Rainbow connection number and complement graphs*

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Abstract

A path in an edge-colored graph, where adjacent edges may be colored the same, is a rainbow path if no two edges of it are colored the same. A nontrivial connected graph G is rainbow connected if there is a rainbow path connecting any two vertices, and the rainbow connection number of G, denoted $rc(G)$, is the minimum number of colors that are needed in order to make G rainbow connected. In this paper, we will derive a sufficient condition to guarantee that $rc(G)$ is a constant (here is 8) by giving constraints to its complement graph: For a connected graph G, if $\overline{G}$ does not belong to the following two cases: (i) $diam(\overline{G}) = 2, 3$,$(ii)$ $\overline{G}$ contains two connected components and one of them is trivial, then $rc(G) \leq 8$, where $\overline{G}$ is the complement graph of G and $diam(G)$ is the diameter of G.

Keywords: edge-colored graph, rainbow path, rainbow connection number, complement graph, diameter

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1 Introduction

All graphs in this paper are finite, undirected and simple. Let G be a nontrivial connected graph on which is defined a coloring $c : E(G) \to \{1, 2, \cdots, n\}$, $n \in \mathbb{N}$, of the edges of G, where adjacent edges may be colored the same. A path is a *rainbow path* if no two edges of it are colored the same. An edge-coloring graph G is *rainbow connected* if any two vertices are connected by a rainbow path. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge-coloring that makes it rainbow connected; just color each edge with a distinct color. Thus, we define the *rainbow connection number* of a connected graph G, denoted $rc(G)$, as the smallest number of colors that are needed in order to make G rainbow connected. If G_1 is a connected spanning subgraph of G, then $rc(G) \leq rc(G_1)$. Chartrand et al. obtained that $rc(G) = 1$ if and only if G is complete, and that $rc(G) = m$ if and only if G is a tree, as well as that a cycle with $k > 3$ vertices has rainbow connection number $\lceil \frac{k}{2} \rceil$, a triangle has rainbow connection number 1 ([2]). Also

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notice that, clearly, $rc(G) \geq diam(G)$ where $diam(G)$ denotes the diameter of G. In an edge-colored graph G, we use $c(e)$ to denote the color of edge e, then for a subgraph G_2 of G, $c(G_2)$ denotes the set of colors of edges in G_2. We use $V(G)$, $E(G)$ for the set of vertices and edges of G, respectively. For any subset X of $V(G)$, let $G[X]$ be the subgraph induced by X, and $E[X]$ the edge set of $G[X]$; For a set S, $|S|$ denote the cardinality of S. A path P_n is a path with n vertices. For a connected graph G, the *distance* between two vertices u and v in G, denoted by $dist(u, v)$ is the length of a shortest path between them in G. The *eccentricity* of a vertex v in G is $ecc_G(v) = \max_{x \in V(G)} dist(v, x)$. We follow the notation and terminology of [1].

In this paper, we'll derive a sufficient condition to guarantee that $rc(G)$ is a constant (here is 8) by giving constraints to its complement graph and our main result (Theorem 3.5) is: For a connected graph G, if $\overline{G}$ doesn't belong to the following two cases: (i) $diam(\overline{G}) = 2, 3$,$(ii)$ $\overline{G}$ contains two connected components and one of them is trivial, then $rc(G) \leq 8$. This provides a new approach to investigate $rc(G)$. We also discuss the remaining cases.

2 Basic results

We now give a necessary condition for an edge-colored graph to be rainbow connected. If G is rainbow connected under some edge-coloring, then for any two cut edges (if exist) $e_1 = u_1u_2$, $e_1 = v_1v_2$, there must exist some $1 \leq i, j \leq 2$, such that any $u_i - v_j$ path must contain edge e_1, e_2. So we have:

Observation 2.1 *If G is rainbow connected under some edge-coloring, e_1 and e_2 are any two cut edges, then*

$$c(e_1) \neq c(e_2).$$

∎

The following lemma will be useful in our discussion.

Lemma 2.2 ([3]) *If G is a connected graph and $H_1, \cdots, H_k$ is a partition of the vertex set of G into connected subgraphs then $rc(G) \leq k - 1 + \sum_{i=1}^{k} rc(H_i)$.* ∎

In [2], the authors derived the precise value of rainbow connection number of complete bipartite graph $K_{s,t}(2 \leq s \leq t)$ and complete k-partite graph.

Theorem 2.3 ([2]) *For integers s and t with $2 \leq s \leq t$,*

$$rc(K_{s,t}) = \min\{\lceil \sqrt[s]{t} \rceil, 4\}.$$

∎

Theorem 2.4 ([2]) *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph, where $k \geq 3$ and $n_1 \leq n_2 \leq \ldots \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then*

$$rc(G) = \begin{cases} 1 & \text{if } n_k = 1, \\ 2 & \text{if } n_k \geq 2 \text{ and } s > t, \\ \min\{\lceil \sqrt[s]{t} \rceil, 3\} & \text{if } s \leq t. \end{cases}$$

∎

3 Main Results

We first investigate the rainbow connection numbers of connected complement graphs of graphs with diameter at least 4.

Theorem 3.1 *Let G be a connected graph with $diam(G) \geq 4$, if $\overline{G}$ is connected, then $rc(\overline{G}) \leq 8$.*

Proof. We choose a vertex x with $ecc_G(x) = diam(G) = d \geq 4$. Let $N_G^i(x) = \{v : dist(x, v) = i\}$ where $0 \leq i \leq d$. Then $\bigcup_{0 \leq i \leq d} N_G^i(x)$ is a vertex partition of $V(G)$ with $|N_G^i(x)| = n_i$. Let $A = \bigcup_{i \text{ is even}} N_G^i(x)$, $B = \bigcup_{i \text{ is odd}} N_G^i(x)$. For example, see figure 3.1, a graph with $diam(G) = 4$.

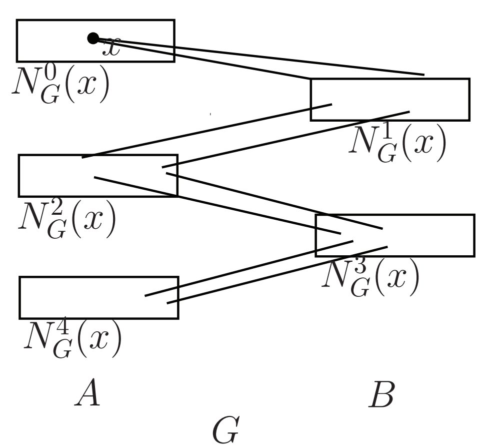

Figure 3.1 Graphs for the example with $d = 4$.

So if $d = 2k (k \geq 2)$, then $A = \bigcup_{0 \leq i \leq d \text{ is even}} N_G^i(x)$, $B = \bigcup_{1 \leq i \leq d-1 \text{ is odd}} N_G^i(x)$; if $d = 2k + 1 (k \geq 2)$, then $A = \bigcup_{0 \leq i \leq d-1 \text{ is even}} N_G^i(x)$, $B = \bigcup_{1 \leq i \leq d \text{ is odd}} N_G^i(x)$. Then by the definition of complement graph, we know $\overline{G}[A] (\overline{G}[B])$ contains a complete k_1-partite spanning subgraph (k_2-partite spanning subgraph) where $k_1 = \lceil \frac{d+1}{2} \rceil (k_2 = \lceil \frac{d}{2} \rceil)$.

Case 1. $d \geq 5$. Then $k_1, k_2 \geq 3$. Then by Lemma 2.2 and Theorem 2.4, we have $rc(\overline{G}) \leq rc(\overline{G}[A]) + rc(\overline{G}[B]) + 1 \leq 3 + 3 + 1 = 7$.

Case 2. $d = 4$, that is, $A = N_G^0(x) \cup N_G^2(x) \cup N_G^4(x)$, $B = N_G^1(x) \cup N_G^3(x)$. So $\overline{G}[A] (\overline{G}[B])$ contains a complete 3-partite spanning subgraph K_{n_0, n_2, n_4} (bipartite spanning subgraph K_{n_1, n_3}). So by Theorem 2.4, we have $rc(\overline{G}[A]) \leq 3$.

Subcase 2.1. $n_1, n_3 \geq 2$. Then by Theorem 2.3, we have $rc(\overline{G}[B]) \leq 4$. Furthermore, by Lemma 2.2, we have $rc(\overline{G}) \leq rc(\overline{G}[A]) + rc(\overline{G}[B]) + 1 \leq 3 + 4 + 1 = 8$.

Subcase 2.2. At least one of n_1, n_3 is 1, say $n_1 = 1$. We give a rainbow coloring to the subgraph $\overline{G}[A]$ using 3 colors. By the definition of complement graph, we know, in $\overline{G}$, there are edges between A and $N_G^1(x)$, we color these edges with a new color a; similarly, there are edges between A and $N_G^3(x)$, we color these edges with a new color b, the edges in $\overline{G}[B]$ receive a new color c. It is easy to show the above coloring is rainbow, and we have $rc(\overline{G}) \leq 6$. Then the conclusion holds. ∎

With a similar argument to that of Theorem 3.1, we have:

Proposition 3.2 *If G is a tree that is not a star, then $rc(\overline{G}) \leq 3$.*

Proof. It is easy to show that if G is a tree that is not a star, then $\overline{G}$ is connected. We now use the same terminology of the argument in Theorem 3.1. Note that A and B are independent sets in G, so $\overline{G}[A]$ and $\overline{G}[B]$ are two disjoint cliques in $\overline{G}$, then by Lemma 2.2, we have $rc(\overline{G}) \leq 3$. ∎

Theorem 3.1 is equivalent to the following result.

Theorem 3.3 *For a connected graph G, if $\overline{G}$ is connected and $diam(\overline{G}) \geq 4$, then $rc(G) \leq 8$.* ∎

For a graph G with $h \geq 2$ connected components, then $\overline{G}$ contains a complete h-partite spanning subgraph, so we have

Proposition 3.4 *G is a graph with $h \geq 2$ connected components G_i and $n_i = n(G_i)(1 \leq i \leq h)$, then $rc(\overline{G}) \leq rc(K_{n_1,\cdots,n_h})$.* ∎

Now we give our main result.

Theorem 3.5 *For a connected graph G, if $\overline{G}$ doesn't belong to the following two cases: (i) $diam(\overline{G}) = 2, 3$,$(ii)$ $\overline{G}$ contains two connected components and one of them is trivial, then $rc(G) \leq 8$.* ∎

Proof. If $\overline{G}$ is connected, as $diam(\overline{G}) \neq 2, 3$ and clearly $diam(\overline{G}) \neq 1$, by Theorem 3.3, we have $rc(G) \leq 8$. If $\overline{G}$ is disconnected, so it has either at least three connected components or two nontrivial components, then by Theorem 2.3, Theorem 2.4 and Proposition 3.4, we have $rc(G) \leq 4$. ∎

4 Discussion for the remaining cases

For the remaining cases, as the complement of the complement graph of a graph G is itself, we need to investigate $rc(\overline{G})$ in two cases:(i) $diam(G) = 2, 3$,(ii) G contains two connected components and one of them is trivial. We use terminology same as that of Theorem 3.1.

Theorem 4.1 *For a vertex x of G satisfying $ecc_G(x) = diam(G) = 3$. We have $rc(\overline{G}) \leq 5$ for three cases (i) $n_1 = n_2 = n_3 = 1$, (ii) $n_1, n_2 = 1, n_3 \geq 2$, (iii) $n_2 = 1, n_1, n_3 \geq 2$. For the remaining cases, $rc(\overline{G})$ may be very large.*

Proof. If $n_1 = n_2 = n_3 = 1$, then G is a 4-path P_4, so $\overline{G} = 3$. So we now consider the following three cases.

Case 1. Two of n_1, n_2, n_3 equal 1.

Subcase 1.1. $n_1, n_2 = 1$. Then it is easy to show that the subgraph $\overline{G}[N_G^0(x) \cup N_G^1(x) \cup N_G^3(x)]$ contains a bipartite spanning subgraph K_{2,n_3}, so by Lemma 2.2 and Theorem 2.3, we have $rc(\overline{G}) \leq rc(K_{2,n_3}) + 1 \leq 5$.

Subcase 1.2. $n_1, n_3 = 1$. Let $n_2' = |\{v \in N_G^2(x) : deg_{\overline{G}}(v) = 1\}|$. Then there are n_2' cut edges in $\overline{G}$, so by Observation 2.1, we have $rc(\overline{G}) \geq n_2'$.

Subcase 1.3. $n_2, n_3 = 1$. With a similar argument to that of **Subcase 1.2**, we have $rc(\overline{G}) \geq n_1'$ where $n_1' = |\{v \in N_G^1(x) : deg_{\overline{G}}(v) = 1\}|$.

Case 2. One of n_1, n_2, n_3 equals 1.

Subcase 2.1. $n_1 = 1$. With a similar argument to that of **Subcase 1.2**, we have $rc(\overline{G}) \geq n_2'$ where $n_2' = |\{v \in N_G^2(x) : deg_{\overline{G}}(v) = 1\}|$.

Subcase 2.2. $n_2 = 1$. Then it is easy to show that the subgraph $\overline{G}[N_G^0(x) \cup N_G^1(x) \cup N_G^3(x)]$ contains a bipartite spanning subgraph K_{1+n_1,n_3}, so by Lemma 2.2 and Theorem 2.3, we have $rc(\overline{G}) \leq rc(K_{1+n_1,n_3}) + 1 \leq 5$.

Subcase 2.3. $n_3 = 1$. With a similar argument to that of **Subcase 1.2**, we have $rc(\overline{G}) \geq n_1' + n_2'$ where $n_i' = |\{v \in N_G^i(x) : deg_{\overline{G}}(v) = 1\}|$ with $i = 1, 2$.

Case 3. $n_1, n_2, n_3 \geq 2$. With a similar argument to that of **Subcase 1.2**, we have $rc(\overline{G}) \geq n_2'$ where $n_2' = |\{v \in N_G^2(x) : deg_{\overline{G}}(v) = 1\}|$.

By the above discussion, we know that $rc(\overline{G}) \leq 5$ for three cases (i) $n_1 = n_2 = n_3 = 1$, (ii) $n_1, n_2 = 1, n_3 \geq 2$, (iii) $n_2 = 1, n_1, n_3 \geq 2$. For the remaining cases, $rc(\overline{G})$ can be very large if $n_i'(i = 1, 2)$ is sufficiently large. ∎

For a graph G with $diam(G) = 2$. Let x be a vertex satisfying $ecc_G(x) = diam(G)$. The two cases: (i) $n_1 = n_2 = n_3 = 1$ and (ii) $n_1 = 1, n_2 \geq 2$ do not hold as in both cases $\overline{G}$ are disconnected and $rc(\overline{G})$ *are undefined*. For the remaining two cases, that is, $n_1 \geq 2, n_2 = 1$, $n_1, n_2 \geq 2$, with a similar argument to that of Theorem 4.1, we have $rc(\overline{G}) \geq n_1' + 1$, $rc(\overline{G}) \geq n_2' + 1$, respectively. So $rc(\overline{G})$ can be very large if n_i' is sufficiently large.

If G contains two connected components, say G_1, G_2. Let $n_1 = |\{v \in G_2 : deg_G(v) = n-2\}|$. Then in $\overline{G}$, there are n_1 pendant vertex and so there are n_1 cut edges, by Observation 2.1, we have $rc(\overline{G}) \geq n_1$. So in this case, we have $rc(\overline{G})$ can be very large if n_1 is sufficiently large.

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