

A CONDITION ON DELAY FOR DIFFERENTIAL EQUATIONS WITH DISCRETE STATE-DEPENDENT DELAY

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Abstract. Parabolic differential equations with discrete state-dependent delay are studied. The approach, based on an additional condition on the delay function introduced in [A.V. Rezounenko, Differential equations with discrete state-dependent delay: uniqueness and well-posedness in the space of continuous functions, *Nonlinear Analysis: Theory, Methods and Applications*, 70 (11) (2009), 3978-3986] is developed. We propose and study a *state-dependent* analogue of the condition which is sufficient for the well-posedness of the corresponding initial value problem on the whole space of continuous functions C . The dynamical system is constructed in C and the existence of a compact global attractor is proved.

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1 Introduction

Delay differential equations is one of the oldest branches of the theory of infinite dimensional dynamical systems - theory which describes qualitative properties of systems, changing in time.

We refer to the classical monographs on the theory of ordinary (O.D.E.) delay equations [11, 12, 8, 2, 20]. The theory of partial (P.D.E.) delay equations is essentially less developed since such equations are infinite-dimensional in both time (as delay equations) and space (as P.D.E.s) variables, which makes the analysis more difficult. We refer to some works which are close to the present research [5, 6, 4, 24] and to the monograph [39].

A new class of equations with delays has recently attracted attention of many researchers. These equations have a delay term that may depend on the state of the system, i.e. the delay is **state-dependent** (SDD). Due to this type of delays such equations are inherently nonlinear and their study has begun in the case of ordinary differential equations [21, 23, 22, 17, 33, 34, 18] (for more details see also a recent survey [13], articles [35, 36] and references therein).

Investigations of these equations essentially differ from the ones of equations with constant or time-dependent delays. The underlying main mathematical difficulty of the theory lies in the fact that delay terms with discrete state-dependent delays are not Lipschitz continuous on the space of continuous functions - the main space, on which the classical theory of equations with delays is developed (see [38] for an explicit example of

the non-uniqueness and [13] for more details). It is a common point of view [13] that the corresponding *initial value problem* (IVP) is not generally well-posed in the sense of J. Hadamard [9, 10] in the space of continuous functions (C). This leads to the search of (particular) classes of equations which may be well-posed in the space of continuous functions (C).

Results for partial differential equations with SDD have been obtained only recently in [25](case of distributed delays, weak solutions), [16] (mild solutions, unbounded discrete delay), and [26] (weak solutions, bounded discrete and distributed delays).

The main goal of the present paper is to develop an alternative approach, based on an additional condition (see (H) below) introduced in [27]. We propose and study a *state-dependent* analogue of the condition which is sufficient for the well-posedness of the corresponding initial value problem in the space C . The presented approach includes the possibility when the state-dependent delay function does not satisfy the condition on a subset of the phase space C , but the IVP still be well-posed in the whole space C . This is our second goal which is to connect the approach developed for ODEs (a restriction to a subset of Lipschitz continuous functions) and the approach [27] of a different nature.

Discussing the meaning of the main assumptions (H) and (\widehat{H}) (see below) for applied problems, we hope that these assumptions are the natural mathematical expression of the fact that many differential equations encountered in modeling real world phenomena have a parameter (time $\eta_{ign} > 0$ or $\Theta^\ell > 0$) which is necessary to take into considerations the time changes in the system. The changes not always can be taken into considerations immediately. To this end, the existence of $\eta_{ign} > 0$ or $\Theta^\ell > 0$ (no matter how small the values of $\eta_{ign} > 0$ or $\Theta^\ell > 0$ are!) makes the corresponding initial value problem well-posed.

Having the well-posedness proved, we study the long-time asymptotic behavior of the correspond dynamical system and prove the existence of a compact global attractor.

2 Formulation of the model with state-dependent discrete delay

Let us consider the following parabolic partial differential equation with delay

$$\frac{\partial}{\partial t}u(t, x) + Au(t, x) + du(t, x) = (F(u_t))(x), \quad x \in \Omega, \quad (1)$$

where A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent, so $A : D(A) \rightarrow L^2(\Omega)$ generates an analytic semigroup, Ω is a smooth bounded domain in R^{m_0} , d is a non-negative constant. As usually for delay equations, we denote by u_t the function of $\theta \in [-r, 0]$ by the formula $u_t \equiv u_t(\theta) \equiv u(t+\theta)$. We denote for short $C \equiv C([-r, 0]; L^2(\Omega))$. The norms in $L^2(\Omega)$ and C are denoted by $\|\cdot\|$ and $\|\cdot\|_C$ respectively.

The (nonlinear) delay term $F : C([-r, 0]; L^2(\Omega)) \rightarrow L^2(\Omega)$ has the form

$$F(\varphi) = B(\varphi(-\eta(\varphi))), \quad (2)$$

where (nonlinear) mapping $B : L^2(\Omega) \rightarrow L^2(\Omega)$ is Lipschitz continuous

$$\|B(v^1) - B(v^2)\| \leq L_B \|v^1 - v^2\|, \quad \forall v^1, v^2 \in L^2(\Omega). \quad (3)$$

The function $\eta(\cdot) : C([-r, 0]; L^2(\Omega)) \rightarrow [0, r] \subset R_+$ represents the **state-dependent discrete delay**. It is important to notice that F is *nonlinear* even in the case of linear B .

We consider equation (1) with the following initial condition

$$u|_{[-r, 0]} = \varphi \in C \equiv C([-r, 0]; L^2(\Omega)). \quad (4)$$

Remark 1. *The results presented in this paper could be easily extended to the case of nonlinearity F of the form $F(\varphi) = \sum_k B^k(\varphi(-\eta^k(\varphi)))$ as well as to O.D.E.s, for example, of the following form [29]*

$$\dot{u}(t) + Au(t) + d \cdot u(t) = b(u(t - \eta(u_t))), \quad u(\cdot) \in R^n, d \geq 0. \quad (5)$$

In the last case one simply needs to substitute $L^2(\Omega)$ by R^n and use $C \equiv C([-r, 0]; R^n)$ instead of $C([-r, 0]; L^2(\Omega))$. The function $b : R^n \rightarrow R^n$ is locally Lipschitz continuous and satisfies $\|b(w)\|_{R^n} \leq C_1\|w\|_{R^n} + C_b$ with $C_1, C_b \geq 0$; A is a matrix.

Remark 2. *As an example we could consider nonlocal delay term F (see (2)) with the following mapping*

$$B(v)(x) \equiv \int_{\Omega} b(v(y))f(x - y)dy, \quad x \in \Omega,$$

where $f : \Omega - \Omega \rightarrow R$ is a bounded and measurable function ($|f(z)| \leq M_f, \forall z \in \Omega - \Omega$) and $b : R \rightarrow R$ is a (locally) Lipschitz mapping, satisfying $|b(w)| \leq C_1|w| + C_b$ with $C_i \geq 0$. In this case equation (1) has the form

$$\frac{\partial}{\partial t}u(t, x) + Au(t, x) + du(t, x) = \int_{\Omega} b(u(t - \eta(u_t), y))f(x - y)dy, \quad x \in \Omega.$$

One can easily check that B satisfies (3) with $L_B \equiv L_b M_f |\Omega|$, where L_b is the Lipschitz constant of b , and $|\Omega| \equiv \int_{\Omega} 1 dx$.

Another example is a (local) delay term F (see (2)) with $B(v)(x) \equiv b(v(x)), x \in \Omega$. Equation (1) has the form

$$\frac{\partial}{\partial t}u(t, x) + Au(t, x) + du(t, x) = b(u(t - \eta(u_t), x)), \quad x \in \Omega.$$

An easy calculation show that (3) is satisfied with $L_B \equiv L_b$.

The methods used in our work can be applied to other types of nonlinear and delay P.D.E.s (as well as O.D.E.s). We choose a particular form of nonlinear delay terms F for simplicity and to illustrate our approach on the diffusive Nicholson's blowflies equation (see the end of the article for more details).

3 The existence of mild solutions

In our study we use the standard

Definition 1. *A function $u \in C([-r, T]; L^2(\Omega))$ is called a mild solution on $[-r, T]$ of the initial value problem (1), (4) if it satisfies (4) and*

$$u(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)} \{F(u_s) - d \cdot u(s)\} ds, \quad t \in [0, T]. \quad (6)$$

Proposition 1[27]. *Assume the mapping B is Lipschitz continuous (see (3)) and delay function $\eta(\cdot) : C([-r, 0]; L^2(\Omega)) \rightarrow [0, r] \subset R_+$ is continuous.*

Then for any initial function $\varphi \in C$, initial value problem (1), (4) has a global mild solution which satisfies $u \in C([-r, +\infty); L^2(\Omega))$.

The existence of a mild solution is a consequence of the continuity of $F : C \rightarrow L^2(\Omega)$ (see (1)) which gives the possibility to use the standard method based on Schauder fixed point theorem (see e.g. [39, theorem 2.1, p.46]). The solution is also global (is defined for all $t \geq -r$) since (3) implies $\|F(\varphi)\| \leq L_B\|\varphi\|_C + \|B(0)\|$ and one can apply, for example, [39, theorem 2.3, p.49].

Remark 3. *It is important to notice that even in the case of ordinary differential equations (even scalar) the mapping of the form $\tilde{F}(\varphi) = \tilde{f}(\varphi(-r(\varphi))) : C([-r_0, 0]; R) \rightarrow R$ has a very unpleasant property. The authors in [19, p.3] write "Notice that the functional \tilde{F} is defined on $C([-r_0, 0]; R)$, but it is clear that it is neither differentiable nor locally Lipschitz continuous, whatever the smoothness of \tilde{f} and r ." As a consequence, the Cauchy problem associated with equations with such a nonlinearity "...is **not** well-posed in the space of continuous functions, due to the non-uniqueness of solutions whatever the regularity of the functions \tilde{f} and r " [19, p.2]. See also a detailed discussion in [13].*

Remark 4. *For a study of solutions to equations with a state-dependent delay in the space $C([-r, 0]; E)$ with E not necessarily finite-dimensional Banach space see e.g. [1]*

In this work we concentrate on conditions for the IVP (1), (4) to be well-posed.

4 Main results: uniqueness, well-posedness and asymptotic behavior

As in the previous section, we assume that $\eta : C \rightarrow [0, r]$ is continuous and B is Lipschitz. Unlike to the existence of solutions, the uniqueness is essentially more delicate question in the presence of discrete state-dependent delay (see a classical example of the non-uniqueness in [38]).

Let us remind an important additional assumption on the delay function η , as it was introduced in [27]:

- $\exists \eta_{ign} > 0$ such that η "ignores" values of $\varphi(\theta)$ for $\theta \in (-\eta_{ign}, 0]$ i.e.

$$\exists \eta_{ign} > 0 : \forall \varphi^1, \varphi^2 \in C : \forall \theta \in [-r, -\eta_{ign}], \Rightarrow \varphi^1(\theta) = \varphi^2(\theta) \quad \Longrightarrow \quad \eta(\varphi^1) = \eta(\varphi^2). \quad (H)$$

For examples of delay functions satisfying (H) and the proof of the *uniqueness* of mild solutions (given by Proposition 1) as well as the well-posedness of the IVP (1), (4) see [27].

Remark 5. *It is important to notice that, discussing the condition (H) and its dependence on the value η_{ign} , we see that in the case $\eta_{ign} > r$, one has that the delay function η ignores all values of $\varphi(\theta), \forall \theta \in [-r, 0]$, so $\eta(\varphi) \equiv const, \forall \varphi \in C$ i.e. equation (1) becomes an equation with constant (!) delay. On the other hand, the analogue of assumption*

(H) with $\eta_{ign} = 0$, is trivial since $\varphi^1(\theta) = \varphi^2(\theta)$ for all $\theta \in [-r, 0]$ means $\varphi^1 = \varphi^2$ in C , so $\eta(\varphi^1) = \eta(\varphi^2)$.

Remark 6. It is worth mentioning that the classical case of **constant** delay (see the previous remark) and the corresponding theory forms the basement for the discussed approach, but could be mixed with the approach of non-vanishing delays. In our case the delay η do may vanish (we do **not** assume the existence of $r_0 > 0$ such that $\eta(\varphi) \geq r_0, \forall \varphi$).

In the above condition (H) the semi-interval $(-\eta_{ign}, 0]$ is fixed (we remind that the value η_{ign} could be arbitrary small).

Our goal is to extend the approach based on the condition (H) to a more wide class of state-dependent delay functions where the value η_{ign} is not a constant any more, but a function of the state. Moreover, as an easy additional extension, we also allow the upper bound of the delayed segment to be state-dependent. More precisely, we consider two functions $\Theta^u, \Theta^\ell : C \rightarrow [0, r]$ (upper and low functions), satisfying

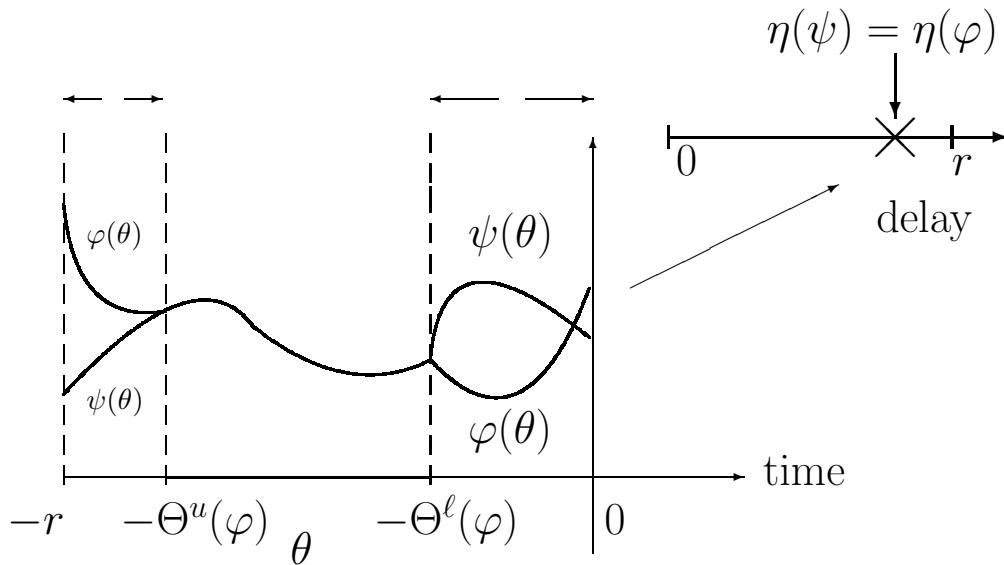
$$\forall \varphi \in C \quad \Rightarrow \quad 0 \leq \Theta^\ell(\varphi) \leq \Theta^u(\varphi) \leq r.$$

Now we are ready to introduce [29] the following *state-dependent* condition for the state-dependent delay function $\eta : C \rightarrow [0, r]$ (c.f. (H)):

- η "ignores" values of $\varphi(\theta)$ for $\theta \notin [-\Theta^u(\varphi), -\Theta^\ell(\varphi)]$ i.e.

$$\forall \psi \in C \text{ such that } \forall \theta \in [-\Theta^u(\varphi), -\Theta^\ell(\varphi)] \Rightarrow \psi(\theta) = \varphi(\theta) \quad \Longrightarrow \quad \eta(\psi) = \eta(\varphi). \quad (\widehat{H})$$

The above condition means that state-dependent delay function η "ignores" all values of its argument φ outside of $[-\Theta^u(\varphi), -\Theta^\ell(\varphi)] \subset [-r, 0]$ and this delayed segment $[-\Theta^u(\varphi), -\Theta^\ell(\varphi)]$ is **state-dependent**. We could illustrate this property on the picture.



Remark 7. One could see that (H) is a particular case of (\widehat{H}) with $\Theta^\ell(\varphi) \equiv \eta_{ign}$ and $\Theta^u(\varphi) \equiv r, \forall \varphi \in C$.

Examples. It is easy to present many examples of (delay) functions η , which satisfy assumption (\widehat{H}) . The simplest one is

$$\eta(\varphi) = p_1(\varphi(-\chi(\varphi(-r)))) \quad \text{with } p_1 : L^2(\Omega) \rightarrow [0, r] \quad (7)$$

and given $\chi : L^2(\Omega) \rightarrow [0, r]$. Here $\Theta^\ell(\varphi) \equiv \chi(\varphi(-r))$ and $\Theta^u(\varphi) = r$. It is easy to see that the above delay function η (7) ignores values of φ at points $\theta \in (-r, -\chi(\varphi(-r))) \cup (-\chi(\varphi(-r)), 0]$ and uses just two values of φ at points $\theta = -r, \theta = -\chi(\varphi(-r))$. In our notations, the delayed segment $[-\Theta^u(\varphi), -\Theta^\ell(\varphi)] = [-r, -\chi(\varphi(-r))]$ is state-dependent.

In the same way, one has

$$\eta(\varphi) = \sum_{k=1}^N p_k(\varphi(-\chi^k(\varphi(-r)))) \quad \text{with } p_k, \chi^k : L^2(\Omega) \rightarrow [0, r].$$

In this case $[-\Theta^u(\varphi), -\Theta^\ell(\varphi)] = [-r, -\min_k \{\chi^k(\varphi(-r))\}]$. A slightly more general example is

$$\eta(\varphi) = \sum_{k=1}^N p_k(\varphi(-\chi^k(\varphi(-r^k)))) \quad \text{with } p_k, \chi^k : L^2(\Omega) \rightarrow [0, r], \quad \min r^k \in (0, r].$$

Here $\Theta^u(\varphi) = \max\{r^1, \dots, r^N, \chi^1(\varphi(-r^1)), \dots, \chi^N(\varphi(-r^N))\}$ and $\Theta^\ell(\varphi) = \min\{r^1, \dots, r^N, \chi^1(\varphi(-r^1)), \dots, \chi^N(\varphi(-r^N))\}$.

Examples of integral delay terms are as follows

$$\eta(\varphi) = \int_{-\chi^2(\varphi(-r^2))}^{-\chi^1(\varphi(-r^1))} p_1(\varphi(\theta))g(\theta) d\theta, \quad \text{and} \quad \eta(\varphi) = p_1 \left(\int_{-\chi^2(\varphi(-r^2))}^{-\chi^1(\varphi(-r^1))} \varphi(\theta)g(\theta) d\theta \right).$$

Similar to the previous example, $\Theta^u(\varphi) = \max\{r^1, r^2, \chi^1(\varphi(-r^1)), \chi^2(\varphi(-r^2))\}$ and $\Theta^\ell(\varphi) = \min\{r^1, r^2, \chi^1(\varphi(-r^1)), \chi^2(\varphi(-r^2))\}$.

Remark 8. It is interesting to notice that an assumption similar to the existence of upper function $\Theta^u(\cdot)$ is used in [37] for ODEs with SDD (locally bounded delay). On the other hand, an assumption similar to (H) is used in [14] for neutral ODEs (see (A4)(ii) in [14]), but together with another assumption on SDD to be bounded from below by a constant $r_0 > 0$ (c.f. remark 6).

Following [27, theorem 1] we have the first result

Theorem 1. Let both upper and low functions $\Theta^u, \Theta^\ell : C \rightarrow [0, r]$ be continuous and $\Theta^\ell(\varphi) > 0, \forall \varphi \in C$. Assume the delay function $\eta : C \rightarrow [0, r] \subset \mathbb{R}_+$ is continuous and satisfies assumption (\widehat{H}) ; the mapping B is Lipschitz continuous (see (3)).

Then for any initial function $\varphi \in C$, initial value problem (1), (4) has an **unique mild solution** $u(t), t \geq 0$ (given by proposition 1).

If we define the **evolution operator** $S_t : C \rightarrow C$ by the formula $S_t \varphi \equiv u_t$, where $u(t)$ is the unique mild solution of (1), (4) with initial function φ , then the pair (S_t, C) constitutes a dynamical system i.e. the following properties are satisfied:

1. $S_0 = Id$ (identity operator in C);
2. $\forall t, \tau \geq 0 \implies S_t S_\tau = S_{t+\tau}$;

3. $t \mapsto S_t$ is a strongly continuous in C mapping;
4. for any $t \geq 0$ the evolution operator S_t is continuous in C i.e. for any $\{\varphi^n\}_{n=1}^\infty \subset C$ such that $\|\varphi^n - \varphi\|_C \rightarrow 0$ as $n \rightarrow \infty$, one has $\|S_t \varphi^n - S_t \varphi\|_C \rightarrow 0$ as $n \rightarrow \infty$.

The proof follows the line of [27, theorem 1] taking into account that condition $\Theta^\ell(\varphi) > 0, \forall \varphi \in C$ implies that for any fixed $\varphi \in C$, due to the continuity of $\Theta^\ell : C \rightarrow [0, r]$, there exists a neighbourhood $U(\varphi) \subset C$ such that for all $\psi \in U(\varphi)$ one has $\Theta^\ell(\psi) \geq \frac{1}{2}\Theta^\ell(\varphi) > 0$. That means that in $U(\varphi) \subset C$ we have the (state-independent) condition (H) with $\eta_{ign} = \frac{1}{2}\Theta^\ell(\varphi) > 0$ and all the arguments presented in [27, theorem 1] could be directly applied to this case. \blacksquare

Remark 9. We do not assume that the upper and low functions Θ^u, Θ^ℓ (which are used in (\widehat{H}) to present the delayed segment $[-\Theta^u(\varphi), -\Theta^\ell(\varphi)]$) are the functions presenting the smallest possible delayed segment. More precisely, it is possible that there exist two other functions $\widetilde{\Theta}^u, \widetilde{\Theta}^\ell$ such that for all $\varphi \in C$ one has $0 \leq \Theta^\ell(\varphi) \leq \widetilde{\Theta}^\ell(\varphi) \leq \widetilde{\Theta}^u(\varphi) \leq \Theta^u(\varphi) \leq r$ and the same delay η satisfies (\widehat{H}) with $\widetilde{\Theta}^u, \widetilde{\Theta}^\ell$ as well.

Our next step in studying the *state-dependent* condition (\widehat{H}) is an attempt to avoid the condition $\Theta^\ell(\varphi) > 0, \forall \varphi \in C$. We are going to consider the general case $\Theta^\ell(\varphi) \geq 0, \forall \varphi \in C$ with a **non-empty** set $Z \equiv \{\varphi \in C : \Theta^\ell(\varphi) = 0\} \neq \emptyset$.

Theorem 2. Assume the mapping B is Lipschitz continuous (see (3)).

Moreover, let the following conditions be satisfied:

- 1) both upper and low functions $\Theta^u, \Theta^\ell : C \rightarrow [0, r]$ are continuous;
- 2) $Z \equiv \{\varphi \in C : \Theta^\ell(\varphi) = 0\} \subset C\mathcal{L}_L \equiv \left\{ \varphi \in C : \sup_{t \neq s} \frac{\|\varphi(t) - \varphi(s)\|}{|t-s|} \leq L \right\}$;
- 3) delay function $\eta : C \rightarrow [0, r] \subset \mathbb{R}_+$ is continuous and satisfies assumption (\widehat{H}) ;
- 4) $\forall \varphi \in Z \Rightarrow \eta(\varphi) > 0$;
- 5) $\exists U_\omega(Z) \equiv \{\chi \in C : \exists \nu \in Z : \|\chi - \nu\|_C \leq \omega\}, \exists L_\eta > 0 : \forall \varphi, \psi \in U_\omega(Z) \Rightarrow$

$$|\eta(\varphi) - \eta(\psi)| \leq L_\eta \cdot \|\varphi - \psi\|_C.$$

Then for any initial function $\varphi \in C$, initial value problem (1), (4) has an **unique mild solution** $u(t), t \geq 0$ (given by proposition 1). Moreover, the pair (S_t, C) constitutes a dynamical system (see thm 1).

Proof of theorem 2. Let us consider $\varphi \in C$ which is an initial condition (see (4)). We start with the simple case $\varphi \notin Z$. By definition of Z , we have $\Theta^\ell(\varphi) > 0$. Hence we apply the same arguments as in the proof of theorem 1 (the state-independent condition (H) is satisfied locally).

The rest of the proof is devoted to the case $\varphi \in Z$. We remind some estimates similar to estimates (6)-(13) in [27]. Denote by $u^k(t)$ any solution of (1),(4) with the initial function φ^k and by $u(t)$ any solution of (1),(4) with the initial function φ .

We use the variation of constants formula for parabolic equation (with $\widetilde{A} \equiv A + d \cdot E$)

$$u(t) = e^{-\widetilde{A}t}u(0) + \int_0^t e^{-\widetilde{A}(t-\tau)}B(u(\tau - \eta(u_\tau))) d\tau, \quad (8)$$

$$u^k(t) = e^{-\tilde{A}t}u^k((0) + \int_0^t e^{-\tilde{A}(t-\tau)}B(u^k(\tau - \eta(u_\tau^k)))d\tau. \quad (9)$$

Using $\|e^{-\tilde{A}t}\| \leq 1$ and $\|e^{-\tilde{A}(t-\tau)}\| \leq 1$, we get

$$\begin{aligned} \|u^k(t) - u(t)\| &\leq \|u^k(0) - u(0)\| + \int_0^t \|B(u^k(\tau - \eta(u_\tau^k))) - B(u(\tau - \eta(u_\tau)))\|d\tau \\ &= \|\varphi^k(0) - \varphi(0)\| + J_1^k(t) + J_2^k(t), \end{aligned} \quad (10)$$

where we denote (for $s \geq 0, x \in \Omega$)

$$J_1^k(s) \equiv J_1^k(s)(x) \equiv \int_0^s \|B(u^k(\tau - \eta(u_\tau^k))) - B(u(\tau - \eta(u_\tau)))\|d\tau, \quad (11)$$

$$J_2^k(s) \equiv J_2^k(s)(x) \equiv \int_0^s \|B(u(\tau - \eta(u_\tau^k))) - B(u(\tau - \eta(u_\tau)))\|d\tau. \quad (12)$$

Using the Lipschitz property (3) of B , one easily gets

$$\begin{aligned} J_1^k(t) &\leq L_B \int_0^t \|u^k(\tau - \eta(u_\tau^k)) - u(\tau - \eta(u_\tau))\|d\tau \\ &\leq L_B t \max_{s \in [-r, t]} \|u^k(s) - u(s)\|. \end{aligned} \quad (13)$$

Estimates (13), (10) and property $J_2^k(s) \leq J_2^k(t)$ for $s \leq t \leq t_0$ give

$$\max_{t \in [0, t_0]} \|u^k(t) - u(t)\| \leq \|\varphi^k(0) - \varphi(0)\| + L_B t_0 \max_{s \in [-r, t_0]} \|u^k(s) - u(s)\| + J_2^k(t_0).$$

Hence

$$\max_{s \in [-r, t_0]} \|u^k(s) - u(s)\| \leq \|\varphi^k - \varphi\|_C + L_B t_0 \max_{s \in [-r, t_0]} \|u^k(s) - u(s)\| + J_2^k(t_0). \quad (14)$$

Now we study properties of J_2^k which essentially differ from the ones in [27] since (H) is not satisfied. The Lipschitz property of B implies

$$J_2^k(t_0) \leq L_B \int_0^{t_0} \|u(\tau - \eta(u_\tau^k)) - u(\tau - \eta(u_\tau))\|d\tau. \quad (15)$$

Since $\varphi \in Z$, property 4) gives $\eta(\varphi) > 0$. Due to the continuity of η (see 3)),

$$\exists U_\alpha(\varphi) \equiv \{\psi \in C : \|\varphi - \psi\|_C \leq \alpha\} : \forall \psi \in U_\alpha(\varphi) \Rightarrow \eta(\psi) \geq \frac{3}{4}\eta(\varphi) > 0. \quad (16)$$

We choose $\alpha < \omega$ (see property 5). By definition, a solution is strongly continuous function (with values in $L^2(\Omega)$), hence for any two solutions $u(t)$ and $u^k(t)$ there exist two time moments $t_\varphi, t_{\varphi^k} > 0$ such that for all $t \in (0, t_\varphi]$ one has $u_t \in U_\alpha(\varphi)$ and for all $t \in (0, t_{\varphi^k}]$ one has $u_t^k \in U_\alpha(\varphi)$.

Remark 10. *More precisely, we assume that $\exists N_\alpha \in \mathbb{N}$ such that for all $k \geq N_\alpha$ one has $\varphi^k \in U_{\alpha/2}(\varphi)$ and hence there exists time moment $t_{\varphi^k} \in (0, t_0]$ such that for all*

$t \in (0, t_{\varphi^k}]$ one has $u_t^k \in U_\alpha(\varphi)$. The last assumption ($\exists N_\alpha \in \mathbb{N} : \forall k \geq N_\alpha \Rightarrow \varphi^k \in U_{\alpha/2}(\varphi)$) is not restrictive since for the uniqueness of solutions we have $\varphi^k = \varphi$ while for the continuity with respect to initial data (see below) we have $\varphi^k \rightarrow \varphi$ in C .

Remark 11. It is important to notice that we take **any** solution from the set of solutions of IVP (1),(4) with the initial function φ (and denote it by $u(t)$) and take **any** solution from the set of solutions of IVP (1),(4) with the initial function φ^k (and denote it by $u^k(t)$) i.e. the values t_φ, t_{φ^k} may depend on the choice of these **two** solutions.

These and (16) imply that for all $\tau \in [0, t_1]$, with $t_1 \leq \min\{t_\varphi; t_{\varphi^k}; \frac{3}{4}\eta(\varphi)\}$ one gets $\tau - \eta(u_\tau) \leq 0, \tau - \eta(u_\tau^k) \leq 0$ and $u(\tau - \eta(u_\tau)) = \varphi(\tau - \eta(u_\tau)), u(\tau - \eta(u_\tau^k)) = \varphi(\tau - \eta(u_\tau^k))$. Hence, see (15) and properties 2), 5),

$$\begin{aligned} J_2^k(t_1) &\leq L_B \int_0^{t_1} \|\varphi(\tau - \eta(u_\tau^k)) - \varphi(\tau - \eta(u_\tau))\| d\tau \leq L_B L \int_0^{t_1} |\eta(u_\tau^k) - \eta(u_\tau)| d\tau \\ &\leq L_B L L_\eta t_1 \max_{s \in [-r, t_1]} \|u^k(s) - u(s)\|. \end{aligned}$$

Finally, we get (see the last estimate and (14))

$$(1 - L_B t_1 [1 + L L_\eta]) \max_{s \in [-r, t_1]} \|u^k(s) - u(s)\| \leq \|\varphi^k - \varphi\|_C.$$

Choosing small enough $t_1 > 0$ (to have $1 - L_B t_1 [1 + L L_\eta] > 0$) i.e.

$$t_1 \equiv \min \left\{ t_\varphi; t_{\varphi^k}; \frac{3}{4}\eta(\varphi); q L_B [1 + L L_\eta]^{-1} \right\} \quad \text{for any fixed } q \in (0, 1), \quad (17)$$

we get

$$\max_{s \in [-r, t_1]} \|u^k(s) - u(s)\| \leq (1 - L_B t_1 [1 + L L_\eta])^{-1} \|\varphi^k - \varphi\|_C. \quad (18)$$

It is easy to see that (18) particularly implies the *uniqueness* of mild solutions to I.V.P. (1),(4) in case when $\varphi^k = \varphi$.

It gives us the possibility to define the **evolution operator** $S_t : C \rightarrow C$ by the formula $S_t \varphi \equiv u_t$, where $u(t)$ is the unique mild solution of (1), (4) with initial function φ .

Our next goal is to prove that pair (S_t, C) constitutes a **dynamical system** (see the properties 1. – 4. as they are formulated in theorem 1). As in [27, p.3981], properties 1, 2 are consequences of the uniqueness of mild solutions. Property 3 is given by Proposition 1 since the solution is a continuous function $u \in C([-r, T]; L^2(\Omega))$.

Let us prove property 4. We consider any sequence $\{\varphi^k\}_{k=1}^\infty \subset C$, which converges (in space C) to φ . Denote by $u^k(t)$ the (unique!) mild solution of (1),(4) with the initial function φ^k and by $u(t)$ the (unique!) mild solution of (1),(4) with the initial function φ .

One could think that (18) already provides the continuity with respect to initial data, but there is an important technical property used in developing (18) i.e. the choice of t_1 (see (17) and remark 11). In contrast to the previous study, now we have infinite set of functions $\{\varphi^k\}_{k=1}^\infty \subset C$, so it may happen that $t_1 = t_1^k \rightarrow 0$ when $k \rightarrow \infty$.

We remind (see the text after (16)) that **two** time moments $t_\varphi, t_{\varphi^k} > 0$ have been chosen such that for all $t \in (0, t_\varphi]$ one has $u_t \in U_\alpha(\varphi)$ and for all $t \in (0, t_{\varphi^k}]$ one has $u_t^k \in U_\alpha(\varphi)$. Now our goal is to show that infinite number of moments $t_\varphi, \{t_{\varphi^k}\}_{k=1}^\infty$ could be chosen in such a way that $t_2 \equiv \inf_{k \in \mathbb{N}} \{t_\varphi, t_{\varphi^k}\} > 0$ and $u_t, u_t^k \in U_\alpha(\varphi)$ for all

$t \in (0, t_2]$. To get this, we use the standard proof of the existence of a mild solution by a fixed point argument (see e.g. [39, p.46, thm 2.1]). More precisely, let U be an open subset of C and $\tilde{F} : [0, b] \times U \rightarrow L^2(\Omega)$ be continuous. For $\varphi \in C$ and any $y \in Y_1 \equiv \{y \in C([-r, t_3]; L^2(\Omega)) : y(0) = \varphi(0)\}$ we consider the extension function \hat{y} as follows

$$\hat{y}(s) \equiv \begin{cases} \varphi(s) & \text{for } s \in [-r, 0]; \\ y(t) & \text{for } s \in (0, t_3] \end{cases}.$$

Let $Y_2 \equiv \{y \in Y_1 : \hat{y}_t \in \overline{B_\delta(\varphi)} \text{ for } t \in [0, t_3]\}$. Consider a mapping G on Y_2 as follows

$$G(y)(t) \equiv e^{-\tilde{A}t}\varphi(0) + \int_0^t e^{-\tilde{A}(t-\tau)}\tilde{F}(\hat{y}_\tau) d\tau.$$

One can check (see [39, p.46,47, thm 2.1]), that G maps Y_2 into Y_2 provided $t_3 \equiv \min\{t'; b; \delta/(3N); \delta\}$. Here we use notations of [39, p.46] chosen as follows. Constants $\delta > 0$ and $N > 0$ are such that $\|\tilde{F}(\psi)\| \leq N$ for all $\psi \in \overline{B_\delta(\varphi)} \equiv \{\psi \in C : \|\psi - \varphi\|_C \leq \delta\}$, $\|e^{-\tilde{A}t}\| \leq M = 1$. The time moment $t' < r$ is chosen so that if $0 \leq t \leq t'$ then $\|\varphi(t + \theta) - \varphi(\theta)\| < \delta/3$ and $\|e^{-\tilde{A}t}\varphi(0) - \varphi(0)\| < \delta/3$. The solution is given by a fixed point $y = G(y)$. For our goal it is sufficient to choose $\delta \leq \alpha$ and $t_2 \leq t_3$ to get $u_t, u_t^k \in U_\alpha(\varphi)$ for all $t \in (0, t_2]$. Here we use φ^k instead of φ when necessary. The crucial point here is the possibility to choose t' (and hence t_3 and t_2) independent of $k \in N$. The choice of $t' < r$ so that if $0 \leq t \leq t'$ then $\|\varphi(t + \theta) - \varphi(\theta)\| < \delta/3$ and $\|\varphi^k(t + \theta) - \varphi^k(\theta)\| < \delta/3$ for all $k \in N$ is possible due to the convergence of φ^k (to φ in C). Since any convergent sequence is a pre-compact set in C , the desired property is the equicontinuity given by the Arzela-Ascoli theorem. Now estimate (18) can be applied to our case and this completes the proof of property 4 and theorem 2. ■

Discussing assumptions of theorem 2, let us present a constructive *example* of low function Θ^ℓ which satisfies assumption 2). Consider any compact and convex set $K_C \subset C\mathcal{L}_L \subset C$. For example, for any compact and convex set $K \in L^2(\Omega)$, the set $\{\varphi \in C : \varphi \in C\mathcal{L}_L, \forall \theta \in [-r, 0] \Rightarrow \varphi(\theta) \in K\}$ is compact (by Arzela-Ascoli theorem) and convex. First, constructing Θ^ℓ , we set $\Theta^\ell(\varphi) = 0$ for all $\varphi \in K_C$. Second, we take any $p \in (0, r]$ and set $\Theta^\ell(\varphi) = p$ for all $\varphi \in C$ such that $\text{dist}_C(\varphi, K_C) \geq 1$. Third, for any $\varphi \in C$ such that $\text{dist}_C(\varphi, K_C) \in (0, 1)$ we find an unique $\hat{\varphi} \in K_C$ such that $\text{dist}_C(\varphi, K_C) = \|\varphi - \hat{\varphi}\|_C$. Such $\hat{\varphi} \in K_C$ exists by the classical Weierstrass theorem since $f(\psi) \equiv \text{dist}_C(\varphi, \psi) : K_C \rightarrow (0, 1)$ is continuous (φ is fixed) and K_C is compact. The uniqueness of $\hat{\varphi}$ follows from the convexity of K_C . Finally, we set $\Theta^\ell(\varphi) = p \cdot \text{dist}_C(\varphi, \hat{\varphi}) \in (0, p)$ for all $\varphi \in C : \text{dist}_C(\varphi, K_C) \in (0, 1)$. By construction, Θ^ℓ satisfies 2).

As for asymptotic behavior, we study of the long-time behavior of the dynamical system (S_t, C) , constructed in theorems 1 and 2. Similar to [27, theorem 2] we have the following result.

Theorem 3. *Assume all the assumptions of theorems 1 or 2 are satisfied and additionally mapping B (see (2)) is bounded. Then the dynamical system (S_t, C) has a compact global attractor \mathcal{A} which is a compact set in all spaces $C_\delta \equiv C([-r, 0]; D(A^\delta)), \forall \delta \in [0, \frac{1}{2})$.*

Lemma. *Let all the assumptions of theorem 2 be satisfied. Then the global attractor \mathcal{A} (see theorem 3) is a subset of $C\mathcal{L}_{\tilde{L}}$ (c.f. condition 2 in theorem 2).*

Remark 12. Lemma gives a possibility to consider system (1), (4) with a state-dependent delay function η which does not ignore values of its argument φ for all points $\varphi \in \mathcal{A}$ i.e. no information is lost on the global attractor \mathcal{A} .

Proof of lemma. Consider any solution $u_t \in \mathcal{A}$. Let us denote $f(t) \equiv F(u_t)$ and prove that f is Hölder continuous.

We will need the following property, proved in [27, estimate (29) with $\delta = 0$]

$$\|u(t_1) - u(t_2)\| \leq L_0 |t_1 - t_2|^{1/2} \quad (19)$$

for any solution, belonging to the ball of dissipation (particularly, for any solution belonging to the attractor). Here L_0 is independent of solution u .

One can check that

$$\|f(t_1) - f(t_2)\| \leq L_B \cdot \|u(t_1 - \eta(u_{t_1})) - u(t_2 - \eta(u_{t_2}))\|. \quad (20)$$

Using (19), the Lipschitz property of η (see 5 in theorem 2), we get from (20) that

$$\begin{aligned} \|f(t_1) - f(t_2)\| &\leq L_B L_0 \cdot |t_1 - \eta(u_{t_1}) - (t_2 - \eta(u_{t_2}))|^{1/2} \\ &\leq L_B L_0 \cdot (|t_1 - t_2| + |\eta(u_{t_1}) - \eta(u_{t_2})|)^{1/2} \leq [\text{using 5 in theorem 2}] \leq \\ &\leq L_B L_0 \cdot (|t_1 - t_2| + L_\eta \|u_{t_1} - u_{t_2}\|)^{1/2} \leq [\text{using (19)}] \leq \\ &\leq L_B L_0 \cdot (|t_1 - t_2| + L_\eta L_0 |t_1 - t_2|^{1/2})^{1/2}. \\ &\leq L_B L_0 \cdot (|t_1 - t_2|^{1/2} + (L_\eta L_0)^{1/2} |t_1 - t_2|^{1/4}). \end{aligned}$$

Finally, for $|t_1 - t_2| < 1$ one has

$$\|f(t_1) - f(t_2)\| \leq L_B L_0 \cdot \{1 + (L_\eta L_0)^{1/2}\} |t_1 - t_2|^{1/4}. \quad (21)$$

Let us consider $\forall \psi \in \mathcal{A}$. It is well-known that the attractor consists of whole trajectories i.e. $u_s \in \mathcal{A}, \forall s \in R$. We take any $t_0 > r > 0$ and get $\varphi \in \mathcal{A}$ such that $S_{t_0} \varphi = \psi$. Consider the variation of constants formula for parabolic equation (with $\tilde{A} \equiv A + d \cdot E$ see (8))

$$u(t) = e^{-\tilde{A}t} \varphi(0) + \int_0^t e^{-\tilde{A}(t-\tau)} F(u_\tau) d\tau. \quad (22)$$

The first term in the above formula (22) is Lipschitz for $t > t_0$ due to the standard estimate $\|e^{-\tilde{A}t_1} v - e^{-\tilde{A}t_2} v\| \leq (t_1 e)^{-1} \|v\| \cdot |t_1 - t_2|$, $0 < t_1 < t_2$. Moreover it is uniformly Lipschitz for any $v = \varphi(0), \varphi \in \mathcal{A}$ since $\|e^{-\tilde{A}t_1} \varphi(0), -e^{-\tilde{A}t_2} \varphi(0)\| \leq (re)^{-1} \|v\| \cdot |t_1 - t_2| \leq (re)^{-1} C(0) \cdot |t_1 - t_2|$, $r < t_1 < t_2$. Here $\|\varphi(0)\| \leq C(0)$ due to the dissipativeness of the dynamical system $(S_t; C)$ (for more details see [27, estimate (23)]).

To prove that the second term in (22) is Lipschitz for $t > t_0$ we need the following

Proposition [15, lemma 3.2.1]. *Let \tilde{A} be a sectorial operator in Banach space X . Assume function $f : (0, T) \rightarrow X$ is locally Hölder continuous and $\int_0^\rho \|f(s)\|_X ds < \infty$ for some $\rho > 0$. Denote by $\Phi(t) \equiv \int_0^t e^{-\tilde{A}(t-s)} f(s) ds$. Then function $\Phi(\cdot)$ is continuous on $[0, T)$, continuously differentiable on $(0, T)$, $\Phi(t) \in D(\tilde{A})$ for $0 < t < T$ and $d\Phi(t)/dt + \tilde{A}\Phi(t) = f(t)$ for $0 < t < T$ and $\Phi(t) \rightarrow 0$ in X as $t \rightarrow 0+$.*

Remark 13. Our operator \tilde{A} is sectorial since any self-adjoint densely defined bounded from below operator in a Hilbert space is sectorial (see e.g. [15, example 2, p.26]).

We apply the above proposition to $f(t) \equiv F(u_t)$ and use (21). The property $\int_0^\rho \|f(s)\|_X ds < \infty$ for some $\rho > 0$ follows from the dissipativeness $\|u(t)\| \leq C(0)$, the continuity of $F : C \rightarrow L^2(\Omega)$ and the strong continuity of mild solution u . One uses the continuous differentiability of Φ on $[t_0 - r, t_0] \subset (0, T)$ which implies that $\max_{t \in [t_0 - r, t_0]} \|\Phi'(t)\| \equiv M_{\Phi;1} < \infty$. In our case Φ represents the second term in (22) which is proved to be Lipschitz continuous with Lipschitz constant $M_{\Phi;1}$ independent of u . The proof of lemma is complete. \blacksquare

Remark 14. One can also easily extend the method developed here to the case of non-autonomous nonlinear delay terms, for example, using another nonlinear function $\hat{b} : R \times R \rightarrow R$ (see remark 2) instead of b to have $(\hat{F}(t, u_t))(x) = \hat{b}(t, u(t - \eta(u_t), x))$ or $(\hat{F}(t, u_t))(x) = \int_\Omega \hat{b}(t, u(t - \eta(u_t), y))f(x - y)dy$ in equation (1).

As an application we can consider the diffusive Nicholson's blowflies equation (see e.g. [31] with state-dependent delays. More precisely, we consider equation (1) where $-A$ is the Laplace operator with the Dirichlet boundary conditions, $\Omega \subset R^n$ is a bounded domain with a smooth boundary, the function f (see remark 2) can be, for example, $f(s) = \frac{1}{\sqrt{4\pi\alpha}}e^{-s^2/4\alpha}$, as in [30] (for the non-local in space variable nonlinearity) or Dirac delta-function to get the local in space variable nonlinearity, the nonlinear function b is given by $b(w) = p \cdot we^{-w}$. Function b is bounded, so for any continuous delay function η , satisfying (\hat{H}) , the conditions of theorems 1,2 are valid. As a result, we conclude that the initial value problem (1),(4) is well-posed in C and the dynamical system (S_t, C) has a global attractor (theorem 3).

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