FUNCTIONAL EQUATIONS FOR WENG'S ZETA FUNCTIONS FOR $(G,P)/\mathbb{Q}$

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Abstract. It is shown that Weng's zeta functions associated with arbitrary semisimple algebraic groups defined over the rational number field and their maximal parabolic subgroups satisfy the functional equations.

1. Introduction

Recently, Lin Weng introduced a new class of abelian zeta functions associated to a pair of reductive algebraic group G and its maximal parabolic subgroup P, which are related with constant terms of Eisenstein series. In this paper, we simply refer to these zeta functions as Weng's zeta functions. These are motivated by and closely related to non-abelian zeta functions called "high rank zeta functions" associated with algebraic number fields, which were also introduced by Weng himself from a viewpoint of Arakelov geometry based on Iwasawa's interpretation and Tate's Fourier analysis on adéles. High rank zeta functions are generalizations of the Dedekind zeta functions and in fact, rank one zeta functions coincide with the Dedekind zeta functions up to constant multiples. Hence the study of Weng's zeta functions is not only interesting itself but also suggestive for the study of the Dedekind zeta functions. The profound background, the path to the discovery, and the development of Weng's zeta functions are detailed in his elaborated papers [10–13].

One of the most significant properties for Weng's zeta functions is the behavior of their zeros. Weng conjectured that for any pair (G,P), Weng's zeta functions satisfy certain functional equations and the Riemann hypothesis, as is expected or shown for various kinds of zeta functions. In fact, in some special cases, it was shown in [4, 6–9] that they satisfy standard functional equations and the Riemann hypothesis.

In this paper, we establish the functional equations in arbitrary semisimple cases in a unified way. We will see that the functional equations are governed by the involutions on the Weyl groups (see the last paragraph of Section 3 and Lemma 5.3).

Since the proofs known so far for the Riemann hypothesis for Weng's zeta functions essentially use the functional equations, our result will be a first and important step toward a comprehensive proof of the general Riemann hypothesis. Furthermore we give the explicit forms of Weng's zeta functions and the precise description of the centers for the functional equations (see (2.3)), by which these zeta functions will become more accessible than before.

This paper is organized as follows. In Section 2, we give basic facts about root systems and state the main results. In Section 3, to explain the idea of the general proof, we demonstrate the proof of the functional equation in a simple example, which also explains the symbols used in the next sections. In Section 4, we show some statements about properties of Weyl groups and subsets of roots. The last section is devoted to the proof of the general functional equations.

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2. Weng's zeta functions and their functional equations

We first fix notation and summarize basic facts about root systems and Weyl groups. See [1–3] for the details. Let V be an r-dimensional real vector space equipped with an inner product $\langle\cdot,\cdot\rangle$. Let $\Phi\subset V$ be a root system of rank r and $\Delta=\{\alpha_1,\dots,\alpha_r\}$, its fundamental system. Let

$\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ be the coroot associated with $\alpha \in \Phi$. Let $\Lambda = \{\lambda_1,\ldots,\lambda_r\}$ be the fundamental weights satisfying $\langle\alpha_i^\vee,\lambda_j\rangle = \delta_{ij}$. Let Φ_+ be the corresponding positive system of Φ and $\Phi_- = -\Phi_+$ so that $\Phi = \Phi_+ \cup \Phi_-$. Let

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha = \sum_{j=1}^{r}\lambda_j \tag{2.1}$$

be the Weyl vector. Let $\operatorname{ht}\alpha^\vee = \langle\rho,\alpha^\vee\rangle$ be the height of $\alpha^\vee$.

Let W be the Weyl group generated by simple reflections $\sigma_j : V \to V$ for α_j. For $w \in W$, let $l(w) = |\Phi_w|$ be the length of w, where $\Phi_w = \Phi_+ \cap w^{-1}\Phi_-$. Let w_0 be the longest element of W. Then we have $w_0^2 = \mathrm{id}$, $w_0\Delta = -\Delta$ and $w_0\Phi_+ = \Phi_-$.

Let $\operatorname{Aut}(\Phi)$ be the group of automorphisms of V which preserves Φ. Then $W \subset \operatorname{Aut}(\Phi)$ and W is a normal subgroup of $\operatorname{Aut}(\Phi)$. Let Γ be the Dynkin diagram of Φ and $\operatorname{Aut}(\Gamma)$, the group of automorphisms of Γ. We identify $\operatorname{Aut}(\Gamma)$ with a group of permutations of indices $\{1,\ldots,r\}$. We also regard $\operatorname{Aut}(\Gamma) \subset \operatorname{Aut}(\Phi)$ in a natural way. For $\varpi \in \operatorname{Aut}(\Gamma)$, we have $\varpi\Delta = \Delta$ and $\varpi\Phi_+ = \Phi_+$. In fact, by use of the simple transitivity of W on positive systems, it is easily shown that $\operatorname{Aut}(\Phi) = \operatorname{Aut}(\Gamma) \ltimes W$. Since $-w_0\Delta = \Delta$, we have $-\mathrm{id} = \varpi_0 w_0$ for some $\varpi_0 \in \operatorname{Aut}(\Gamma)$. We see that $\varpi_0^2 = \mathrm{id}$.

In the following, we fix p with $1 \le p \le r$. Let Φ_p be the root system normal to λ_p. A fundamental system of Φ_p is given by $\Delta_p = \Delta \setminus \{\alpha_p\}$. Let $\Phi_{p+} = \Phi_p \cap \Phi_+ \subset \Phi_+$ be the corresponding positive system of Φ_p. Let

$$\rho_p = \frac{1}{2}\sum_{\alpha\in\Phi_{p+}}\alpha. \tag{2.2}$$

Note that $\rho_p \neq \sum_{j\neq p}^{r}\lambda_j$ in general. Let W_p be the Weyl group of Φ_p. Let w_p be the longest element of W_p. Then we have $w_p^2 = \mathrm{id}$, $w_p\Delta_p = -\Delta_p$ and $w_p\Phi_{p+} = \Phi_{p-}$.

Let $\mathbb{N}$ be the set of all positive integers. Throughout this paper, we use the constants

$$c_p = 2\langle\lambda_p - \rho_p, \alpha_p^\vee\rangle \in \mathbb{N}, \tag{2.3}$$

which are important quantities describing the critical lines of Weng's zeta functions. Note that

$$c_p = c_q \tag{2.4}$$

for $q \in \operatorname{Aut}(\Gamma)p$.

Following [12, 13], we introduce Weng's zeta function associated with a semisimple algebraic group G of rank r defined over the rational number field $\mathbb{Q}$ and its maximal parabolic subgroup P. Let Φ be the root system of G, and p be the index for which a simple root $\alpha_p \in \Delta$ corresponds to P. Similarly we use the index q corresponding to another maximal parabolic subgroup Q. For the details of Weng's zeta functions, see [12, 13] and the references therein.

Let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, where ζ is the Riemann zeta function. The poles of $\xi(s)$ are simple and on $s = 0, 1$ with their residues being $-1, 1$ respectively. Moreover we have the functional equation $\xi(1-s) = \xi(s)$. Then the period $\omega_{\mathbb{Q}}^G(\lambda;T)$ for G over $\mathbb{Q}$ is defined as follows.

Definition 2.1 (Periods [13, p.12, Fact E′])**.** For $\lambda, T \in V$,

$$\omega_{\mathbb{Q}}^G(\lambda;T) = \sum_{w\in W} e^{\langle w\lambda-\rho,T\rangle}\Bigl(\prod_{\alpha\in\Delta}\frac{1}{\langle w\lambda-\rho,\alpha^\vee\rangle}\Bigr)\Bigl(\prod_{\alpha\in\Phi_w}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\Bigr), \tag{2.5}$$

$$\omega_{\mathbb{Q}}^G(\lambda) = \omega_{\mathbb{Q}}^G(\lambda;0). \tag{2.6}$$

Put $\Delta_p = \Delta \setminus \{\alpha_p\} = \{\beta_1,\ldots,\beta_{r-1}\}$ and $s = \langle\lambda-\rho,\alpha_p^\vee\rangle$. Let

$$\omega_{\mathbb{Q}}^{G/P}(s;T) = \operatorname*{Res}_{\langle\lambda-\rho,\beta_1^\vee\rangle=0}\cdots\operatorname*{Res}_{\langle\lambda-\rho,\beta_{r-1}^\vee\rangle=0}\omega_{\mathbb{Q}}^G(\lambda;T). \tag{2.7}$$

Then we have the explicit form of $\omega_{\mathbb{Q}}^{G/P}(s;T)$.

Proposition 2.2. *$\omega_{\mathbb{Q}}^{G/P}(s;T)$ is independent of the ordering of Δ_p and is given by*

$$\begin{aligned}\omega_{\mathbb{Q}}^{G/P}(s;T) = &\sum_{\substack{w\in W\\ \Delta_p\subset w^{-1}(\Delta\cup\Phi_-)}} e^{\langle w(\rho+s\lambda_p)-\rho,T\rangle}\Bigl(\prod_{\alpha\in(w^{-1}\Delta)\setminus\Delta_p}\frac{1}{\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee-1}\Bigr)\\ &\times\Bigl(\prod_{\alpha\in\Phi_w\setminus\Delta_p}\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee)\Bigr)\Bigl(\prod_{\alpha\in(-\Phi_w)}\frac{1}{\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee)}\Bigr),\end{aligned} \tag{2.8}$$

From this proposition, we see that $\omega_{\mathbb{Q}}^{G/P}(s;T)$ is a sum of rational functions of ξ functions. Weng's zeta function $\xi_{\mathbb{Q};o}^{G/P}(s;T)$ is defined by multiplying the minimal numbers of ξ functions such that all the denominators are cancelled. To describe the minimal ξ factor, we need the following: for $(k,h)\in\mathbb{Z}^2$,

$$\begin{aligned}M_p(k,h) = \max_{\substack{w\in W\\ \Delta_p\subset w^{-1}(\Delta\cup\Phi_-)}} \bigl(&\sharp\{\alpha\in w^{-1}\Phi_-\mid\langle\lambda_p,\alpha^\vee\rangle=k,\operatorname{ht}\alpha^\vee=h-1\}\\ &-\sharp\{\alpha\in w^{-1}\Phi_-\mid\langle\lambda_p,\alpha^\vee\rangle=k,\operatorname{ht}\alpha^\vee=h\}\bigr).\end{aligned} \tag{2.9}$$

Theorem 2.3.

$$\xi_{\mathbb{Q};o}^{G/P}(s;T) = \omega_{\mathbb{Q}}^{G/P}(s;T)\prod_{k=0}^{\infty}\prod_{h=2}^{\infty}\xi(ks+h)^{M_p(k,h)}. \tag{2.10}$$

Note that $M_p(k,h)\neq 0$ for only finitely many pairs (k,h) and the infinite products in this theorem should be understood as finite products.

Now we have the following functional equations for $\xi_{\mathbb{Q};o}^{G/P}(s;T)$.

Theorem 2.4 (Functional Equations)**.**

$$\begin{aligned}\xi_{\mathbb{Q};o}^{G/P}(-c_p-s;\varpi_0T) &= \xi_{\mathbb{Q};o}^{G/P}(s;T)\\ &= \xi_{\mathbb{Q};o}^{G/Q}(s;\varpi T),\end{aligned} \tag{2.11}$$

where $\varpi\in\operatorname{Aut}(\Gamma)$ with $q=\varpi p$.

From the view point of the classical symmetry $s\leftrightarrow 1-s$, we arrive at the following normalization and functional equations, which immediately follow from Theorem 2.4.

Definition 2.5 (Normalized Weng's zeta function)**.**

$$\xi_{\mathbb{Q}}^{G/P}(s) = \xi_{\mathbb{Q};o}^{G/P}(s-(c_p+1)/2;0). \tag{2.12}$$

Theorem 2.6 (Functional Equations)**.**

$$\begin{aligned}\xi_{\mathbb{Q}}^{G/P}(1-s) &= \xi_{\mathbb{Q}}^{G/P}(s)\\ &= \xi_{\mathbb{Q}}^{G/Q}(s),\end{aligned} \tag{2.13}$$

where $q\in\operatorname{Aut}(\Gamma)p$.

Conjecture 2.7 (Riemann Hypothesis [12,13])**.** All zeros of the zeta function $\xi_{\mathbb{Q}}^{G/P}(s)$ lie on the central line $\Re s=\dfrac{1}{2}$.

In the cases A_1, A_2, B_2 and G_2, this conjecture was already confirmed in [6–9].

Remark 2.8. In [5], a weak version of Conjecture 2.7 is proved in arbitrary root systems. Furthermore in [5, Corollary 8.7], a case-by-case investigation shows that the maximum in the definition (2.9) is attained by the longest element w_0, and hence we have

$$M_p(k,h)=\sharp\{\alpha\in\Phi_+\mid\langle\lambda_p,\alpha^\vee\rangle=k,\operatorname{ht}\alpha^\vee=h-1\}-\sharp\{\alpha\in\Phi_+\mid\langle\lambda_p,\alpha^\vee\rangle=k,\operatorname{ht}\alpha^\vee=h\}. \tag{2.14}$$

In particular,

$$M_p(0,h)=\sharp\{\alpha\in\Phi_{p+}\mid\ \operatorname{ht}\alpha^\vee=h-1\}-\sharp\{\alpha\in\Phi_{p+}\mid\ \operatorname{ht}\alpha^\vee=h\}. \tag{2.15}$$

Thus we obtain

$$\xi_{\mathbb{Q};o}^{G/P}(s;T)=\omega_{\mathbb{Q}}^{G/P}(s;T)\prod_{j=1}^{r-1}\xi(d_j)\prod_{k=1}^{\infty}\prod_{h=2}^{\infty}\xi(ks+h)^{M_p(k,h)}, \tag{2.16}$$

where d_j $(1\leq j\leq r-1)$ are the degrees of the Weyl group W_p (see [3] for the details).

3. Example

To explain the idea and to clarify the roles of the symbols appearing in this paper, we give an example in the case of type A_2 (i.e. $G=\mathrm{SL}(3)$) and $p=1$.

Let $\Delta=\{\alpha_1,\alpha_2\}$ be a fundamental system and $\Phi_+=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2\}$, the corresponding positive system, and $\rho=\alpha_1+\alpha_2$. Let $\{\lambda_1,\lambda_2\}$ be the fundamental weights. The Weyl group is given by

$$W=\{\mathrm{id},\sigma_1,\sigma_2,\sigma_1\sigma_2,\sigma_2\sigma_1,\sigma_1\sigma_2\sigma_1=\sigma_2\sigma_1\sigma_2=w_0\}, \tag{3.1}$$

where w_0 is the longest element. We have $\Phi_{1+}=\{\alpha_2\}$ and the longest element $w_1=\sigma_2$ of the Weyl group of Φ_1.

Put $\lambda=\rho+s_1\lambda_1+s_2\lambda_2$.

	$w^{-1}\Delta$	$\Phi_w=\Phi_+\cap w^{-1}\Phi_-$	w_0ww_1
id	$\{\alpha_1,\alpha_2\}$	$\emptyset$	$\sigma_2\sigma_1$
σ_1	$\{-\alpha_1,\alpha_1+\alpha_2\}$	$\{\alpha_1\}$	σ_1
σ_2	$\{\alpha_1+\alpha_2,-\alpha_2\}$	$\{\alpha_2\}$	$\sigma_1\sigma_2\sigma_1$
$\sigma_2\sigma_1$	$\{\alpha_2,-\alpha_1-\alpha_2\}$	$\{\alpha_1,\alpha_1+\alpha_2\}$	id
$\sigma_1\sigma_2$	$\{-\alpha_1-\alpha_2,\alpha_1\}$	$\{\alpha_2,\alpha_1+\alpha_2\}$	$\sigma_1\sigma_2$
$\sigma_1\sigma_2\sigma_1=w_0$	$\{-\alpha_1,-\alpha_2\}$	$\{\alpha_1,\alpha_2,\alpha_1+\alpha_2\}=\Phi_+$	σ_2

From the above table, we obtain

$$\begin{aligned}\omega_{\mathbb{Q}}^{G}(\lambda)&=\frac{1}{s_1s_2}+\frac{1}{(-s_1-2)(s_1+s_2+1)}\frac{\xi(s_1+1)}{\xi(s_1+2)}+\frac{1}{(s_1+s_2+1)(-s_2-2)}\frac{\xi(s_2+1)}{\xi(s_2+2)}\\&+\frac{1}{s_2(-s_1-s_2-3)}\frac{\xi(s_1+1)\xi(s_1+s_2+2)}{\xi(s_1+2)\xi(s_1+s_2+3)}+\frac{1}{(-s_1-s_2-3)s_1}\frac{\xi(s_2+1)\xi(s_1+s_2+2)}{\xi(s_2+2)\xi(s_1+s_2+3)}\\&+\frac{1}{(-s_2-2)(-s_1-2)}\frac{\xi(s_1+1)\xi(s_2+1)\xi(s_1+s_2+2)}{\xi(s_1+2)\xi(s_2+2)\xi(s_1+s_2+3)}.\end{aligned} \tag{3.2}$$

By putting $s_1=s$ and taking the residue at $s_2=0$, we obtain

$$\begin{aligned}\omega_{\mathbb{Q}}^{G/P}(s)=\operatorname*{Res}_{s_2=0}\omega_{\mathbb{Q}}^{G}(\lambda)&=\frac{1}{s}+0+\frac{1}{(s+1)(-2)}\frac{1}{\xi(2)}\\&+\frac{1}{(-s-3)}\frac{\xi(s+1)\xi(s+2)}{\xi(s+2)\xi(s+3)}+\frac{1}{(-s-3)s}\frac{\xi(s+2)}{\xi(2)\xi(s+3)}\\&+\frac{1}{(-2)(-s-2)}\frac{\xi(s+1)\xi(s+2)}{\xi(s+2)\xi(2)\xi(s+3)}.\end{aligned} \tag{3.3}$$

By multiplying the formal common ξ factor

$$F_1(s)=\xi(-s-1)\xi(-1)\xi(-s-2)=\xi(s+2)\xi(2)\xi(s+3), \tag{3.4}$$

we define

$$\begin{aligned}Z_1(s)&=F_1(s)\omega_{\mathbb{Q}}^{G/P}(s)\\&=\frac{1}{s}\xi(s+2)\xi(2)\xi(s+3)+0+\frac{1}{(s+1)(-2)}\xi(s+2)\xi(s+3)\\&+\frac{1}{(-s-3)}\xi2\xi(s+1)\xi(s+2)+\frac{1}{(-s-3)s}\xi(s+2)^2\\&+\frac{1}{(-2)(-s-2)}\xi(s+1)\xi(s+2).\end{aligned} \tag{3.5}$$

It can be directly checked that

$$Z_1(-3-s)=Z_1(s), \tag{3.6}$$

where the term corresponding to w is exchanged for that corresponding to w_0ww_1. Note that $2\rho_1=\alpha_2$ and

$$c_1=2\langle\lambda_1-\rho_1,\alpha_1^\vee\rangle=3. \tag{3.7}$$

We have shown that $Z_1(s)$ satisfies the functional equation. It is, however, not Weng's zeta function because $F_1(s)$ is not the minimal ξ factor. To obtain Weng's zeta function, we need the minimal ξ factor such that all the true denominators are cancelled in $\omega_{\mathbb{Q}}^{G/P}(s)$. It is read off from (3.3) as

$$\xi(2)\xi(s+3)=\frac{F_1(s)}{D_1(s)}, \tag{3.8}$$

where $D_1(s)=\xi(s+2)$, which itself satisfies the functional equation

$$D_1(-3-s)=D_1(s). \tag{3.9}$$

Due to the symmetries (3.6) and (3.9), we conclude that Weng's zeta function

$$\xi_{\mathbb{Q};o}^{G/P}(s)=\Big(\frac{F_1(s)}{D_1(s)}\Big)\omega_{\mathbb{Q}}^{G/P}(s)=\frac{Z_1(s)}{D_1(s)} \tag{3.10}$$

satisfies the functional equation

$$\xi_{\mathbb{Q};o}^{G/P}(-3-s)=\xi_{\mathbb{Q};o}^{G/P}(s). \tag{3.11}$$

Note that in (3.8), we see that $\xi(2)=\xi(d_1)$, where $d_1=2$ is the degree of the Weyl group of type A_1.

In general cases, this procedure works well and we prove the functional equations in the following sections in this strategy. As we remarked in the introduction, we see that the map $\iota: W\to W$ defined by $w\mapsto w_0ww_p$ plays an important role in (3.6); ι is an involution, namely $\iota^2=\mathrm{id}$, and governs the functional equations at the level of the Weyl group.

4. Preliminaries

In this section, we prove some statements about root systems which is used in the proof of the functional equations.

Lemma 4.1.

$$c_p\lambda_p-w_p\rho=\rho. \tag{4.1}$$

Proof. For $\alpha\in\Phi_{p+}$, we have $w_p\alpha\in\Phi_{p-}\subset\Phi_-$ by the definition of w_p. For $\alpha\in\Phi_+\setminus\Phi_{p+}$, we have $w_p\alpha\in\Phi_+$ since α is of the form $a_p\alpha_p+\cdots$ with $a_p>0$ and $w_p\alpha=a_p\alpha_p+\cdots$ remains positive. Hence we obtain

$$\Phi_{w_p}=\Phi_+\cap w_p^{-1}\Phi_-=\Phi_+\cap(-w_p\Phi_+)=\Phi_{p+} \tag{4.2}$$

and

$$w_p\rho=\rho-\sum_{\alpha\in\Phi_{w_p}}\alpha=\rho-\sum_{\alpha\in\Phi_{p+}}\alpha=\rho-2\rho_p. \tag{4.3}$$

By the property $\sigma_k\Phi_{p+}=(\Phi_{p+}\setminus\{\alpha_k\})\cup\{-\alpha_k\}$ for $k\neq p$, we have

$$\sigma_k\rho_p=\rho_p-\alpha_k=\rho_p-\langle\rho_p,\alpha_k^\vee\rangle\alpha_k, \tag{4.4}$$

which implies $\langle\rho_p,\alpha_k^\vee\rangle=1$. Therefore

$$\rho_p=\sum_{k=1}^r\langle\rho_p,\alpha_k^\vee\rangle\lambda_k=\sum_{k\neq p}\lambda_k+\langle\rho_p,\alpha_p^\vee\rangle\lambda_p=\rho+\langle\rho_p-\lambda_p,\alpha_p^\vee\rangle\lambda_p. \tag{4.5}$$

Combining (4.3) and (4.5), we have

$$c_p\lambda_p-w_p\rho=\rho+(c_p+2\langle\rho_p-\lambda_p,\alpha_p^\vee\rangle)\lambda_p=\rho. \tag{4.6}$$

□

Lemma 4.2. (1) *For $w \in W$, $\Delta_p \subset w^{-1}(\Delta \cup \Phi_-)$ if and only if $\Delta_p \subset w_p w^{-1} w_0(\Delta \cup \Phi_-)$.*

(2) *For $w \in W$ and $\varpi \in \mathrm{Aut}(\Gamma)$ with $q = \varpi p$, $\Delta_p \subset w^{-1}(\Delta \cup \Phi_-)$ if and only if $\Delta_q \subset \varpi w^{-1} \varpi^{-1}(\Delta \cup \Phi_-)$.*

Proof. (1) We see that $\Delta_p \subset w_p w^{-1} w_0(\Delta \cup \Phi_-)$ is equivalent to $-\Delta_p \subset w^{-1}(-\Delta \cup \Phi_+)$ and hence to $\Delta_p \subset w^{-1}(\Delta \cup \Phi_-)$.

(2) It follows from $\varpi\Delta_p = \Delta_q$, $\varpi\Delta = \Delta$ and $\varpi\Phi_- = \Phi_-$.

□

For $w \in W$ and $(k,h) \in \mathbb{Z}^2$, let

$$\begin{aligned} N_{p,w}(k,h) &= \sharp\{\alpha \in w^{-1}\Phi_- \mid \langle \lambda_p, \alpha^\vee \rangle = k, \operatorname{ht}\alpha^\vee = h\}, \\ N_p(k,h) &= \sharp\{\alpha \in \Phi \mid \langle \lambda_p, \alpha^\vee \rangle = k, \operatorname{ht}\alpha^\vee = h\}. \end{aligned} \tag{4.7}$$

We note that $N_{p,w}(k,h) \neq 0$ for finite numbers of pairs $(k,h) \in \mathbb{Z}^2$ and that for $(k,h) \in \mathbb{Z}^2$ with $k \geq 1$ or $h \geq 1$,

$$\begin{aligned} N_{p,w}(k,h) &= \sharp\{\alpha \in \Phi_+ \cap w^{-1}\Phi_- \mid \langle \lambda_p, \alpha^\vee \rangle = k, \operatorname{ht}\alpha^\vee = h\}, \\ N_p(k,h) &= \sharp\{\alpha \in \Phi_+ \mid \langle \lambda_p, \alpha^\vee \rangle = k, \operatorname{ht}\alpha^\vee = h\} \end{aligned} \tag{4.8}$$

because $\alpha \in \Phi$ is either $\alpha \in \Phi_+$ or $\alpha \in \Phi_-$.

Consider the character of the dual Lie algebra ignoring the Cartan subalgebra

$$X(\nu) = \sum_{\alpha \in \Phi} e^{\alpha^\vee}(\nu) \tag{4.9}$$

for $\nu \in V$, where

$$e^{\alpha^\vee}(\nu) = e^{\langle \nu, \alpha^\vee \rangle} \tag{4.10}$$

as usual. Then

$$X(t\lambda_p + \rho) = \sum_{\alpha \in \Phi} e^{\langle \lambda_p, \alpha^\vee \rangle t + \operatorname{ht}\alpha^\vee} = \sum_{k=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} N_p(k,h) e^{kt+h}. \tag{4.11}$$

Note that for $\nu \in V$ and $w \in \mathrm{Aut}(\Phi)$,

$$X(\nu) = X(w\nu). \tag{4.12}$$

Lemma 4.3. (1) *For $(k,h) \in \mathbb{Z}^2$,*

$$N_p(k, kc_p - h) = N_p(k,h). \tag{4.13}$$

(2) *For $(k,h) \in \mathbb{Z}^2$ and $q \in \mathrm{Aut}(\Gamma)p$,*

$$N_p(k,h) = N_q(k,h). \tag{4.14}$$

Proof. (1) We have

$$X((c_p + t)\lambda_p - \rho) = X(t\lambda_p + w_p\rho) = X(w_p(t\lambda_p + \rho)) = X(t\lambda_p + \rho) \tag{4.15}$$

by Lemma 4.1. Hence (4.13) by comparing the coefficients.

(2) Since for $\varpi \in \mathrm{Aut}(\Phi)$ such that $q = \varpi p$,

$$X(t\lambda_p + \rho) = X(\varpi(t\lambda_p + \rho)) = X(t\lambda_q + \rho), \tag{4.16}$$

we have (4.14).

□

Lemma 4.4. (1) *For $(k,h) \in \mathbb{Z}^2$,*

$$N_p(k,h) - N_{p,w_0 w w_p}(k, kc_p - h) = N_{p,w}(k,h). \tag{4.17}$$

(2) *For $(k,h) \in \mathbb{Z}^2$ and $\varpi \in \mathrm{Aut}(\Gamma)$ with $q = \varpi p$,*

$$N_{p,w}(k,h) = N_{q,\varpi w \varpi^{-1}}(k,h). \tag{4.18}$$

Proof. (1) Since

$$\begin{aligned}\Phi &= w^{-1}\Phi_- \cup w^{-1}\Phi_+ \\ &= w^{-1}\Phi_- \cup w_p(w_p w^{-1} w_0)\Phi_-,\end{aligned} \tag{4.19}$$

we have

$$\begin{aligned}X(t\lambda_p+\rho) &= \sum_{\alpha\in w^{-1}\Phi_-} e^{\alpha^\vee}(t\lambda_p+\rho) + \sum_{\alpha\in w_p(w_0ww_p)^{-1}\Phi_-} e^{\alpha^\vee}(t\lambda_p+\rho) \\ &= \sum_{\alpha\in w^{-1}\Phi_-} e^{\alpha^\vee}(t\lambda_p+\rho) + \sum_{\alpha\in (w_0ww_p)^{-1}\Phi_-} e^{\alpha^\vee}((c_p+t)\lambda_p-\rho).\end{aligned} \tag{4.20}$$

By comparing this with (4.11), we obtain (4.17).

(2) We have

$$\sum_{\alpha\in w^{-1}\Phi_-} e^{\alpha^\vee}(t\lambda_p+\rho) = \sum_{\alpha\in \varpi w^{-1}\varpi^{-1}\Phi_-} e^{\alpha^\vee}(\varpi(t\lambda_p+\rho)) = \sum_{\alpha\in (\varpi w\varpi^{-1})^{-1}\Phi_-} e^{\alpha^\vee}(t\lambda_q+\rho), \tag{4.21}$$

which implies (4.18).

□

5. Proof of the functional equations

Proof of Proposition 2.2. Put the coordinate

$$\lambda = \sum_{k=1}^r (1+s_k)\lambda_k = \rho + \sum_{k=1}^r s_k\lambda_k, \tag{5.1}$$

so that for $\alpha^\vee = \sum_{k=1}^r a_k\alpha_k^\vee$,

$$\langle \lambda-\rho,\alpha^\vee\rangle = \sum_{k=1}^r a_k s_k. \tag{5.2}$$

For $w\in W$, the corresponding term in (2.5) besides the exponential factor is calculated as

$$\begin{aligned}A_w &= \Bigl(\prod_{\alpha\in\Delta}\frac{1}{\langle w\lambda-\rho,\alpha^\vee\rangle}\Bigr)\Bigl(\prod_{\alpha\in\Phi_w}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\Bigr) \\ &= \Bigl(\prod_{\alpha\in\Delta}\frac{1}{\langle w\lambda,\alpha^\vee\rangle-1}\Bigr)\Bigl(\prod_{\alpha\in\Phi_w\cap\Delta_p}\frac{1}{\langle\lambda,\alpha^\vee\rangle-1}\Bigr) \\ &\qquad\times\Bigl(\prod_{\alpha\in\Phi_w\cap\Delta_p}\frac{(\langle\lambda,\alpha^\vee\rangle-1)\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\Bigr)\Bigl(\prod_{\alpha\in\Phi_w\setminus\Delta_p}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\Bigr) \\ &= \Bigl(\prod_{\alpha\in(w^{-1}\Delta\cup\Phi_w)\cap\Delta_p}\frac{1}{\langle\lambda,\alpha^\vee\rangle-1}\Bigr)\Bigl(\prod_{\alpha\in(w^{-1}\Delta)\setminus\Delta_p}\frac{1}{\langle\lambda,\alpha^\vee\rangle-1}\Bigr) \\ &\qquad\times\Bigl(\prod_{\alpha\in\Phi_w\cap\Delta_p}\frac{(\langle\lambda,\alpha^\vee\rangle-1)\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\Bigr)\Bigl(\prod_{\alpha\in\Phi_w\setminus\Delta_p}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\Bigr).\end{aligned} \tag{5.3}$$

In order to put $s_p=s$ and take all the residues at $s_k=0$ for $k\neq p$ in (5.3), first we consider the third factor of the last member of (5.3). For $\alpha_k\in\Phi_w\cap\Delta_p$, we have

$$\frac{(\langle\lambda,\alpha_k^\vee\rangle-1)\xi(\langle\lambda,\alpha_k^\vee\rangle)}{\xi(\langle\lambda,\alpha_k^\vee\rangle+1)} = \frac{s_k\xi(s_k+1)}{\xi(s_k+2)} = \frac{1}{\xi(2)}+o(s_k) \tag{5.4}$$

when $s_k\to 0$.

In the last factor, for $\alpha\in\Phi_w\setminus\Delta_p$, we have

$$\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)} = \frac{\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee)}{\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee+1)} \tag{5.5}$$

when $s_k=0$ for $k\neq p$ and $s_p=s$. If $\langle\lambda_p,\alpha^\vee\rangle=0$, then $\alpha\in\Phi_{p+}\setminus\Delta_p$ and hence $\operatorname{ht}\alpha^\vee\geq 2$. Thus we see that (5.5) is finite if $\langle\lambda_p,\alpha^\vee\rangle=0$, due to $\operatorname{ht}\alpha^\vee\geq 2$. Moreover it is also finite for appropriate $s\in\mathbb{C}$ if $\langle\lambda_p,\alpha^\vee\rangle\neq 0$.

We consider the second factor of the last member of (5.3). When $s_k=0$ for $k\neq p$ and $s_p=s$, we have

$$\langle\lambda,\alpha^\vee\rangle-1=\langle\lambda-\rho,\alpha^\vee\rangle+\operatorname{ht}\alpha^\vee-1=\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee-1. \tag{5.6}$$

Since for $\alpha\in(w^{-1}\Delta)\setminus\Delta_p$, $\operatorname{ht}\alpha^\vee\neq 1$ or $\langle\lambda_p,\alpha^\vee\rangle\neq 0$ holds, (5.6) does not vanish identically.

The first factor is calculated as

$$\prod_{\alpha\in(w^{-1}\Delta\cup\Phi_w)\cap\Delta_p}\frac{1}{\langle\lambda,\alpha^\vee\rangle-1}=\prod_{\alpha_k\in(w^{-1}\Delta\cup\Phi_w)\cap\Delta_p}\frac{1}{s_k}. \tag{5.7}$$

Hence from (5.4), (5.5), (5.6) and (5.7), we see that when we take all the residues, only the terms with $\Delta_p\subset w^{-1}\Delta\cup\Phi_w$ survive and the others vanish. In the former cases, we obtain

$$\begin{aligned}\operatorname*{Res}_{\substack{s_k=0\\k\neq p}}A_w=&\Biggl(\prod_{\alpha\in(w^{-1}\Delta)\setminus\Delta_p}\frac{1}{\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee-1}\Biggr)\\&\times\Biggl(\prod_{\alpha\in\Phi_w\cap\Delta_p}\frac{1}{\xi(2)}\Biggr)\Biggl(\prod_{\alpha\in\Phi_w\setminus\Delta_p}\frac{\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee)}{\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee+1)}\Biggr),\end{aligned} \tag{5.8}$$

which does not depend on the ordering of Δ_p.

Note that $\Delta_p\subset w^{-1}\Delta\cup\Phi_w$ if and only if $\Delta_p\subset w^{-1}(\Delta\cup\Phi_-)$. If we put $s_p=s$ and take all the residues at $s_k=0$ for $k\neq p$ in (2.5), we get

$$\begin{aligned}\omega_{\mathbb{Q}}^{G/P}(s;T)=&\sum_{\substack{w\in W\\\Delta_p\subset w^{-1}(\Delta\cup\Phi_-)}}e^{\langle w(\rho+s\lambda_p)-\rho,T\rangle}\Biggl(\prod_{\alpha\in(w^{-1}\Delta)\setminus\Delta_p}\frac{1}{\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee-1}\Biggr)\\&\times\Biggl(\prod_{\alpha\in\Phi_w\cap\Delta_p}\frac{1}{\xi(2)}\Biggr)\Biggl(\prod_{\alpha\in\Phi_w\setminus\Delta_p}\frac{\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee)}{\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee+1)}\Biggr)\\=&\sum_{\substack{w\in W\\\Delta_p\subset w^{-1}(\Delta\cup\Phi_-)}}e^{\langle w(\rho+s\lambda_p)-\rho,T\rangle}\Biggl(\prod_{\alpha\in(w^{-1}\Delta)\setminus\Delta_p}\frac{1}{\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee-1}\Biggr)\\&\times\Biggl(\prod_{\alpha\in\Phi_w\setminus\Delta_p}\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee)\Biggr)\Biggl(\prod_{\alpha\in\Phi_w}\frac{1}{\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee+1)}\Biggr)\\=&\sum_{\substack{w\in W\\\Delta_p\subset w^{-1}(\Delta\cup\Phi_-)}}e^{\langle w(\rho+s\lambda_p)-\rho,T\rangle}\Biggl(\prod_{\alpha\in(w^{-1}\Delta)\setminus\Delta_p}\frac{1}{\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee-1}\Biggr)\\&\times\Biggl(\prod_{\alpha\in\Phi_w\setminus\Delta_p}\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee)\Biggr)\Biggl(\prod_{\alpha\in(-\Phi_w)}\frac{1}{\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee)}\Biggr),\end{aligned} \tag{5.9}$$

where in the last equality, we used the functional equation for $\xi(s)$. □

Let

$$F_p(s)=\prod_{\alpha\in\Phi_-}\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee), \tag{5.10}$$

and define

$$Z_p(s;T)=F_p(s)\omega_{\mathbb{Q}}^{G/P}(s;T). \tag{5.11}$$

Then we have the following proposition.

Proposition 5.1 (Functional Equations)**.**

$$\begin{aligned} Z_p(-c_p - s; \varpi_0 T) &= Z_p(s;T) \\ &= Z_q(s;\varpi T), \end{aligned} \tag{5.12}$$

where $\varpi \in \mathrm{Aut}(\Gamma)$ *with* $q = \varpi p$.

To show this proposition, we need the explicit form of $Z_p(s;T)$.

Proposition 5.2.

$$\begin{aligned} Z_p(s;T) = \sum_{\substack{w\in W \\ \Delta_p \subset w^{-1}(\Delta\cup\Phi_-)}} & e^{\langle w(\rho+s\lambda_p)-\rho, T\rangle} \Biggl(\prod_{\alpha\in(w^{-1}\Delta)\setminus\Delta_p} \frac{1}{\langle\lambda_p,\alpha^\vee\rangle s + \mathrm{ht}\,\alpha^\vee - 1}\Biggr) \\ & \times \Biggl(\prod_{\alpha\in(w^{-1}\Phi_-)\setminus\Delta_p} \xi(\langle\lambda_p,\alpha^\vee\rangle s + \mathrm{ht}\,\alpha^\vee)\Biggr). \end{aligned} \tag{5.13}$$

Proof. Since

$$\begin{aligned} \Phi_- \setminus (-\Phi_w) &= \Phi_- \setminus (\Phi_- \cap w^{-1}\Phi_+) \\ &= \Phi_- \setminus w^{-1}\Phi_+ \\ &= \Phi_- \cap w^{-1}\Phi_-, \end{aligned} \tag{5.14}$$

we have

$$\begin{aligned} Z_p(s;T) = \sum_{\substack{w\in W \\ \Delta_p \subset w^{-1}(\Delta\cup\Phi_-)}} & e^{\langle w(\rho+s\lambda_p)-\rho, T\rangle} \Biggl(\prod_{\alpha\in(w^{-1}\Delta)\setminus\Delta_p} \frac{1}{\langle\lambda_p,\alpha^\vee\rangle s + \mathrm{ht}\,\alpha^\vee - 1}\Biggr) \\ & \times \Biggl(\prod_{\alpha\in(\Phi_w\setminus\Delta_p)\cup(\Phi_-\cap w^{-1}\Phi_-)} \xi(\langle\lambda_p,\alpha^\vee\rangle s + \mathrm{ht}\,\alpha^\vee)\Biggr). \end{aligned} \tag{5.15}$$

Using

$$\begin{aligned} (\Phi_w \setminus \Delta_p) \cup (\Phi_- \cap w^{-1}\Phi_-) &= ((\Phi_+ \cap w^{-1}\Phi_-) \setminus \Delta_p) \cup (\Phi_- \cap w^{-1}\Phi_-) \\ &= ((\Phi_+ \cap w^{-1}\Phi_-) \cup (\Phi_- \cap w^{-1}\Phi_-)) \setminus \Delta_p \\ &= w^{-1}\Phi_- \setminus \Delta_p, \end{aligned} \tag{5.16}$$

we arrive at (5.13). □

Let

$$f_{p,w}(s) = \prod_{\alpha\in(w^{-1}\Delta)\setminus\Delta_p} \frac{1}{\langle\lambda_p,\alpha^\vee\rangle s + \mathrm{ht}\,\alpha^\vee - 1}, \tag{5.17}$$

$$g_{p,w}(s) = \prod_{\alpha\in(w^{-1}\Phi_-)\setminus\Delta_p} \xi(\langle\lambda_p,\alpha^\vee\rangle s + \mathrm{ht}\,\alpha^\vee), \tag{5.18}$$

so that

$$Z_p(s;T) = \sum_{\substack{w\in W \\ \Delta_p \subset w^{-1}(\Delta\cup\Phi_-)}} e^{\langle w(\rho+s\lambda_p)-\rho, T\rangle} f_{p,w}(s) g_{p,w}(s). \tag{5.19}$$

Lemma 5.3. *For* $w \in W$ *and* $\varpi \in \mathrm{Aut}(\Gamma)$ *with* $q = \varpi p$,

$$f_{p,w}(-c_p - s) = f_{p,w_0 w w_p}(s), \qquad g_{p,w}(-c_p - s) = g_{p,w_0 w w_p}(s), \tag{5.20}$$

$$f_{p,\varpi^{-1} w \varpi}(s) = f_{q,w}(s), \qquad g_{p,\varpi^{-1} w \varpi}(s) = g_{q,w}(s). \tag{5.21}$$

Proof. Fix $w \in W$. Then for a subset $A \subset \Phi$ with $A = \Delta$ or Φ_-, we have $w_0 A = -A$ and

$$\begin{aligned} -w_p(w^{-1}A \setminus \Delta_p) &= (w_p w^{-1}(-A)) \setminus (w_p(-\Delta_p)) \\ &= (w_p w^{-1} w_0 A) \setminus \Delta_p. \end{aligned} \tag{5.22}$$

Let

$$S(A; s; p, w) = \{\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht} \alpha^\vee \mid \alpha \in (w^{-1}A) \setminus \Delta_p\} \tag{5.23}$$

be a set of affine linear functionals of the form $as+b$ with $a, b \in \mathbb{N} \cup \{0\}$ which admits duplications. Then we have

$$f_{p,w}(s) = \prod_{as+b \in S(\Delta; s; p, w)} \frac{1}{as+b-1}, \tag{5.24}$$

$$g_{p,w}(s) = \prod_{as+b \in S(\Phi_-; s; p, w)} \frac{1}{\xi(as+b)}. \tag{5.25}$$

Using the formula (5.22) and Lemma 4.1, we have

$$\begin{aligned} S(A; -c_p - s; p, w) &= \{\langle \lambda_p, \alpha^\vee \rangle (-c_p - s) + \operatorname{ht} \alpha^\vee \mid \alpha \in (w^{-1}A) \setminus \Delta_p\} \\ &= \{\langle \lambda_p, -w_p \alpha^\vee \rangle s + \langle c_p \lambda_p - w_p \rho, -w_p \alpha^\vee \rangle \mid \alpha \in (w^{-1}A) \setminus \Delta_p\} \\ &= \{\langle \lambda_p, \beta^\vee \rangle s + \langle \rho, \beta^\vee \rangle \mid \beta \in (w_p w^{-1} w_0 A) \setminus \Delta_p\} \\ &= S(A; s; p, w_0 w w_p), \end{aligned} \tag{5.26}$$

which implies (5.20).

For (5.21), using $\varpi \Delta_p = \Delta_q$ and $\varpi A = A$, we have

$$\begin{aligned} S(A; s; p, \varpi^{-1} w \varpi) &= \{\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht} \alpha^\vee \mid \alpha \in (\varpi^{-1} w^{-1} \varpi A) \setminus \Delta_p\} \\ &= \{\langle \varpi \lambda_p, \varpi \alpha^\vee \rangle s + \operatorname{ht} \varpi \alpha^\vee \mid \alpha \in (\varpi^{-1} w^{-1} \varpi A) \setminus \Delta_p\} \\ &= \{\langle \lambda_q, \beta^\vee \rangle s + \operatorname{ht} \beta^\vee \mid \beta \in (w^{-1}A) \setminus \Delta_q\} \\ &= S(A; s; q, w). \end{aligned} \tag{5.27}$$

□

Proof of Proposition 5.1. For $w \in W$, we have by Lemma 4.1,

$$\begin{aligned} \langle w(\rho + (-c_p - s)\lambda_p) - \rho, \varpi_0 T \rangle &= \langle w(\rho - c_p \lambda_p) - w s \lambda_p - \rho, -w_0 T \rangle \\ &= \langle -w w_p \rho - w w_p s \lambda_p - \rho, -w_0 T \rangle \\ &= \langle w_0 w w_p(\rho + s\lambda_p) - \rho, T \rangle. \end{aligned} \tag{5.28}$$

Hence using Proposition 5.2 and Lemma 5.3, we obtain

$$\begin{aligned} &Z_p(-c_p - s; \varpi_0 T) \\ &= \sum_{\substack{w \in W \\ \Delta_p \subset w^{-1}(\Delta \cup \Phi_-)}} e^{\langle w(\rho + (-c_p - s)\lambda_p) - \rho, \varpi_0 T \rangle} f_{p,w}(-c_p - s) g_{p,w}(-c_p - s) \\ &= \sum_{\substack{w \in W \\ \Delta_p \subset w^{-1}(\Delta \cup \Phi_-)}} e^{\langle w_0 w w_p(\rho + s\lambda_p) - \rho, T \rangle} f_{p, w_0 w w_p}(s) g_{p, w_0 w w_p}(s) \\ &= \sum_{\substack{v \in W \\ \Delta_p \subset w_p v^{-1} w_0(\Delta \cup \Phi_-)}} e^{\langle v(\rho + s\lambda_p) - \rho, T \rangle} f_{p,v}(s) g_{p,v}(s), \end{aligned} \tag{5.29}$$

which implies the first equality of (5.12) by Lemma 4.2 (1).

As for the second equality, we have by Lemmas 5.3 and 4.2 (2),

$$
\begin{aligned}
Z_q(s;\varpi T) &= \sum_{\substack{w\in W\\ \Delta_q\subset w^{-1}(\Delta\cup\Phi_-)}} e^{\langle \varpi^{-1}w(\rho+s\lambda_q)-\rho,T\rangle} f_{q,w}(s)g_{q,w}(s)\\
&= \sum_{\substack{w\in W\\ \Delta_q\subset w^{-1}(\Delta\cup\Phi_-)}} e^{\langle \varpi^{-1}w\varpi(\rho+s\lambda_p)-\rho,T\rangle} f_{p,\varpi^{-1}w\varpi}(s)g_{p,\varpi^{-1}w\varpi}(s)\\
&= \sum_{\substack{v\in W\\ \Delta_q\subset \varpi v^{-1}\varpi^{-1}(\Delta\cup\Phi_-)}} e^{\langle v(\rho+s\lambda_p)-\rho,T\rangle} f_{p,v}(s)g_{p,v}(s).
\end{aligned}
\tag{5.30}
$$

□

From Proposition 5.2, we see that $Z_p(s;T)$ has no ξ functions in the denominator of each term. In fact, it is too much multiplied; $Z_p(s;T)$ can be factorized by some ξ functions and should be divided by them in order to obtain Weng's zeta function $\xi_{\mathbb{Q};o}^{G/P}(s;T)$.

Let

$$
H_{p,w}(s) = \Bigg(\prod_{\alpha\in\Phi_w\setminus\Delta_p} \xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee)\Bigg)\Bigg(\prod_{\alpha\in\Phi_w}\frac{1}{\xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee+1)}\Bigg),
\tag{5.31}
$$

which is the term corresponding to w in $\omega_{\mathbb{Q}}^{G/P}(s;T)$ (see Proposition 2.2). Since

$$
\begin{aligned}
\prod_{\alpha\in\Phi_w\setminus\Delta_p} \xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee) &= \xi(s+1)^{N_{p,w}(1,1)}\prod_{\alpha\in\Phi_w\setminus\Delta} \xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee)\\
&= \xi(s+1)^{N_{p,w}(1,1)}\prod_{k=0}^{\infty}\prod_{h=2}^{\infty}\xi(ks+h)^{N_{p,w}(k,h)},
\end{aligned}
\tag{5.32}
$$

and

$$
\begin{aligned}
\prod_{\alpha\in\Phi_w} \xi(\langle\lambda_p,\alpha^\vee\rangle s+\operatorname{ht}\alpha^\vee+1) &= \prod_{k=0}^{\infty}\prod_{h=1}^{\infty}\xi(ks+h+1)^{N_{p,w}(k,h)}\\
&= \prod_{k=0}^{\infty}\prod_{h=2}^{\infty}\xi(ks+h)^{N_{p,w}(k,h-1)},
\end{aligned}
\tag{5.33}
$$

we have the expression

$$
H_{p,w}(s) = \xi(s+1)^{N_{p,w}(1,1)}\prod_{k=0}^{\infty}\prod_{h=2}^{\infty}\xi(ks+h)^{N_{p,w}(k,h)-N_{p,w}(k,h-1)}.
\tag{5.34}
$$

From this expression, we see that if $N_{p,w}(k,h)-N_{p,w}(k,h-1)<0$, then $\xi(ks+h)^{N_{p,w}(k,h-1)-N_{p,w}(k,h)}$ appears in the denominator of the term $H_{p,w}(s)$. Let $\delta(a)=a$ if $a>0$, and $\delta(a)=0$ otherwise. In order to describe the minimal ξ factor that cancels all the denominators of $\omega_{\mathbb{Q}}^{G/P}(s;T)$, we introduce

$$
\tilde{M}_p(k,h) = \max_{\substack{w\in W\\ \Delta_p\subset w^{-1}(\Delta\cup\Phi_-)}} \delta(N_{p,w}(k,h-1)-N_{p,w}(k,h))
\tag{5.35}
$$

for $(k,h)\in\mathbb{Z}^2$ and we define $D_p(s)$ by

$$
D_p(s) = \prod_{k=0}^{\infty}\prod_{h=2}^{\infty}\xi(ks+h)^{N_p(k,h-1)-\tilde{M}_p(k,h)}.
\tag{5.36}
$$

By use of the definition (5.10), we see

$$\begin{aligned} F_p(s) &= \prod_{\alpha\in\Phi_+} \xi(\langle\lambda_p,\alpha^\vee\rangle s + \operatorname{ht}\alpha^\vee + 1) \\ &= \prod_{k=0}^{\infty}\prod_{h=1}^{\infty} \xi(ks+h+1)^{N_p(k,h)} \\ &= \prod_{k=0}^{\infty}\prod_{h=2}^{\infty} \xi(ks+h)^{N_p(k,h-1)} \end{aligned} \tag{5.37}$$

and

$$D_p(s) = F_p(s)\prod_{k=0}^{\infty}\prod_{h=2}^{\infty} \xi(ks+h)^{-\tilde{M}_p(k,h)}, \tag{5.38}$$

so that $F_p(s)/D_p(s)$ is the minimal ξ factor. Note that (2.9) is rewritten as

$$M_p(k,h) = \max_{\substack{w\in W\\ \Delta_p\subset w^{-1}(\Delta\cup\Phi_-)}} (N_{p,w}(k,h-1) - N_{p,w}(k,h)). \tag{5.39}$$

Lemma 5.4. (1) *For* $(k,h)\in\mathbb{Z}^2$ *with* $h\geq 1$,

$$M_p(k,h) = \tilde{M}_p(k,h). \tag{5.40}$$

(2) *For* $(k,h)\in\mathbb{Z}^2$,

$$N_p(k,kc_p-h) - M_p(k,kc_p-h+1) = N_p(k,h-1) - M_p(k,h). \tag{5.41}$$

(3) *For* $(k,h)\in\mathbb{Z}^2$,

$$M_p(k,h) = M_q(k,h), \tag{5.42}$$

where $q\in\operatorname{Aut}(\Gamma)p$.

Proof. (1) Note that $\Delta_p\subset(\Delta\cup\Phi_-)$. For $l\geq 0$, we have

$$N_{p,\mathrm{id}}(k,l) = \sharp\{\alpha\in\Phi_- \mid \langle\lambda_p,\alpha^\vee\rangle = k, \operatorname{ht}\alpha^\vee = l\} = 0, \tag{5.43}$$

which implies

$$N_{p,\mathrm{id}}(k,h-1) - N_{p,\mathrm{id}}(k,h) = 0. \tag{5.44}$$

Hence $\delta(x)$ can be replaced by x in $\tilde{M}_p(k,h)$.

(2) From Lemma 4.4 (1), we have

$$\begin{aligned} &M_p(k,kc_p-h+1) \\ &\quad= \max_{\substack{w\in W\\ \Delta_p\subset w^{-1}(\Delta\cup\Phi_-)}} (N_{p,w}(k,kc_p-h) - N_{p,w}(k,kc_p-h+1)) \\ &\quad= \max_{\substack{w\in W\\ \Delta_p\subset w^{-1}(\Delta\cup\Phi_-)}} (N_p(k,h) - N_p(k,h-1) - N_{p,w_0ww_p}(k,h) + N_{p,w_0ww_p}(k,h-1)). \end{aligned} \tag{5.45}$$

Hence by Lemmas 4.2 (1) and 4.3 (1),

$$\begin{aligned} &N_p(k,kc_p-h) - M_p(k,kc_p-h+1) \\ &= N_p(k,h) - \max_{\substack{w\in W\\ \Delta_p\subset w^{-1}(\Delta\cup\Phi_-)}} (N_p(k,h) - N_p(k,h-1) - N_{p,w_0ww_p}(k,h) + N_{p,w_0ww_p}(k,h-1)) \\ &= N_p(k,h-1) - \max_{\substack{w\in W\\ \Delta_p\subset w^{-1}(\Delta\cup\Phi_-)}} (N_{p,w_0ww_p}(k,h-1) - N_{p,w_0ww_p}(k,h)) \\ &= N_p(k,h-1) - M_p(k,h). \end{aligned} \tag{5.46}$$

(3) By Lemmas 4.2 (2) and 4.4 (2), we have the result.

∎

Proof of Theorem 2.3. Since by (5.38), $F_p(s)/D_p(s)$ is the minimal ξ factor for $\omega_{\mathbb{Q}}^{G/P}(s;T)$, we have (2.10) by Lemma 5.4 (1) and the expression

$$\xi_{\mathbb{Q};o}^{G/P}(s;T) = \Big(\frac{F_p(s)}{D_p(s)}\Big)\omega_{\mathbb{Q}}^{G/P}(s;T). \tag{5.47}$$

□

Lemma 5.5.

$$\begin{aligned} D_p(-c_p - s) &= D_p(s) \\ &= D_q(s), \end{aligned} \tag{5.48}$$

where $q \in \mathrm{Aut}(\Gamma)p$.

Proof. We show the first equality. We use

$$D^{(0)} = \prod_{h=2}^{\infty} \xi(h)^{N_p(0,h-1)-M_p(0,h)}, \tag{5.49}$$

$$\begin{aligned} D^{(1)}(s) &= \prod_{k=1}^{\infty}\prod_{h=2}^{\infty} \xi(ks+h)^{N_p(k,h-1)-M_p(k,h)} \\ &= \prod_{k=1}^{\infty}\prod_{h=-\infty}^{\infty} \xi(ks+h)^{N_p(k,h-1)-M_p(k,h)}, \end{aligned} \tag{5.50}$$

since $N_{p,w}(k,h-1) = 0$ and $M_p(k,h) = 0$ for $k \geq 1$ and $h \leq 1$. Note that $D_p(s) = D^{(0)}D^{(1)}(s)$. It is sufficient to show $D^{(1)}(-c_p - s) = D^{(1)}(s)$. We have

$$\begin{aligned} D^{(1)}(-c_p - s) &= \prod_{k=1}^{\infty}\prod_{h=-\infty}^{\infty} \xi(-kc_p - ks + h)^{N_p(k,h-1)-M_p(k,h)} \\ &= \prod_{k=1}^{\infty}\prod_{h=-\infty}^{\infty} \xi(ks + kc_p - h + 1)^{N_p(k,h-1)-M_p(k,h)} \\ &= \prod_{k=1}^{\infty}\prod_{h=-\infty}^{\infty} \xi(ks+h)^{N_p(k,kc_p-h)-M_p(k,kc_p-h+1)}. \end{aligned} \tag{5.51}$$

By Lemma 5.4 (2), we obtain

$$D^{(1)}(-c_p - s) = \prod_{k=1}^{\infty}\prod_{h=-\infty}^{\infty} \xi(ks+h)^{N_p(k,h-1)-M_p(k,h)} = D^{(1)}(s) \tag{5.52}$$

and hence the result.

The second equality of (5.48) follows from the definition (5.36) and Lemmas 4.3 (2) and 5.4 (3). □

Proof of Theorem 2.4. By (5.47) and (5.11), we rewrite

$$\xi_{\mathbb{Q};o}^{G/P}(s;T) = \Big(\frac{F_p(s)}{D_p(s)}\Big)\omega_{\mathbb{Q}}^{G/P}(s;T) = \frac{Z_p(s;T)}{D_p(s)}. \tag{5.53}$$

Then the functional equation follows from Proposition 5.1 and Lemma 5.5. □

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