}^{N-1} V(j)_{t_j} - V(N)_{t_{N-1}} - \sum_{j=1}^{N-1} V(j)_{t_j} \\ &= \int_{t_{N-1}}^{t} h(s, Y(N)_s, Z(N)_s) ds, \end{aligned}$$

and

$$\begin{aligned} X_t - X_{t_{N-1}} &= X(N)^{t_{N-1},X_{t_{N-1}}}_t - X_{t_{N-1}} \\ &= \int_{t_{N-1}}^{t} f(s, Y(N)_s, Z(N)_s) ds + \int_{t_{N-1}}^{t} dB_s, \end{aligned}$$

so

$$\Psi_t - \Psi_{t_{N-1}} = \int_{t_{N-1}}^{t} F(s, Y(N)_s, Z(N_s)) ds + \int_{t_{N-1}}^{t} \chi_1 dB_s,$$

where

$$\begin{aligned}
Y(N)_t &= E\left[\phi\left(X(N)_T^{t_{N-1},X_{t_{N-1}}}\right) + V(N)_T|\mathcal{F}_t\right] - V(N)_t \\
&= E\left[\phi(X_T) + V(N)_T + \sum_{j=1}^{N-1} V(j)_{t_j}|\mathcal{F}_t\right] \\
&\quad - V(N)_t - \sum_{j=1}^{N-1} V(j)_{t_j} \\
&= E[\phi(X_T) + V_T|\mathcal{F}_t] - V_t,
\end{aligned}$$

and

$$\begin{aligned}
\int_{t_{N-1}}^{T} Z(N)_s dB_s &= \phi(X(N)_T^{t_{N-1},X_{t_{N-1}}}) + V(N)_T \\
&\quad - E\left[\phi\left(X(N)_T^{t_{N-1},X_{t_{N-1}}}\right) + V(N)_T|\mathcal{F}_{t_{N-1}}\right] \\
&= \phi(X_T) + V(N)_T + \sum_{j=1}^{N-1} V(j)_{t_j} \\
&\quad - E\left[\phi(X_T) + V(N)_T + \sum_{j=1}^{N-1} V(j)_{t_j}|\mathcal{F}_{t_{N-1}}\right] \\
&= \phi(X_T) + V_T - E[\phi(X_T) + V_T|\mathcal{F}_{t_{N-1}}].
\end{aligned}$$

Hence (V, X) with (Y, Z) defined in the lemma satisfy (1.6) on $[t_{N-1}, t_N]$.

In general for $1 \leq i \leq N-1$, by the backward induction, it is easy to verify $\Psi_t = (V_t, X_t)^*$ with (Y_t, Z_t) also satisfy (1.6) for $t \in [t_{i-1}, t_i]$. ■

2.3 Uniform integrability of stochastic exponential

In this subsection we will verify the Doléans-Dade exponential $\mathscr{E}(N)$ is a uniform-integrable martingale. To prove this we need an appropriate martingale space. It turns out the martingale space we need is *bounded-mean-oscillation* (BMO)-martingale space. As the BMO-martingale theory is already quite standard in the quadratic BSDE study (e.g. [3] and [21]), we only recall some basic facts that are necessary in the following. For the further details and proofs, we refer to He et al [18].

Let M be a continuous local martingale on $[0, T]$. For $p \geq 1$, define the martingale space $\mathcal{H}^p$ equipped with the norm $||M||_{\mathcal{H}^p} = E\{[M, M]_T^{p/2}\}^{1/p}$. For $p > 1$, $\mathcal{H}^p$ is the dual space of $\mathcal{H}^q$ with q being the conjugate of p, i.e. $1/p+1/q = 1$. However for $p = 1$, the dual space of $\mathcal{H}^1$ is strictly larger than $\mathcal{H}^\infty$, the class of all martingales with bounded quadratic variation. By the Fefferman's inequality, the dual of $\mathcal{H}^1$ is in fact BMO_2, which is the subspace of $\mathcal{H}^2$ and such that there exists a constant C_5,

$$E\left\{|M_T - M_\tau|^2|\mathcal{F}_\tau\right\} \leq C_5^2$$

for any stopping time $\tau \leq T$, and C_5 is defined to be the BMO_2-norm. Similarly we can also define BMO_p-space for any $p \geq 1$, which are equivalent to each other. In fact for $p \geq 1$ and $M \in \mathcal{H}^2$, there exists a constant $C_6(p)$ depending on p such that

$$||M||_{BMO_1} \leq ||M||_{BMO_p} \leq C_6(p)||M||_{BMO_1}$$

by the Jensen's inequality and the John-Nirenberg inequality respectively. So from now on we will simply write BMO without specifying p.

Lemma 12 *If the coefficients satisfy Condition 1, then*

$$N = -\int_0^{\cdot} \langle f(s, Y_s, Z_s), dB_s \rangle_d$$

is a BMO-martingale under $\mathbf{Q}$*, and therefore the Doléans-Dade exponential* $\mathscr{E}(N)$ *is a uniform-integrable martingale under* $\mathbf{Q}$*.*

Proof. For any stopping time $\tau \leq T$, by Itô's isometry and the linear growth condition $|f(t,y,z)| \leq C_1(t + |y| + |z|)$, we obtain

$$\begin{aligned}
& \sup_\tau E\left[|N_T - N_\tau|^2 | \mathcal{F}_\tau\right] \\
= & \sup_\tau E\left[\int_\tau^T |f(t, Y_t, Z_t)|^2 dt \mid \mathcal{F}_\tau\right] \\
\leq & C_1^2 T^3 + 3C_1^2 \sup_\tau E\left[\int_\tau^T |Y_t|^2 dt \mid \mathcal{F}_\tau\right] + 3C_1^2 \sup_\tau E\left[\int_\tau^T |Z_t|^2 dt \mid \mathcal{F}_\tau\right], \quad a.s..
\end{aligned} \tag{2.3}$$

Since Y is uniformly bounded, we only need to control the last term of (2.3). By applying Itô's formula to $(Y_t)^2$ from τ to T and taking conditional expectation on $\mathcal{F}_\tau$, we obtain

$$\begin{aligned}
& |Y_\tau|^2 + E\left[\int_\tau^T |Z_t|^2 dt \mid \mathcal{F}_\tau\right] \\
= & E[\phi(X_T)^2 | \mathcal{F}_\tau] + 2E\left[\int_\tau^T \langle Y_t, h(t, Y_t, Z_t)\rangle_n dt \mid \mathcal{F}_\tau\right] \\
\leq & M^2 + \lambda^2 E\left[\int_\tau^T |Y_t|^2 dt \mid \mathcal{F}_\tau\right] + \frac{1}{\lambda^2} E\left[\int_\tau^T |h(t, Y_t, Z_t)|^2 dt \mid \mathcal{F}_\tau\right] \\
\leq & M^2 + \lambda^2 E\left[\int_\tau^T |Y_t|^2 dt \mid \mathcal{F}_\tau\right] + \frac{C_1^2 T^3}{\lambda^2} + \frac{3C_1^2}{\lambda^2} E\left[\int_\tau^T |Y_t|^2 dt \mid \mathcal{F}_\tau\right] \\
& + \frac{3C_1^2}{\lambda^2} E\left[\int_\tau^T |Z_t|^2 dt \mid \mathcal{F}_\tau\right], \quad a.s.,
\end{aligned}$$

where we used the elementary inequality $2ab \leq \lambda^2 + b^2/\lambda^2$. By choosing λ large enough such that $1 - 3C_1^2/\lambda^2 > 0$, and by the uniform boundedness of Y, there exists a constant C_7 such that

$$\sup_\tau E\left[\int_\tau^T |Z_t|^2 dt \mid \mathcal{F}_\tau\right] \leq C_7, \quad a.s.,$$

and the conclusion follows by plugging the above estimate into (2.3). ■

3 Uniqueness and strong solutions for $n = 1$

As in the classical SDE theory, there are also several notions of uniqueness for BSDEs as well. In this section we discuss the uniqueness of BSDE (1.7). We assume the following condition on the coefficients:

Condition 2 *Condition 1 is assumed to be satisfied. Moreover $n = 1$, i.e. BSDE (1.7) is a scalar BSDE; $F = F(t, z)$ with $F = (h, f)^*$, i.e. both of the coefficients h and f only depend on t and z; and $f^j = f^j(t, z^j)$ for $j = 1, \cdots, d$, i.e. there is no mixture terms of z in f.*

The weak solution can be regarded as probability distribution on the sample path space, so we will specify the sample path space of (1.7) firstly. By $\mathbf{W}^m$ we denote the space of continuous functions $C([0, T]; R^m)$. Define the coordinate mapping $X_t : \mathbf{W}^m \to R^m$ by

$$X_t(x) = x_t, \quad \text{for } x \in \mathbf{W}^m,$$

and on $\mathbf{W}^m$, define the following σ-algebras:

$$\mathcal{B}_t^X = \sigma(x_s : s \leq t); \quad \mathcal{B}_{\hat{t}}^X = \sigma(x_u - x_t : t \leq u \leq T);$$

and $\mathcal{B}^X = \vee_{t \in [0,T]} \mathcal{B}_t^X$. Obviously we have the relationship $\mathcal{B}^X = \mathcal{B}_t^X \vee \mathcal{B}_{\hat{t}}^X$ for any $t \in [0, T]$. By Definition 2, Y and W must be continuous. However it is not obvious at all that the density representation Z has any path regularity. Fortunately under Condition 2, Imkeller and Dos Reis [25] already did this work for us, which states that there is a continuous modification of Z. We will choose such continuous version of Z from now on.

If $(\Omega, \mathcal{F}, \mathbf{P})$, $\{\mathcal{F}_t\}$, and (Y, Z, W) is one weak solution of (1.7), we can consider the image measure of $\mathbf{P}$ under the mapping $(Y, Z, W) : \Omega \to \mathbf{W}^{1+d} \times \mathbf{W}^d$ defined by

$$\bar{\omega} \mapsto ((Y(\bar{\omega}), Z(\bar{\omega})), W(\bar{\omega})), \quad \text{for } \bar{\omega} \in \Omega,$$

which is denoted by $\mathbf{Q}$. Since the projection of $\mathbf{Q}$ on the third component W is a Wiener measure on $(\mathbf{W}^d, \mathcal{B}^W)$, denoted by $\mathbf{Q}^*$ from now on, and all the spaces are Polish under the uniform topology, there exists a unique regular conditional probability $\mathbf{Q}\{\cdot|\omega\}$ such that:

(1) for $\omega \in \mathcal{B}^W$, $\mathbf{Q}\{\cdot|\omega\}$ is a probability measure on $(\mathbf{W}^{1+d}, \mathcal{B}^Y \otimes \mathcal{B}^Z)$;

(2) for $A \in \mathcal{B}^Y \otimes \mathcal{B}^Z$, the map $\omega \mapsto \mathbf{Q}\{A|\omega\}$ is $\mathcal{B}^W$-measurable;

(3) for $A \in \mathcal{B}^Y \otimes \mathcal{B}^Z$ and $B \in \mathcal{B}^W$, we have

$$\mathbf{Q}(A \times B) = \int_B \mathbf{Q}\{A|\omega\}\mathbf{Q}^*(d\omega).$$

Definition 13 *The weak solution to (1.7) is called unique in law if for any two weak solutions $(\Omega, \mathcal{F}, \mathbf{P})$, $\{\mathcal{F}_t\}$, (Y, Z, W) and $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}})$, $\{\bar{\mathcal{F}}_t\}$, $(\bar{Y}, \bar{Z}, \bar{W})$, the probability distributions of (Y, Z) and $(\bar{Y}, \bar{Z})$ are equal. i.e. $\mathbf{P}_{(Y,Z)} = \bar{\mathbf{P}}_{(\bar{Y},\bar{Z})}$.*

The weak solution to (1.7) is called pathwise unique if for any two weak solutions (Y, Z) and $(\bar{Y}, \bar{Z})$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with the filtration $\{\mathcal{F}_t\}$ and the same Brownian motion W, (Y, Z) is a continuous modification of $(\bar{Y}, \bar{Z})$.

For a SDE, the celebrated Yamada-Watanabe theorem states that the weak existence and pathwise uniqueness of the solutions to a SDE implies the existence of a strong solution. As Kurtz [24] pointed out: strong solution is a consequence of measurable selection, and such result has little to do with the equation, but really a consequence of the convexity of collections of the probability distributions of solutions. If the compatibility constraint (See Definition 2) is satisfied, we further have the adapteness of solutions.

Theorem 14 *If the coefficients satisfy Condition 2, then the weak solution to BSDE (1.7) is pathwise unique, and the strong solution also exists.*

We first prove the pathwise uniqueness. By employing the Girsanov's theorem reversely, we have the following pathwise uniqueness result.

Lemma 15 *If the coefficients satisfy Condition 2, then the weak solution to (1.7) is pathwise unique.*

Proof. Suppose (Y, Z) and $(\bar{Y}, \bar{Z})$ are two weak solutions on $(\Omega, \mathcal{F}, \mathbf{P})$ with $\{\mathcal{F}_t\}$ and Brownian motion W. By applying Itô's formula to $e^{\alpha t}(Y_t - \bar{Y}_t)^2$ for some α to be determined, we obtain

$$\begin{aligned}
& e^{\alpha t}(Y_t - \bar{Y}_t)^2 \\
=& -2\int_t^T e^{\alpha s}(Y_s - \bar{Y}_s)d(Y_s - \bar{Y}_s) - \int_t^T e^{\alpha s} d[Y - \bar{Y}, Y - \bar{Y}]_s \\
& - \int_t^T \alpha e^{\alpha s}(Y_s - \bar{Y}_s)^2 ds \\
=& \, 2\int_t^T e^{\alpha s}(Y_s - \bar{Y}_s)\left\{\sum_{j=1}^d (Z_s^j f^j(s, Z_s^j) - \bar{Z}_s^j f^j(s, \bar{Z}_s^j)) + (h(s, Z_s) - h(s, \bar{Z}_s))\right\} ds \\
& - 2\int_t^T e^{\alpha s}(Y_s - \bar{Y}_s)\sum_{j=1}^d (Z_s^j - \bar{Z}_s^j) dW_s^j - \int_t^T e^{\alpha s}\sum_{j=1}^d |Z_s^j - \bar{Z}_s^j|^2 ds \\
& - \int_t^T \alpha e^{\alpha s}(Y_s - \bar{Y}_s)^2 ds. \qquad (3.1)
\end{aligned}$$

Note that for $s \in [0, T]$, and $z^j, \bar{z}^j \in R$ for $j = 1, \cdots, d$,

$$\begin{aligned}
& \left|z^j f^j(s, z^j) - \bar{z}^j f^j(s, \bar{z}^j)\right| \\
\leq & \left|z^j f^j(s, z^j) - \bar{z}^j f^j(s, z^j)\right| + \left|\bar{z}^j f^j(s, z^j) - \bar{z}^j f^j(s, \bar{z}^j)\right| \\
\leq & \, C_1(T + |z^j|)|z^j - \bar{z}^j| + C_1|\bar{z}^j||z^j - \bar{z}^j| \\
\leq & \, C_1(T + |z^j| + |\bar{z}^j|)|z^j - \bar{z}^j|.
\end{aligned}$$

Now if we set

$$\beta_s^j = \frac{Z_s^j f^j(s, Z_s^j) - \bar{Z}_s^j f^j(s, \bar{Z}_s^j)}{Z_s^j - \bar{Z}_s^j}, \quad \text{for } s \in [0, T],$$

when $Z_s^j - \bar{Z}_s^j \neq 0$, and $\beta_s^j = 0$ for $s \in [0, T]$ otherwise, then there exists a constant C_8 such that $|\beta_s^j|^2 \leq C_8(1 + |Z_s^j|^2 + |\bar{Z}_s^j|^2)$. Based on such β^j, we define a new probability measure $\mathbf{Q}$ by $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathscr{E}(N)$ where

$$N = \sum_{j=1}^{d} \int_0^{\cdot} \beta_s^j dW_s^j,$$

and under $\mathbf{Q}$ define a new Brownian motion B by $B = W - [W, N]$. Then under the probability measure $\mathbf{Q}$, (3.1) reduces to

$$\begin{aligned}
& e^{\alpha t}(Y_t - \bar{Y}_t)^2 \\
= & -2\int_t^T e^{\alpha s}(Y_s - \bar{Y}_s)\sum_{j=1}^{d}(Z_s^j - \bar{Z}_s^j)dB_s^j + 2\int_t^T e^{\alpha s}(Y_s - \bar{Y}_s)(h(s, Z_s) - h(s, \bar{Z}_s))ds \\
& - \int_t^T e^{\alpha s}\sum_{j=1}^{d}|Z_s^j - \bar{Z}_s^j|^2 ds - \int_t^T \alpha e^{\alpha s}(Y_s - \bar{Y}_s)^2 ds.
\end{aligned}$$

By taking expectation under $\mathbf{Q}$ we have

$$\begin{aligned}
E^{\mathbf{Q}}[e^{\alpha t}(Y_t - \bar{Y}_t)^2] = & E^{\mathbf{Q}}\left\{\int_t^T e^{\alpha s}2(Y_s - \bar{Y}_s)(h(s, Z_s) - h(s, \bar{Z}_s))ds\right\} \\
& - E^{\mathbf{Q}}\left\{\int_t^T e^{\alpha s}|Z_s - \bar{Z}_s|^2 ds\right\} - E^{\mathbf{Q}}\left\{\int_t^T \alpha e^{\alpha s}(Y_s - \bar{Y}_s)^2 ds\right\}.
\end{aligned}$$

By the elementary inequality $2ab \leq \lambda^2 a^2 + b^2/\lambda^2$,

$$2(Y_s - \bar{Y}_s)(h(s, Z_s) - h(s, \bar{Z}_s)) \leq \lambda^2(Y_s - \bar{Y}_s)^2 + \frac{C_1^2}{\lambda^2}|Z_s - \bar{Z}_s|^2.$$

By choosing $\lambda^2 = \alpha$ and $\alpha = 2C_1^2$, we obtain

$$E^{\mathbf{Q}}[e^{2C_1^2 t}(Y_t - \bar{Y}_t)^2] \leq -\frac{1}{2}E^{\mathbf{Q}}\left\{\int_t^T e^{2C_1^2 s}|Z_s - \bar{Z}_s|^2 ds\right\} \leq 0,$$

so $Y_t = \bar{Y}_t$ for $t \in [0, T]$, *a.s.*, and $Z_t = \bar{Z}_t$ for *a.e.* $t \in [0, T]$, *a.s.*. Now the only step left is to verify $\mathscr{E}(N)$ is a uniformly-integrable martingale, and we need to verify N is a BMO-martingale under $\mathbf{P}$. In fact for any stopping time $\tau \leq T$,

$$\begin{aligned}
E\left\{|N_T - N_\tau|^2 | \mathcal{F}_\tau\right\} = & \sum_{j=1}^{d} E\left\{\int_\tau^T |\beta_s^j|^2 ds | \mathcal{F}_\tau\right\} \\
\leq & C_8 E\left\{\int_\tau^T (d + |Z_s|^2 + |\bar{Z}_s|^2)ds | \mathcal{F}_\tau\right\}, \quad a.s..
\end{aligned}$$

The way to control the integral term involving Z and $\bar{Z}$ has already been presented in the proof of Lemma 12. Therefore $\mathbf{Q}$ defined above is indeed a probability measure. ■

The idea for the following lemma is standard: to transfer the structure of a weak solution such that $\mathbf{W}^{1+d}$ becomes the sample path space for (Y, Z) and $\mathbf{W}^d$ that for W. What allows us to carry it through is the regular conditional probability introduced above.

Lemma 16 *For BSDE (1.7) with the coefficients satisfying Condition 2, pathwise uniqueness implies uniqueness in law, and furthermore if the weak solution exists, then the strong solution also exists.*

Proof. Since the proof is quite standard, we only present the basic steps.

Let $(\Omega, \mathcal{F}, \mathbf{P})$, $\{\mathcal{F}_t\}$, (Y, Z, W) and $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}})$, $\{\bar{\mathcal{F}}_t\}$, $(\bar{Y}, \bar{Z}, \bar{W})$ be two weak solutions. Let $\mathbf{W}^{1+d}$ and $\bar{\mathbf{W}}^{1+d}$ be two copies of $C([0,T]; R) \times C([0,T]; R^d)$. By using the regular conditional probability $\mathbf{Q}\{\cdot|\omega\}$ and $\bar{\mathbf{Q}}\{\cdot|\omega\}$, we define a probability measure π on the probability space $(\Theta, \mathcal{B}(\Theta))$ by

$$\pi\left((dy, dz), (d\bar{y}, d\bar{z}), d\omega\right) = \mathbf{Q}\{(dy, dz)|\omega\}\bar{\mathbf{Q}}\{(d\bar{y}, d\bar{z})|\omega\}\mathbf{Q}^*(d\omega)$$

where

$$(\Theta, \mathcal{B}(\Theta)) = (\mathbf{W}^{1+d} \times \bar{\mathbf{W}}^{1+d} \times \mathbf{W}^d, \mathcal{B}^Y \otimes \mathcal{B}^Z \otimes \mathcal{B}^{\bar{Y}} \otimes \mathcal{B}^{\bar{Z}} \otimes \mathcal{B}^W).$$

On $(\Theta, \mathcal{B}(\Theta), \pi)$, we further define the filtration $\{\mathcal{G}_t\}$ which is generated by $\sigma((y_s, z_s), (\bar{y}_s, \bar{z}_s), \omega_s : s \leq t)$ augmented by the π-null sets in $\mathcal{B}(\Theta)$. Then under π and $\{\mathcal{G}_t\}$, ω is still a Brownian motion. In fact by the compatibility constraint in Definition 3, $\mathcal{B}_t^Y \otimes \mathcal{B}_t^Z$ is independent of $\mathcal{B}_t^W$. Hence for $A_t \in \mathcal{B}_t^Y \otimes \mathcal{B}_t^Z$,

$$\mathbf{Q}\{A_t|\omega\} = \mathbf{Q}\{A_t|\omega_t\}.$$

Likewise we also have $\bar{\mathbf{Q}}\{\bar{A}_t|\omega\} = \bar{\mathbf{Q}}\{\bar{A}_t|\omega_t\}$ for $\bar{A}_t \in \mathcal{B}_t^{\bar{Y}} \otimes \mathcal{B}_t^{\bar{Z}}$. Based on above relationships, for $B_t \in \mathcal{B}_t^W$, $u \in [t, T]$ and $\xi \in R^d$,

$$\begin{aligned}
&E^\pi\left[e^{i\langle \xi, \omega_u - \omega_t\rangle_d} 1_{A_t} 1_{\bar{A}_t} 1_{B_t}\right] \\
=&\int_{B_t} e^{i\langle \xi, \omega_u - \omega_t\rangle_d} \mathbf{Q}\{A_t|\omega\}\bar{\mathbf{Q}}\{\bar{A}_t|\omega\}\mathbf{Q}^*(d\omega) \\
=&\int_{\mathbf{W}^d} e^{i\langle \xi, \omega_u - \omega_t\rangle_d} \mathbf{Q}^*(d\omega) \int_{B_t} \mathbf{Q}\{A_t|\omega_t\}\bar{\mathbf{Q}}\{\bar{A}_t|\omega_t\}\mathbf{Q}^*(d\omega) \\
=&E^\pi\left[e^{i\langle \xi, \omega_u - \omega_t\rangle_d}\right] E^\pi\left[1_{A_t} 1_{\bar{A}_t} 1_{B_t}\right],
\end{aligned}$$

which means $\{\omega_u - \omega_t : t \leq u \leq T\}$ is independent of the filtration $\{\mathcal{G}_t\}$.

Therefore (y, z, ω) and $(\bar{y}, \bar{z}, \omega)$ are two weak solutions on the same filtered probability space $(\Theta, \mathcal{B}(\Theta), \{\mathcal{G}_t\}, \pi)$. Pathwise uniqueness means

$$\pi\left(\{(y, z), (\bar{y}, \bar{z}), \omega\} \in \Theta : (y, z) = (\bar{y}, \bar{z})\right) = 1, \tag{3.2}$$

so for any $A \in \mathcal{B}^Y \otimes \mathcal{B}^Z$, the probability distribution $\mathbf{P}_{(Y,Z)}(\bar{\omega} \in \Omega : (Y, Z) \in A)$ equals

$$\pi\left(\{(y, z), (\bar{y}, \bar{z}), \omega\} \in \Theta : (y, z) \in A\right) = \pi\left(\{(y, z), (\bar{y}, \bar{z}), \omega\} \in \Theta : (\bar{y}, \bar{z}) \in A\right)$$

which is equal to the probability distribution $\bar{\mathbf{P}}_{(\bar{Y},\bar{Z})}(\bar{\omega} \in \bar{\Omega} : (\bar{Y}, \bar{Z}) \in A)$.

To prove the second claim, we firstly show $\mathbf{Q}\{\cdot|\omega\}$ and $\bar{\mathbf{Q}}\{\cdot|\omega\}$ assign full measure to the same singleton. By (3.2) and the definition of π, we have

$$\int_{\mathbf{W}^d} \int_{(y,z)=(\bar{y},\bar{z})} \mathbf{Q}\{(dy, dz)|\omega\}\bar{\mathbf{Q}}\{(\bar{y}, \bar{z})|\omega\}\mathbf{Q}^*(d\omega) = 1,$$

so there exists $N \in \mathcal{B}^W$ with $\mathbf{Q}^*(N) = 0$ such that

$$\int_{(y,z)=(\bar{y},\bar{z})} \mathbf{Q}\{(dy,dz)|\omega\}\bar{\mathbf{Q}}\{(\bar{y},\bar{z})|\omega\} = 1, \quad \text{for } \omega \in N^c.$$

This can only occur if there exists a $\mathcal{B}^W/\mathcal{B}^Y \otimes \mathcal{B}^Z$-measurable map $\Phi = (\Phi^Y, \Phi^Z) : \mathbf{W}^d \to \mathbf{W}^{1+d}$ such that

$$\mathbf{Q}\{(y,z)|\omega\} = \bar{\mathbf{Q}}\{(y,z)|\omega\} = \delta_{\Phi(\omega)}(y,z), \quad \text{for } \omega \in N^c.$$

It then follows $(y,z) = (\Phi^Y(\omega), \Phi^Z(\omega))$ for $\omega \in N^c$. But recalling $\omega \mapsto \mathbf{Q}\{A_t|\omega\}$ is $\mathcal{B}^W_t$-measurable for $A_t \in \mathcal{B}^Y_t \otimes \mathcal{B}^Z_t$, then by the standard Dynkin arguments, Φ is in fact also $\mathcal{B}^W_t/\mathcal{B}^Y_t \otimes \mathcal{B}^Z_t$-measurable, and on any given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ satisfying the usual conditions with W being a Brownian motion on it,

$$\begin{aligned}\Phi^Y(W_t) = \phi(W_T) &+ \int_t^T h(s, \Phi^Z(W_s))ds \\ &+ \int_t^T \sum_{j=1}^d \Phi^{Z,j}(W_s) f^j(s, \Phi^{Z,j}(W_s))ds - \int_t^T \sum_{j=1}^d \Phi^{Z,j}(W_s)dW^j_s,\end{aligned}$$

so Φ is a strong solution. ■

4 Optimal portfolio in incomplete markets

In this section our main aim is to demonstrate how the functional differential equation approach and the nonlinear Girsanov's transformation can be used in finance. Specifically we consider an example of optimal portfolio problems in incomplete markets which is often used in the indifference valuation.

Assumption 1 *(the probability space) Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space which is to be determined, and $\{\mathcal{F}_t\}$ be its associated filtration satisfying the usual conditions, which is also to be determined.*

Assumption 2 *(the market) The market is built with three assets: a risk-free bond with zero interest rate, a tradeable asset and a nontradeable asset. The pricing dynamic of the tradeable asset satisfies the following SDE on the above given probability space:*

$$\begin{cases} dS_t/S_t = \mu^S_t dt + \bar{\sigma}^S_t d\bar{W}_t, \\ S_0 = s, \end{cases} \tag{4.1}$$

and the dynamic of the nontradeable asset follows

$$\begin{cases} dV_t/V_t = \mu^V_t dt + \sigma^V_t dW^V_t + \bar{\sigma}^V_t d\bar{W}_t, \\ V_0 = v \end{cases} \tag{4.2}$$

on the same given probability space, where $\mathbf{W} = (W^V, \bar{W})$ is a two-dimensional Brownian motion to be determined. σ^V and $\mu^i, \bar{\sigma}^i$ for $i = S, V$, as the market coefficients, are bounded and continuous functions.

Assumption 3 *(the investor) The investor has an exponential utility function depending on his/her terminal wealth, which has the form:*

$$U(x) = -e^{-\gamma x}, \quad \text{for } x \in R,$$

where $\gamma \geq 0$ representing the degree of the investor's risk aversion.

Assumption 4 *(the trading strategy) The investor, with initial wealth x, invests in the tradeable asset and the risk-free bond during the time period $[0, T]$. Let π be the amount of money invested in the tradeable asset. We assume π is taken from the following admissible set, which of course depends on the above given probability space.*

$$\mathcal{A}_{ad} := \{\pi : [0,T]\times\Omega \to R : \ \pi \text{ is } \mathcal{F}_t\text{-adapted, self-financing and } ||\pi||_{H^2[0,T]} < \infty.\}$$

The dynamic of the investor's wealth process, denoted by $X^x(\pi)$, follows

$$\begin{cases} dX_t^x(\pi) = \pi_t(\mu_t^S dt + \bar{\sigma}_t^S d\bar{W}_t), \\ X_0^x(\pi) = x. \end{cases} \tag{4.3}$$

Assumption 5 *(the cost functional) At time $t = T$, the investor gets the total amount $X_T^x(\pi)$ plus a random endowment $g(V_T, S_T)$, where g is assumed to be Lipschitz continuous and uniform bounded. The investor decides the optimal trading strategy to maximize the following cost functional:*

$$\sup_{\pi \in \mathcal{A}_{ad}} E^{\mathbf{P}} \left[-e^{-\gamma(X_T^x(\pi) + g(V_T, S_T))} \right].$$

Here we use the superscript $\mathbf{P}$ to emphasize the expectation is taken under the probability measure $\mathbf{P}$, which is to be determined.

The random endowment g depends not only on the nontradeable asset V but also on the tradeable asset S, which distinguishes the current problem from the ones usually considered in the literature. In the book edited by Carmona [9], this problem is even called an open problem ([19]). Such form of random endowment actually appears naturally when one wants to consider the credit risk of options traded in OTC markets (see Henderson and Liang [20]). We also emphasize the well known Cole-Hopf transformation does not help to deduce the closed-from solutions in our setting.

In the following we give the definition of weak admissible trading strategy and the corresponding weak formulation of optimal portfolio problems. For the weak formulation of general stochastic control problems, we refer to Yong and Zhou [41].

Definition 17 *A triple $(\Omega, \mathcal{F}, \mathbf{P})$ $\{\mathcal{F}_t\}$ and $(\pi, \mathbf{W})$ is called a weak admissible trading strategy if*
***(1)** $(\Omega, \mathcal{F}, \mathbf{P})$ is a complete probability space with the filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions;*
***(2)** $\mathbf{W}$ is a Brownian motion, and the increment $\{\mathbf{W}_u - \mathbf{W}_t : t \leq u \leq T\}$ must be independent of σ-algebra $\mathcal{F}_t$;*
***(3)** π is taken from the admissible set $\mathcal{A}_{ad}$.*

The set of all weak admissible trading strategies is denoted as $\mathcal{A}_{ad}^W$, and a generic element in such weak admissible set $\mathcal{A}_{ad}^W$ is denoted as Π. The investor decides the optimal weak admissible trading strategy Π in order to maximize his/her cost functional:

$$\sup_{\Pi\in\mathcal{A}_{ad}^W} E^{\mathbf{P}}\left[-e^{-\gamma(X_T^x(\pi)+g(V_T,S_T))}\right]. \tag{4.4}$$

Remark 18 *The motivation of introducing the above weak formulation of optimal portfolio problems is more from mathematics rather than finance. Later We will use the martingale optimality principle to deduce a quadratic BSDE as the characterization of the optimal portfolio, and we will look for the weak solution of such quadratic BSDE. The probability space will be chosen from the weak solution of the associated quadratic BSDE.*

Next we use the martingale optimality principle to characterize the optimal portfolio. For a given filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbf{P})$ with a two-dimensional Brownian motion $\mathbf{W}=(W^V,\bar{W})$, all of which are to be determined, we want to construct a family of stochastic processes

$$(-e^{-\gamma(X_t^x(\pi)+Y_t)})_{t\in[0,T]}$$

such that

(1) the process $(-e^{-\gamma(X_t^x(\pi)+Y_t)})_{t\in[0,T]}$ is a supermartingale for any $\pi\in\mathcal{A}_{ad}$, and there exists an optimal $\pi^*\in\mathcal{A}_{ad}$ such that $(-e^{-\gamma(X_t^x(\pi^*)+Y_t)})_{t\in[0,T]}$ is a martingale;

(2) the auxiliary process $(Y_t)_{t\in[0,T]}$ has the terminal value $Y_T=g(V_T,S_T)$.

If such auxiliary process $(Y_t)_{t\in[0,T]}$ and the optimal π^* exist, then we have

$$E^{\mathbf{P}}\left[-e^{-\gamma(X_T^x(\pi)+Y_T)}\right]\leq -e^{-\gamma(x+Y_0)},\quad \text{for any } \pi\in\mathcal{A}_{ad},$$

and

$$E^{\mathbf{P}}\left[-e^{-\gamma(X_T^x(\pi^*)+Y_T)}\right]= -e^{-\gamma(x+Y_0)},\quad \text{for optimal } \pi^*\in\mathcal{A}_{ad}.$$

Therefore

$$\begin{aligned}\sup_{\pi\in\mathcal{A}_{ad}} E^{\mathbf{P}}\left[-e^{-\gamma(X_T^x(\pi)+Y_T)}\right] &= E^{\mathbf{P}}\left[-e^{-\gamma(X_T^x(\pi^*)+Y_T)}\right]\\ &= -e^{-\gamma(x+Y_0)}.\end{aligned}$$

Note that the filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbf{P})$ and the Brownian motion $\mathbf{W}$ are still to be determined. Next we use the weak solution of a quadratic BSDE to characterize the auxiliary processes Y and π^*, which also provides us with the filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbf{P})$ and the Brownian motion $\mathbf{W}$.

Theorem 19 *Let $(\Omega,\mathcal{F},\mathbf{P})$, $\{\mathcal{F}_t\}$ and $(Y,\mathbf{Z},\mathbf{W})$ (with $\mathbf{Z}=(Z^V,\bar{Z})$) be the weak solution to the following quadratic BSDE:*

$$Y_t = g(V_T,S_T)-\int_t^T f_s ds-\int_t^T (Z_s^V dW_s^V+\bar{Z}_s d\bar{W}_s) \tag{4.5}$$

with

$$f_t = \frac{\gamma}{2}(Z_t^V)^2 + \frac{\mu_t^S}{\bar{\sigma}_t^S}\bar{Z}_t - \frac{(\mu_t^S)^2}{2\gamma(\sigma_t^S)^2}.$$

Then the value function of the optimal portfolio problem (4.4) is given by

$$-e^{-\gamma(x+Y_0)},$$

and the optimal weak admissible trading strategy Π^* *is the triple* $(\Omega, \mathcal{F}, \mathbf{P})$, $\{\mathcal{F}_t\}$ *and* $(\pi^*, \mathbf{W})$ *with*

$$\pi_t^* = -\bar{Z}_t + \frac{\mu_t^S}{\gamma(\bar{\sigma}_t^S)^2}. \tag{4.6}$$

Proof. On a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ with the Brownian motion $\mathbf{W}$, we suppose $(Y_t)_{t\in[0,T]}$ satisfies the following BSDE:

$$Y_t = g(V_T, S_T) - \int_t^T f_s ds - \int_t^T (Z_s^V dW_s^V + \bar{Z}_s d\bar{W}_s),$$

where the driver f is to be determined. By applying Itô's formula to $e^{-\gamma(X_t^x(\pi)+Y_t)}$, we obtain

$$\begin{aligned} & de^{-\gamma(X_t^x(\pi)+Y_t)} \\ = {} & e^{-\gamma(X_t^x(\pi)+Y_t)}\left\{-\gamma(dX_t^x(\pi) + dY_t) + \frac{\gamma^2}{2}d[X^x(\pi)+Y, X^x(\pi)+Y]_t\right\} \\ = {} & e^{-\gamma(X_t^x(\pi)+Y_t)}\Big\{-\gamma\mu_t^S\pi_t - \gamma f_t + \frac{\gamma^2}{2}\left[(\bar{\sigma}_t^S)^2\pi_t^2 + (Z_t^V)^2 + (\bar{Z}_t)^2\right. \\ & \left.+2\bar{\sigma}_t^S\bar{Z}_t\pi_t\right]\Big\}\,dt + \text{martingale term.} \end{aligned}$$

Since $(-e^{-\gamma(X_t^x(\pi)+Y_t)})_{t\in[0,T]}$ is supermartingale for any $\pi \in \mathcal{A}_{ad}$, and a martingale for optimal $\pi^* \in \mathcal{A}_{ad}$, we must have f_t and π^* such that

$$\frac{\gamma^2}{2}(\bar{\sigma}_t^S)^2\left(\pi_t + \bar{Z}_t - \frac{\mu_t^S}{\gamma(\bar{\sigma}_t^S)^2}\right)^2 + \frac{\gamma^2}{2}(Z_t^V)^2 + \frac{\gamma\mu_t^S}{\bar{\sigma}_t^S}\bar{Z}_t - \frac{(\mu_t^S)^2}{2(\sigma_t^S)^2} - \gamma f_t \geq 0$$

for any $\pi \in \mathcal{A}_{ad}$, and equality holds for optimal π^*. Therefore by solving the above variational inequality, we obtain

$$f_t = \frac{\gamma}{2}(Z_t^V)^2 + \frac{\mu_t^S}{\bar{\sigma}_t^S}\bar{Z}_t - \frac{(\mu_t^S)^2}{2\gamma(\sigma_t^S)^2},$$

and

$$\pi_t^* = -\bar{Z}_t + \frac{\mu_t^S}{\gamma(\bar{\sigma}_t^S)^2}.$$

■

In the following we will employ the functional differential equation approach and the nonlinear Girsanov's transformation to find the weak solution of BSDE (4.5). Since the coefficients satisfy Condition 2, by Theorem 14, the weak solution we will find is pathwise unique, and moreover the strong solution also exists.

The idea is to use the strong solution of the following FBSDE (4.7) to construct the weak solution of BSDE (4.5). Let's start with a Brownian motion $\mathbf{B} = (B^V, \bar{B})$ on $(\Omega, \mathcal{F}, \mathbf{Q})$ with the filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, and consider the following FBSDE:

$$\begin{cases} d\ln V_t &= \left\{\mu_t^V - \frac{1}{2}[(\sigma_t^V)^2 + (\bar{\sigma}_t^V)^2]\right\} dt - \sigma_t^V \frac{\gamma}{2} Z_t^V dt \\ & \quad -\bar{\sigma}_t^V \frac{\mu_t^S}{\bar{\sigma}_t^S} dt + \sigma_t^V dB_t^V + \bar{\sigma}_t^V d\bar{B}_t, \\ \ln V_0 &= \ln v, \\ d\ln S_t &= -\frac{1}{2}(\bar{\sigma}_t^S)^2 dt + \bar{\sigma}_t^S d\bar{B}_t, \\ \ln S_0 &= \ln s, \\ dY_t &= -\frac{(\mu_t^S)^2}{2\gamma(\bar{\sigma}_t^S)^2} dt + Z_t^V dB_t^V + \bar{Z}_t d\bar{B}_t, \\ Y_T &= g(e^{\ln V_T}, e^{\ln S_T}). \end{cases} \tag{4.7}$$

Note that FBSDE (4.7) is linear and the coefficients satisfy Condition 1, so by Lemma 8, we know there exists a unique solution $(Y, \mathbf{Z}, \ln V, \ln S) \in \mathcal{C}([0,T]; R) \times H^2([0,T]; R^2) \times \mathcal{C}([0,T]; R) \times \mathcal{C}([0,T]; R)$.

Based on the solution $(Y, \mathbf{Z})$, we define a new probability measure $\mathbf{P}$ by

$$\frac{d\mathbf{P}}{d\mathbf{Q}} = \mathscr{E}(N),$$

where $\mathscr{E}(N)$ is the Doléans-Dade exponential of N with

$$N = \int_0^\cdot \frac{\gamma}{2} Z_t^V dB_t^V + \int_0^\cdot \frac{\mu_t^S}{\bar{\sigma}_t^S} d\bar{B}_t.$$

By Lemma 12, we know $\mathbf{P}$ is indeed a probability measure. Under the new probability measure $\mathbf{P}$, by Girsanov's theorem, $\mathbf{W} = \mathbf{B} - [\mathbf{B}, N]$ is a Brownian motion with

$$\begin{cases} W^V = B^V - \int_0^\cdot \frac{\gamma}{2} Z_t^V dt, \\ \bar{W} = \bar{B} - \int_0^\cdot \frac{\mu_t^S}{\bar{\sigma}_t^S} dt. \end{cases}$$

Under the probability measure $\mathbf{P}$ and with the Brownian motion $\mathbf{W}$, let's rewrite the backward equation in FBSDE (4.7):

$$\begin{aligned} dY_t &= -\frac{(\mu_t^S)^2}{2\gamma(\bar{\sigma}_t^S)^2} dt + Z_t^V \left(dW_t^V + \frac{\gamma}{2} Z_t^V dt\right) + \bar{Z}_t \left(d\bar{W}_t + \frac{\mu_t^S}{\bar{\sigma}_t^S} dt\right) \\ &= f_t dt + Z_t^V dW_t^V + \bar{Z}_t d\bar{W}_t \end{aligned}$$

with $Y_T = g(V_T, S_T)$, and rewrite the forward equations in FBSDE (4.7):

$$dV_t / V_t = \mu_t^V dt + \sigma_t^V dW_t^V + \bar{\sigma}_t^V d\bar{W}_t,$$

with

$$dS_t/S_t = \mu_t^S dt + \bar{\sigma}_t^S d\bar{W}_t.$$

Therefore the triple $(\Omega, \mathcal{F}, \mathbf{P})$, $\{\mathcal{F}_t\}$ and $(Y, \mathbf{Z}, \mathbf{W})$ is just one weak solution we want to find.

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References

[1] Antonelli, F., Backward-forward stochastic differential equations, *The Annals of Applied Probability*, 3(3), 1993, 777–793.

[2] Antonelli, F. and Ma, J., Weak solutions of forward-backward SDEs, *Stochastic Analysis and Applications*, 21(3), 2003, 493–514.

[3] Barrieu, P., Cazanave, N. and El Karoui, N., Closedness results for BMO semi-martingales and application to quadratic BSDEs, *Comptes Rendus Mathématique. Académie des Sciences. Paris*, 346(15-16), 2008, 881–886.

[4] Bielecki, T. and Jeanblanc, M., Indifference pricing of defaultable claims, *Indifference pricing: theory and applications, Princeton University Press*, 2009, 211-240.

[5] Briand, P. and Hu, Y., BSDE with quadratic growth and unbounded terminal value, *Probability Theory and Related Fields*, 136(4), 2006, 604–618.

[6] Briand, P. and Hu, Y., Quadratic BSDEs with convex generators and unbounded terminal conditions, *Probability Theory and Related Fields*, 141(3-4), 2008, 543–567.

[7] Buckdahn, R., Engelbert, H.-J. and Răşcanu, A., On weak solutions of backward stochastic differential equations, *Rossiĭskaya Akademiya Nauk. Teoriya Veroyatnosteĭ i ee Primeneniya*, 49(1), 2004, 70–108.

[8] Buckdahn, R. and Engelbert, H.-J., On the continuity of weak solutions of backward stochastic differential equations, *Rossiĭskaya Akademiya Nauk. Teoriya Veroyatnosteĭ i ee Primeneniya*, 52(1), 2007, 190–199.

[9] Carmona, R. (editor), Indifference pricing, theory and applications, *Princeton University Press* (2009).

[10] Delarue, F., On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case, *Stochastic Processes and Their Applications*, 99, 2002, 209-286.

[11] Delarue, F., Estimates of the solutions of a system of quasi-linear PDEs: A probabilistic scheme, *Séminaire de Probabilités XXXVII, Lecture Notes in Math., Spring*, 1832, 2003, 290–332.

[12] Delarue, F. and Guatteri, G., Weak existence and uniqueness for forward-backward SDEs, Stochastic Processes and their Applications, 116(12), 2006, 1712–1742.

[13] Duffie, D. and Epstein, L., Stochastic differential utility, *Econometrica*, 60, 1992, 353–394.

[14] El Karoui, N., Hamadene, S. and Matoussi, A., BSDEs and applications, *Indifference pricing: theory and applications, Princeton University Press*, 2009, 267–320.

[15] El Karoui, N. and Mazliak, L. (editors), Backward stochastic differential equations, *Pitman Research Notes in Mathematics Series, (Paris, 1995–1996)*, 364, 1997.

[16] El Karoui, N., Peng, S. and Quenez, M. C., Backward stochastic differential equations in finance, *Mathematical Finance*, 7(1), 1997, 1–71.

[17] Frei, C., Malamud, S. and Schweizer, M., Convexity bounds for BSDE solutions, with applications to indifference valuation, *working paper*, 2009.

[18] He, S., Wang, J. and Yan, J., Semimartingale Theory and Stochastic Calculus, *CRC press*, 1992.

[19] Henderson, V. and Hobson. D, Utility indifference pricing: an overview, *Indifference Pricing, Princeton University Press* (2009) 44–74.

[20] Henderson, V. and Liang, G., Hedging of vulnerable contingent claims subject to risk aversion, *working paper*, 2010.

[21] Hu, Y., Imkeller, P. and Müller, M., Utility maximization in incomplete markets, *The Annals of Applied Probability*, 15(3), 2005, 1691–1712.

[22] Hu, Y. and Peng, S., Solution of forward-backward stochastic differential equations, *Probability Theory and Related Fields*, 103(2), 1995, 273–283.

[23] Kobylanski, M., Backward stochastic differential equations and partial differential equations with quadratic growth, *The Annals of Probability*, 28(2), 2000, 558–602.

[24] Kurtz, T., The Yamada-Watanabe-Engelbert theorem for general stochastic equations and inequalities, *Electronic Journal of Probability*, 12, 2007, 951–965.

[25] Imkeller, P. and Dos Reis, G., Path regularity and explicit convergence rate for BSDE with truncated quadratic growth, *Stochastic Processes and Their Applications*, 120, 2010, 348-379.

[26] Liang, G., Lyons, T. and Qian, Z., Backward stochastic dynamics on a filtered probability space, *to appear in the Annals of Probability*, 2009.

[27] Liang, G. , Lionnet, A., and Qian, Z., On Girsanov's transform for backward stochastic differential equations, *working paper,* 2010.

[28] Ma, J., Protter, P. and Yong, J., Solving forward-backward stochastic differential equations explicitly–a four step scheme, *Probability Theory and Related Fields*, 98(3), 1994, 339–359.

[29] Ma, J. and Yong, J., Forward-backward stochastic differential equations and their applications, *Lecture Notes in Mathematics, Springer-Verlag*, Berlin, 1999.

[30] Ma, J., Zhang, J. and Zheng, Z., Weak solutions for forward-backward SDEs: a martingale problem approach, *The Annals of Probability*, 36(6), 2008, 2092–2125.

[31] Ma, J. and Zhang, J., On Weak solutions of forward-backward SDEs, *working paper*, 2009.

[32] Mania, M. and Santacroce, M., Exponential utility maximization under partial information, *Finance and Stochastic*, 14(3), 2010, 419–448.

[33] Morlais, M.-A., Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem, *Finance and Stochastics*, 13(1), 2009, 121–150.

[34] Pardoux, É. and Peng, S. G., Adapted solution of a backward stochastic differential equation, *Systems & Control Letters*, 14(1), 1990, 55–61,

[35] Pardoux, É. and Peng, S., Backward stochastic differential equations and quasilinear parabolic partial differential equations, *Stochastic partial differential equations and their applications (Charlotte, NC, 1991), Lecture Notes in Control and Inform. Sci.*, 176, 1992, 200–217.

[36] Pardoux, E. and Tang, S., Forward-backward stochastic differential equations and quasilinear parabolic PDEs, *Probability Theory and Related Fields*, 114(2), 1999, 123–150.

[37] Peng, S. G., Probabilistic interpretation for systems of quasilinear parabolic partial differential equations, *Stochastics and Stochastics Reports*, 37(1-2), 1991, 61–74.

[38] Peng, S. and Wu, Z., Fully coupled forward-backward stochastic differential equations and applications to optimal control, *SIAM Journal on Control and Optimization*, 37(3), 1999, 825–843.

[39] Rouge, R. and El Karoui, N., Pricing via utility maximization and entropy, *Mathematical Finance*, 10(2), 2000, 259–276.

[40] Yong, J., Finding adapted solutions of forward-backward stochastic differential equations: method of continuation, *Probability Theory and Related Fields*, 107(4), 1997, 537–572.

[41] Yong, J. and Zhou, X. Y., Stochastic controls: Hamiltonian systems and HJB equations, *Springer-Verlag*, New York, 1999.

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