ON THE ZEROS OF WENG ZETA FUNCTIONS
FOR CHEVALLEY GROUPS

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ABSTRACT. We prove that all but finitely many zeros of Weng's zeta function for a Chevalley group defined over $\mathbb{Q}$ are simple and on the critical line.

1. INTRODUCTION

Weng zeta functions are new objects in the theory of zeta functions arose from the theory of (periods of) Eisenstein series of reductive algebraic groups G defined over $\mathbb{Q}$. They are defined for every standard maximal parabolic subgroup P of G. Roughly Weng zeta functions are linear combinations of products of the complete Riemann zeta-function with rational function coefficients, but they have amazing structures that may come from their origin. We start from the most simple example.

Let $G = SL(2)$, $P = B = P_{1,1} = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \subset G$ and $K = SO(2)$. (In this case, the Borel subgroup B coincides with the maximal parabolic subgroup P.) Let $\mathcal{H}$ be the upper half plane. Usual non-holomorphic Eisenstein series attached to this pair (G, P) is defined by the series

$$E(s, z) = \sum_{\gamma \in P(\mathbb{Z}) \backslash G(\mathbb{Z})} \Im\,(\gamma z)^s$$

for $z \in \mathcal{H}$, $\mathrm{Re}(s) > 1$, and then it is continued meromorphically to the whole complex $\mathbb{C}$.

To understand the integral

$$\int_{\mathcal{F}} E(s, z)\, d\mu(z)$$

in a suitable sense is a fundamental problem for the theory of harmonic analysis on $SL(2, \mathbb{Z}) \backslash \mathcal{H}$, where $\mathcal{F}$ is a standard fundamental domain of the action of $G(\mathbb{Z}) = SL(2, \mathbb{Z})$ on $\mathcal{H}$. Usually the above integral is studied by using the analytic truncation

$$\Lambda^\tau E(s, z) = \begin{cases} E(s, z), & y \leqslant e^\tau,\ z \in \mathcal{F}, \\ E(s, z) - (y^s + \frac{\hat{\zeta}(2s-1)}{\hat{\zeta}(2s)}\, y^{1-s}), & y > e^\tau,\ z \in \mathcal{F}. \end{cases}$$

For every $\tau \geqslant 0$ we have

$$\int_{\mathcal{F}} \Lambda^\tau E(s, z)\, d\mu(z) = \frac{e^{\tau(s-1)}}{s-1} - \frac{e^{-\tau s}}{s} \frac{\hat{\zeta}(2s-1)}{\hat{\zeta}(2s)}.$$

(Therefore $\int_{\mathcal{F}} E(s, z)\, d\mu(z) = 0$ if $0 < \Re(s) < 1$.) Multiplying by $s(s-1)\hat{\zeta}(2s)$ on both sides, we define

$$Z(s, \tau) = s(s-1) \cdot \hat{\zeta}(2s) \cdot \int_{\mathcal{F}} \Lambda^\tau E(s, z)\, d\mu(z)$$

$$= e^{\tau(s-1)}\, s\, \hat{\zeta}(2s) - e^{-\tau s}\, (s-1)\, \hat{\zeta}(2s-1).$$

for $\tau \geqslant 0$. The functional equation of $E(z,s)$ derives $Z(s,\tau) = Z(1-s,\tau)$ automatically, but we mention that the functional equation of $E(z,s)$ coming from the symmetry of the Weyl group of $SL(2)$.

A remarkable fact is that all zeros of $Z(s,\tau)$ lie on the line $\Re(s) = 1/2$ and simple for every fixed $\tau \geqslant 0$ (see [11, 14]). Weng zeta functions are of a generalization of the specialization of $Z(s,\tau)$ for $\tau = 0$.

Let G be a connected semisimple algebraic group defined over $\mathbb{Q}$ endowed with a maximal ($\mathbb{Q}$-)split torus T. Let Φ be the root system with respect to (G,T). Fix a Borel subgroup B of G containing T. Then it determines the fundamental system Δ of Φ.

Let $X^*(T)$ be the group of characters of T defined over $\mathbb{Q}$ that is a free module of rank $r = \dim T$. Let $\mathfrak{a}_0^* = X^*(T) \otimes \mathbb{R}$ and $\mathfrak{a}_0 = \mathrm{Hom}(X^*(T), \mathbb{R})$. Then $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$ are real vector spaces of dimension r. The root system Φ is a finite subset of $X^*(T)$, therefore it is canonically embedded in $\mathfrak{a}_0^*$. For every simple root $\alpha \in \Delta$, we have the coroot $\alpha^\vee \in \mathfrak{a}_0$.

Let $E(\lambda, g) = E^{G/B}(\mathbf{1}, \lambda, g)$ be Langlands' Eisenstein series for $\lambda \in \mathfrak{a}_0^* \otimes \mathbb{C} \simeq \mathbb{C}^r$ and $g \in G(\mathbb{A})$. By using J. Arthur's truncation operators Λ^τ acting on a space of automorphic forms (see [2] and also [1]), we obtain the well-defined integral

$$\int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \Lambda^\tau E(\lambda, g)\, dg$$

for sufficiently regular $\tau \in \mathfrak{a}_0$. It is often called the period of $E(\lambda, g)$. The integral can be calculated explicitly by using the root system Φ and its Weyl group W of Φ:

$$\int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \Lambda^\tau E(\lambda, g)\, dg$$
$$= v \sum_{w \in W} e^{\langle w\lambda - \rho, \tau \rangle} \prod_{\alpha \in \Delta} \frac{1}{\langle w\lambda - \rho, \alpha^\vee \rangle} M(w, \lambda),$$

where v is the volume of $\{\sum_{\alpha \in \Delta} a_\alpha \alpha^\vee \mid 0 \leqslant a_\alpha < 1\}$, ρ is the half sum of all positive roots and $M(w, \lambda)$ are usual (unnormalized) intertwining operators $M(w, \lambda): I_B(\lambda) \to I_B(w\lambda)$ for $\lambda \in \mathfrak{a}_0^* \otimes \mathbb{C}$ and $w \in W$. If G is a split semisimple group, we have

$$M(w, \lambda) = \prod_{\alpha \in \Phi_w} \frac{\hat{\zeta}(\langle \lambda, \alpha^\vee \rangle)}{\hat{\zeta}(\langle \lambda, \alpha^\vee \rangle + 1)},$$

where $\Phi_w = \Phi^+ \cap w^{-1}\Phi^-$ (see Chapter 5 of Langlands [15], Jaquet-Lapid-Rogawski [8]).

Note that we assumed that G is split. Standing on the above formula for the period of $E(\lambda, g)$, Weng introduced the "period" $\omega_{\mathbb{Q}, B}^{(G,T)}: \mathfrak{a}_0^* \otimes \mathbb{C} \simeq \mathbb{C}^r \to \mathbb{C}$ by

$$\omega_{\mathbb{Q}, B}^{(G,T)}(\lambda) = \sum_{w \in W} \prod_{\alpha \in \Delta} \frac{1}{\langle w\lambda - \rho, \alpha^\vee \rangle} \prod_{\alpha \in \Phi_w} \frac{\hat{\zeta}(\langle \lambda, \alpha^\vee \rangle)}{\hat{\zeta}(\langle \lambda, \alpha^\vee \rangle + 1)}.$$

In this definition, we ignored the volume v, and took $\tau = 0$ in the right hand side of the above formula of periods of Eisenstein series. We do not know $\tau = 0$ is regular or not in general. Therefore we should note that $\omega_{\mathbb{Q}, B}^{(G,T)}$ may not be a period of Eisenstein series in a rigorous meaning. However, we will find that $\omega_{\mathbb{Q}, B}^{(G,T)}$ itself is an interesting object apart from its origin. For instance, a priori, there is no obvious reason for the restriction $\tau = 0$. But we find that $\tau = 0$ is a special point for the distribution of zeros of Weng zeta functions after a study of τ-version $\omega_{\mathbb{Q}, B}^{(G,T)}(\lambda; \tau)$ (see **A.4** of [24] for details).

From the point of view for the theory of periods of automorphic functions, it is natural to consider more general "period" by replacing $\hat{\zeta}$ by (completed) L-functions attached to Dirichlet characters or automorphic representations. However we restrict our interest only on the above $\omega_{\mathbb{Q}, B}^{(G,T)}$ for the simplicity of arguments.

By definition, the period $\omega_{\mathbb{Q},B}^{(G,T)}(\lambda)$ has a large symmetry with respect to the Weyl group, but it is a r-variable function in general ($r = \dim T$). A priori there is no obvious way to derive a single-variable function from $\omega_{\mathbb{Q},B}^{(G,T)}(\lambda)$ preserving the symmetry of the Weyl group. A discovery of Weng was that there is a canonical way to derive a (conjecturally) nice single-variable function from $\omega_{\mathbb{Q},B}^{(G,T)}(\lambda)$ if we use a standard maximal parabolic subgroup of G. The key fact is the bijection

$$\Delta = \{\alpha_1, \cdots, \alpha_\ell\} \quad \rightarrow \quad \{\text{maximal parabolic subgroups } P\}$$
$$\alpha_j \quad \mapsto \quad P_j$$

We denote by α_P the simple root corresponding to the standard maximal parabolic subgroup P. Following Weng, we define

$$\omega_{\mathbb{Q},P/B}^{(G,T)}(s) = \underset{\lambda}{\mathrm{Res}}\, \omega_{\mathbb{Q},B}^{(G,T)}(\lambda) \quad (s = \langle \lambda - \rho, \alpha_P^\vee \rangle),$$

where $\underset{\lambda}{\mathrm{Res}}$ means taking residues along $(r-1)$ many hyperplanes

$$\langle \lambda - \rho, \beta^\vee \rangle = 0, \quad \beta \in \Delta \setminus \{\alpha_P\}.$$

Then zeta functions for $((G,T); P/B)/\mathbb{Q}$ are defined by

$$\hat{\zeta}_P(s) := \hat{\zeta}_{\mathbb{Q},P/B}^{(G,T)}(s) := \prod_{k,h} \hat{\zeta}(ks+h) \cdot \omega_{\mathbb{Q},P/B}^{(G,T)}(s),$$

where $\prod_{k,h} \hat{\zeta}(ks+h)$ is the "minimal" finite product of $\hat{\zeta}(ks+h)$ in the sense that $\hat{\zeta}_P(s)$ has no $\hat{\zeta}(as+b)$ or $\hat{\zeta}(c)$ in its denominators.

Remark The functions $\omega_{\mathbb{Q},P/B}^{(G,T)}(\lambda)$, $\omega_{\mathbb{Q},P/B}^{(G,T)}(s)$, $\hat{\zeta}_P(s)$ are written as $\omega_{\mathbb{Q}}^G(\lambda)$, $\omega_{\mathbb{Q}}^{G/P}(s)$, $\xi_{\mathbb{Q},o}^{G/P}(s)$ respectively in [24] and [13].

Here we mention several examples of $\hat{\zeta}_P(s)$. Let $P_{n_1,n_2} \subset SL(n)$ be the maximal parabolic subgroup attached to $n = n_1 + n_2$. Then we find that

$$\hat{\zeta}_{\mathbb{Q},P_{n-1,1}}^{SL(n)}(s) = \hat{\zeta}_{\mathbb{Q},P_{1,n-1}}^{SL(n)}(s) = \sum_{h=1}^{n} \frac{Q_h(s)}{P_h(s)} \cdot \hat{\zeta}(s+h),$$

where P_h, Q_h are polynomials satisfying $\deg P_h \leqslant n-1$, $\deg Q_h < \deg P_h$ (see also section 4). For example,

$$\hat{\zeta}_{\mathbb{Q},P_{1,1}}^{SL(2)}(s) = \frac{\hat{\zeta}(s+2)}{s} - \frac{\hat{\zeta}(s+1)}{s+2},$$

$$\hat{\zeta}_{\mathbb{Q},P_{2,1}}^{SL(3)}(s) = \left(\frac{\hat{\zeta}(2)}{s} - \frac{1}{2(s+1)}\right)\hat{\zeta}(s+3) - \frac{\hat{\zeta}(s+2)}{s(s+3)} - \left(\frac{\hat{\zeta}(2)}{s+3} - \frac{1}{2(s+2)}\right)\hat{\zeta}(s+1).$$

In the following examples, each term of $\hat{\zeta}_P(s)$ consists of the product of several $\hat{\zeta}(s)$:

$$\hat{\zeta}_{\mathbb{Q},P_{2,2}}^{SL(4)/P_{2,2}}(s) = R_1(s)\,\hat{\zeta}(s+3)\hat{\zeta}(s+4) + R_2(s)\,\hat{\zeta}(s+3)^2$$
$$+ R_3(s)\,\hat{\zeta}(s+1)\hat{\zeta}(s+3) + R_4(s)\,\hat{\zeta}(s+2)^2 + R_5(s)\,\hat{\zeta}(s+2)\hat{\zeta}(s+3),$$

$$\hat{\zeta}_{\mathbb{Q},P_{\text{short}}}^{Sp(4)}(s) = R_1(s)\,\hat{\zeta}(s+3)\hat{\zeta}(2s+2) + R_2(s)\,\hat{\zeta}(s+2)\hat{\zeta}(2s+2)$$
$$+ R_3(s)\,\hat{\zeta}(s+2)\hat{\zeta}(2s+1) + R_4(s)\,\hat{\zeta}(s+1)\hat{\zeta}(2s+1),$$

$$\hat{\zeta}_{\mathbb{Q},P_{\text{long}}}^{G_2}(s) = R_1(s)\,\hat{\zeta}(s+3)\hat{\zeta}(2s+2)\hat{\zeta}(3s+3) + R_2(s)\,\hat{\zeta}(s+2)\hat{\zeta}(2s+1)\hat{\zeta}(3s+1)$$
$$+ R_3(s)\,\hat{\zeta}(s+1)\hat{\zeta}(2s+2)\hat{\zeta}(3s+2) + R_4(s)\,\hat{\zeta}(s+1)\hat{\zeta}(2s+1)\hat{\zeta}(3s+2)$$
$$+ R_5(s)\,\hat{\zeta}(s+1)\hat{\zeta}(2s+1)\hat{\zeta}(3s+1) + R_6(s)\,\hat{\zeta}(s+1)\hat{\zeta}(2s+2)\hat{\zeta}(3s+3),$$

where R_i's are certain rational functions. For more examples, see Appendix B of [24]

By definition, $\hat{\zeta}_P(s)$ are meromorphic functions in $\mathbb{C}$ having only finitely many poles. In addition Weng observed that $\hat{\zeta}_P(s)$ satisfy standard functional equations for several examples of $((G,T); P/B)$, and conjectured that it holds for general pairs (if G is a classical semisimple group at least):

Conjecture 1 There exists $c = c((G,T); P/B) \in \mathbb{Q}$ such that $\hat{\zeta}_P(-c-s) = \hat{\zeta}_P(s)$.

The conjectural functional equation derives the corresponding Riemann hypothesis:

Conjecture 2 All zeros of $\hat{\zeta}_P(s)$ lie on the critical line $\Re(s) = -c/2$.

As for Conjecture 1, initially, Weng proved it for $SL(n)$ ($n = 2, 3, 4, 5$), $Sp(4)$, $SO(8)$ and G_2. Successively, H. Kim-Weng proved the case of $(SL(n), P_{n-1,1})$ for arbitrary $n \geqslant 2$ (unpublished). Finally, the second author established the conjectural functional equations of $\hat{\zeta}_P(s)$ for general $((G,T); P/B)$, and determined the value $c((G,T); P/B)$ explicitly in [13]. As for Conjecture 2, its validity was known for ten examples of $((G,T); P/B)$, namely, $G = SL(n)$ ($n = 2, 3, 4, 5$), $Sp(4)$, G_2 and their arbitrary maximal parabolic subgroup P, and further, the simplicity of zeros were also known for these pairs ([11,12,14,19–21]). This is surprising, because a linear combination of zeta functions has a lot of off-line zeros in general even if it has a functional equation (e.g. [5,6,17]).

Roughly, the known proof of Conjecture 2 for the above special cases consists of two parts. The first one is to show that all but finitely many zeros of $\hat{\zeta}_P(s)$ lie on the critical line, and the second one is to remove the possibility of off-line zeros. Methods for these two parts have different nature. We may say that the first part is of the problem for the zeros of higher position, and the other part is of the problem for low-lying zeros. The latter problem is difficult and interesting than the former one as well as in the theory of classical zeta/L-functions. In fact, the proof of the first part for the above ten examples were improved and unified in [12], unfortunately we still need a numerical computation for the latter part. In the present paper, we prove the following result for the first part of Conjecture 2 by generalizing the method of [12] and by using the theory of [13]:

Theorem 1.1 (Weak Riemann Hypothesis) *Let G be a Chevalley group defined over $\mathbb{Q}$, in other words, G is a connected semisimple algebraic group defined over $\mathbb{Q}$ endowed with a maximal ($\mathbb{Q}$-)split torus T. Let B be a Borel subgroup of G containing T. Let P be a maximal parabolic subgroup of G defined over $\mathbb{Q}$ containing B.*

Then all but finitely many zeros of $\hat{\zeta}_{\mathbb{Q},P/B}^{(G,T)}(s)$ are simple and on the critical line of its functional equation.

Remark Similar results hold if we replace $\hat{\zeta}(s)$ by a suitable (completed) L-function, because we use only standard analytic properties of $\hat{\zeta}(s)$ in the proof.

Remark It is expected that $\hat{\zeta}_{\mathbb{Q},P/B}^{(G,T)}(s)$ has no zeros outside the critical line as well as in known cases $G = SL(n)$ ($n = 2, 3, 4, 5$), $Sp(4)$, G_2 that correspond to root systems of type A_n ($n = 1, 2, 3, 4$), $C_2(\simeq B_2)$ and G_2. However we have no idea how to prove it in general.

We note that the known proof of Conjecture 2 about ten examples mentioned just before Theorem 1.1 depends on explicit formulas of $\hat{\zeta}_{\mathbb{Q},P/B}^{(G,T)}(s)$ presented by Weng (see [24]). On the other hand, the method of the proof of the functional equation in [13] is completely abstract, and hence it does not depend on an individual root system. In this paper, we will prove the above main theorem in such a way that we refine the proof of the Riemann hypothesis of $\hat{\zeta}_{\mathbb{Q},P/B}^{(G,T)}(s)$ demonstrated in [12, 14, 19–21] by using terms of the abstract root system as well as in the proof of the functional equations in [13].

The paper is organized as follows. In section 2, we rewrite zeta functions $\hat{\zeta}_{\mathbb{Q},P/B}^{(G,T)}(s)$ in terms of abstract root systems, and define zeta functions more rigorously. In section

3, we sketch the outline of the proof of Theorem 1.1. In section 4, we briefly review the story of the proof of Theorem 1.1 restricting on the most simple cases $(SL(n), P_{n-1,1})$. In sections 5 to 7, we carry out the scheme of section 3 without proofs of lemmas and propositions. The section that the proof of each lemma or proposition is accomplished is mentioned here. In section 8, we prepare further notations and auxiliary lemmas for the proofs of results in sections 5 to 7. Finally, in sections 9 to 15, we fill the details of proofs of lemmas and propositions in sections 5 to 7, and complete the proof of Theorem 1.1.

2. DEFINITION OF WENG ZETA FUNCTIONS FOR (G, P)

2.1. **Root system and the Weyl group.** Let V be a r-dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. Let $\Phi \subset V$ be a (reduced) root system of rank r and $\Delta = \{\alpha_1, \cdots, \alpha_r\}$, its fundamental system. Let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ be the coroot associated with $\alpha \in \Phi$. Let $\Lambda = \{\lambda_1, \cdots, \lambda_r\}$ be the set of fundamental weights satisfying $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$. Let Φ^+ be the corresponding positive system of Φ and $\Phi^- = -\Phi^+$ so that $\Phi = \Phi^+ \cup \Phi^-$. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{j=1}^r \lambda_j$$

be the Weyl vector. Let $\operatorname{ht} \alpha^\vee = \langle \rho, \alpha^\vee \rangle$ be the height of $\alpha^\vee$.

Let W be the Weyl group generated by simple reflections $\sigma_j = \sigma_{\alpha_j} : V \to V$ attached to simple roots $\alpha_j \in \Delta$. We denote the identity of W by id. For $w \in W$, we put

$$\Phi_w = \Phi^+ \cap w^{-1}\Phi^-$$

and let $l(w) = |\Phi_w|$ be the length of w. Let w_0 be the longest element of W. Then we have $w_0^2 = \operatorname{id}$, $w_0\Delta = -\Delta$ and $w_0\Phi^+ = \Phi^-$.

We fix an integer p with $1 \leqslant p \leqslant r$. Let Φ_p be the root system normal to the fundamental weight λ_p. A fundamental system of Φ_p is given by $\Delta_p = \Delta \setminus \{\alpha_p\}$. Let $\Phi_p^+ = \Phi^+ \cap \Phi_p \; (\subset \Phi^+)$ be the corresponding positive system of Φ_p. Then $\Phi_p = \Phi_p^+ \cup \Phi_p^-$ with $\Phi_p^- = -\Phi_p^+$. Let

$$\rho_p = \frac{1}{2} \sum_{\alpha \in \Phi_p^+} \alpha.$$

Let W_p be the subgroup of W generated by simple reflections $\{\sigma_j = \sigma_{\alpha_j} \mid \alpha_j \in \Delta_p\}$. Let w_p be the longest element of W_p. Then $w_p^2 = \operatorname{id}$, $w_p\Delta_p = -\Delta_p$ and $w_p\Phi_p^+ = \Phi_p^-$.

Definition 2.1 *Define the subset $\mathfrak{W}_p$ of W by*

$$\mathfrak{W}_p := \{w \in W \mid \Delta_p \subset w^{-1}(\Delta \cup \Phi^-)\}.$$

Clearly id, w_0, w_p belong to $\mathfrak{W}_p$. The condition

$$\Delta_p \subset w^{-1}(\Delta \cup \Phi^-) \tag{2.1}$$

plays an important role in several places of the proof of Theorem 1.1.

To describe the functional equation, we introduce the constants

$$c_p = 2 \langle \lambda_p - \rho_p, \alpha_p^\vee \rangle \tag{2.2}$$

as well as [13]. Then c_p is a positive integer for every $1 \leqslant r \leqslant p$.

2.2. **Definition of Weng zeta functions in terms of abstract root system.**

Let G be a connected semisimple algebraic group and let $\mathfrak{g}$ be its Lie algebra of G. Let T be a maximal torus of G, and let $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \,|\, \mathrm{Ad}(t)X = \alpha(t)X\}$ for each character $\alpha \in X^*(T)$. Then the set $\Phi = \Phi(G, T) = \{\alpha \in X^*(T) \,|\, \mathfrak{g}_\alpha \neq \emptyset\}$ is finite, and makes a root system (in the vector space $X^*(T) \otimes \mathbb{R}$). Conversely, for a given root system Φ, there exists a connected semisimple algebraic group $G = G(\Phi)$ defined over a prime field having Φ as its root system with respect to a split maximal torus T of G by the fundamental theorem of Chevalley. The group $G(\Phi)$ is called a Chevalley group of type Φ (or split group, since it has a maximal torus which is split over the prime field).

Therefore we can deal with Weng zeta functions for Chevalley groups defined over $\mathbb{Q}$ by using the language of abstract root systems only. Now we define Weng zeta functions again in terms of abstract root systems.

Let $\zeta(s)$ be the Riemann zeta function, and let

$$\hat{\zeta}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

$$\xi(s) = s(s-1)\hat{\zeta}(s) = s(s-1)\pi^{-s/2}\Gamma(s)\zeta(s).$$

Note that Weng [24] and Komori [13] use the notation $\xi(s)$ to indicate our $\hat{\zeta}(s)$ as well as Langlands [15] et al.

Definition 2.2 (Periods for (Φ, Δ)) *Let Φ be an irreducible root system and let Δ be a fundamental system of Φ. For $\lambda \in V$, we define*

$$\omega_\Delta^\Phi(\lambda) = \sum_{w \in W} \prod_{\alpha \in \Delta} \frac{1}{\langle w\lambda - \rho, \alpha^\vee \rangle} \prod_{\alpha \in \Phi_w} \frac{\hat{\zeta}(\langle \lambda, \alpha^\vee \rangle)}{\hat{\zeta}(\langle \lambda, \alpha^\vee \rangle + 1)}.$$

Here we understand that the second product equals 1 if $\Phi_w = \Phi^+ \cap w^{-1}\Phi^- = \emptyset$.

Definition 2.3 (Periods for (Φ, Δ, p)) *Let Φ be an irreducible root system of rank r with a fundamental system $\Delta = \{\alpha_1, \cdots, \alpha_r\}$. Let $1 \leqslant p \leqslant r$. Take the coordinate of V as*

$$\lambda = \sum_{j=1}^{r}(1 + s_j)\lambda_j = \rho + \sum_{j=1}^{r} s_j\lambda_j \quad (\lambda \in V)$$

so that $\langle \lambda - \rho, \alpha^\vee \rangle = \sum_{j=1}^{r} a_j s_j$ for $\alpha^\vee = \sum_{j=1}^{r} a_j\alpha_j^\vee$, and write $\Delta_p = \{\beta_1, \cdots, \beta_{r-1}\}$. Then we define $\omega_p : \mathbb{C} \to \mathbb{C}$ by

$$\omega_p(s) = \omega_{\Delta, p}^\Phi(s) = \operatorname*{Res}_{\langle \lambda - \rho, \beta_1^\vee \rangle = 0} \cdots \operatorname*{Res}_{\langle \lambda - \rho, \beta_{r-1}^\vee \rangle = 0} \omega_\Delta^\Phi(\lambda)$$

$$= \operatorname*{Res}_{s_1=0} \cdots \operatorname*{Res}_{s_{p-1}=0} \operatorname*{Res}_{s_{p+1}=0} \cdots \operatorname*{Res}_{s_r=0} \omega_\Delta^\Phi(\lambda),$$

(2.3)

where the variable s_p is written as s.

Remark The function $\omega_p(s)$ is well defined, since it does not depend on the ordering of the set of simple roots Δ_p by Proposition 2.2 of [13].

Remark Let $G = G(\Phi)$ be a Chevalley group, and let B a Borel subgroup containing the maximal split torus T. Let $P (\supset B)$ be a maximal parabolic subgroup of G corresponding to the simple root α_p. Then functions $\omega_\Delta^\Phi(\lambda)$ and $\omega_{\Delta, p}^\Phi(s)$ are the periods $\omega_{\mathbb{Q}, B}^{(G,T)}(\lambda)$ and $\omega_{\mathbb{Q}, P/B}^{(G,T)}(s)$ respectively.

Definition 2.4 *For $w \in \mathfrak{W}_p$ and $(k, h) \in \mathbb{Z}^2$, we define*

$$N_{p,w}(k, h) = \sharp\{\alpha \in w^{-1}\Phi^- \,|\, \langle \lambda_p, \alpha^\vee \rangle = k, \ \mathrm{ht}\,\alpha^\vee = h\},$$

$$N_p(k, h) = \sharp\{\alpha \in \Phi \,|\, \langle \lambda_p, \alpha^\vee \rangle = k, \ \mathrm{ht}\,\alpha^\vee = h\}$$

and

$$M_p(k, h) = \max_{w \in \mathfrak{W}_p} \big(N_{p,w}(k, h-1) - N_{p,w}(k, h) \big).$$

If $k \geqslant 1$ or $h \geqslant 1$, we have

$$N_{p,w}(k, h) = \sharp\{\alpha \in \Phi_w \mid \langle \lambda_p, \alpha^\vee \rangle = k, \ \mathrm{ht}\,\alpha^\vee = h\},$$
$$N_p(k, h) = \sharp\{\alpha \in \Phi^+ \mid \langle \lambda_p, \alpha^\vee \rangle = k, \ \mathrm{ht}\,\alpha^\vee = h\},$$

because a root $\alpha \in \Phi$ is either $\alpha \in \Phi^+$ or $\alpha \in \Phi^-$.

Definition 2.5 (Weng zeta function for (Φ, Δ, p)) *Using the above notation, we define*

$$\hat{\zeta}_p(s) = \hat{\zeta}^\Phi_{\Delta,p}(s) = \omega_p(s) \prod_{k=0}^\infty \prod_{h=2}^\infty \hat{\zeta}(ks+h)^{M_p(k,h)}.$$

Remark Note that $M_p(k, h) = 0$ except for finitely many pairs of integers (k, h). The function $\hat{\zeta}_p(s)$ coincides with the zeta function $\xi_{\mathbb{Q},o}^{G/P}(s)$ of [24] if Φ is the root system attached to (G, T) and α_p corresponds to the maximal parabolic P. Moreover, we find that the numbers $I(G/P)$ and $J(G/P)$ defined in [24, section 2] are given by

$$I(G/P) = \sum_{(k,h) \in \mathbb{Z}^2, \, k \neq 0} M_p(k, h), \quad J(G/P) = \sum_{h \in \mathbb{Z}} M_p(0, h),$$

because the product $\prod_{k=0}^\infty \prod_{h=2}^\infty \hat{\zeta}(ks+h)^{M_p(k,h)}$ is minimal in the sense that it eliminates all $\hat{\zeta}(as+b)$ and $\hat{\zeta}(c)$ appearing in the denominators of $\omega_p(s)$ ([13]).

Remark We will have $M_p(k, h) = N_{p,w_0}(k, h-1) - N_{p,w_0}(k, h)$ in Corollary 8.7 below.

Theorem 2.6 (Komori, Theorem 2.4 of [13]) *Let c_p be the constant defined in (2.2). Then $\hat{\zeta}_p(s)$ satisfies the functional equation*

$$\hat{\zeta}_p(s) = \hat{\zeta}_p(-c_p - s).$$

Remark See Appendix 2 for table of numbers c_p. The numbers c_p can be interpreted geometrically in terms of the first Chern class of tangent bundles on a homogeneous space or an index of a homogeneous space. See Appendix 3 for details.

Now the reformulation of Theorem 1.1 in terms of $\hat{\zeta}_p(s)$ is obvious, thus we omit such a direct reformulation of Theorem 1.1. See Theorem 7.4 and Theorem 7.5 that are reformulations of Theorem 1.1 in terms of entire functions $\xi_p(s)$ defined below.

3. Outline of the proof

Define the entire function $\xi_p(s) = Q(s)\,\hat{\zeta}_p(s)$ by multiplying a suitable polynomial.

1. At first we construct an entire function $\varepsilon_p(s)$ satisfying

$$(\star) \qquad \xi_p(s) = \varepsilon_p(s) \pm \varepsilon_p(-c_p - s).$$

Here the sign $\pm$ depends on the degree of $Q(s)$. The formula $(\star)$ plays a central role on the current line of the proof. More precisely, Theorem 1.1 is reduced to a study of the location of zeros of $\varepsilon_p(s)$ by $(\star)$, and fortunately, it is less hard to investigate the zeros of $\varepsilon_p(s)$ than $\hat{\zeta}_p(s)$. A kind of the formula $(\star)$ was used in every (known) proof of the Riemann hypothesis of $\hat{\zeta}_p(s)$ for $G = SL(n)$ (n=2,3,4,5), $Sp(4)$, G_2. This step is described more precisely in section 5.

2. Successively we investigate the location of the zeros of $\varepsilon_p(s)$. The aim of this second step is to show that

    (i) the number of zeros of $\varepsilon_p(s)$ in $\Re(s) \geqslant -c_p/2$ is finitely many,
    (ii) in a left half-plane, $\varepsilon_p(s)$ has no zero in a region $\Re(s) \leqslant -\kappa \log(|\Im(s)| + 10)$.

This step is described more precisely in section 6.

3. Finally we prove Theorem 1.1 by using the results of the second step. The main tool of this step is the Hadamard product formula of $\varepsilon_p(s)$. Essentially this part is a modification of the method which was established in [12] by the first author. This step is described more precisely in section 7.

The most essential part of our proof of Theorem 1.1 is in the second step. To complete the proof of the second step, we need a detailed study of a structure of $\mathbb{Z}$-grading of root systems Φ (or $\mathrm{Lie}(G)$) induced from fundamental weights λ_P. We review the flow from the first step to the second step in the next section by taking up the cases of $(SL(n), P_{n-1,1})$ as an example.

The first step is established by an algebraic way via the general theory of root systems and Weyl groups attached to maximal parabolic subgroups P. The argument of the final step is achieved by a purely analytic way. Frequently, basic analytic properties of $\zeta(s)$ will be used in demonstrating Theorem 1.1.

4. Cases of $(SL(n), P_{n-1,1})$

Let $P = P_{n-1,1} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in SL(n) \,\middle|\, A \in GL(n-1),\, D \in GL(1) \right\}$ be the standard maximal parabolic subgroup of $G = SL(n)$ attached to the partition $n = (n-1)+1$. For these special cases, Theorem 1.1 was established by Weng (unpublished) after the work of the first author [12]. As a review of the proof of general cases, we sketch the proof of Theorem 1.1 for these special cases restricting it on the first and the second step.

Let B be the standard Borel subgroup consisting of upper triangular matrices, and let T be the standard split torus in B, that is, the torus consisting of diagonal matrices. The root system Φ associated with T is of type A_{n-1}, and is realized as $\Phi^+ = \{e_i - e_j \mid 1 \leqslant i < j \leqslant n\}$, where $\{e_i \mid 1 \leqslant i \leqslant n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. The set of simple roots Δ associated with B is $\Delta = \{\alpha_1 := e_1 - e_2, \cdots, \alpha_{n-1} := e_{n-1} - e_n\}$, and the half sum of positive roots is

$$\rho = \frac{1}{2}\big((n-1)e_1 + (n-3)e_2 + \cdots - (n-3)e_{n-1} - (n-1)e_n\big).$$

The Weyl group W is identified with the symmetric group on n letters S_n by the convention $w(e_i - e_j) = e_{w(i)} - e_{w(j)}$ $(w \in S_n)$. The longest element w_0 of W is given by the permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$, and $-w_0\rho = \rho$. The maximal parabolic subgroup $P = P_{n-1,1}$ corresponds to the simple root $\alpha_{n-1} = e_{n-1} - e_n$, and has the Levi decomposition $P = MN$ with $M \simeq GL(n-1)$. We have

$$\Phi_P^+ = \Phi_{n-1}^+ = \{e_i - e_j \mid 1 \leqslant i < j \leqslant n-1\}, \quad \Delta_P = \Delta_{n-1} = \{\alpha_1, \cdots, \alpha_{n-2}\},$$

$$\rho_P = \rho_{n-1} = \frac{1}{2}\big((n-2)e_1 + (n-4)e_2 + \cdots - (n-4)e_{n-2} - (n-2)e_{n-1}\big),$$

and the fundamental weight corresponding to P is

$$\lambda_P = \lambda_{n-1} = \frac{1}{n}(e_1 + \cdots + e_{n-1} - (n-1)e_n).$$

The Weyl group $W_P = W_{n-1}$ is the subgroup of W which corresponds to the symmetric group on $(n-1)$ letters $\{1, 2, \cdots, n-1\}$. The longest element $w_P = w_{n-1}$ of W_P is given by the permutation $\begin{pmatrix} 1 & 2 & \cdots & n-1 \\ n-1 & n-2 & \cdots & 1 \end{pmatrix}$, and $-w_P\rho_P = \rho_P$.

Comparing with general cases, structures of $(\Phi^+ \setminus \Phi_P^+) \cap w^{-1}\Phi^\pm$ and $\Phi_P^+ \cap w^{-1}\Phi^\pm$ $(w \in W)$ in the present case are rather simple, therefore it is not hard to find that $\prod_{h=2}^{n-1} \hat{\zeta}(h) \cdot \hat{\zeta}(s+n)$ is the minimal product of zeta functions and zeta values which

eliminates all zeta functions and zeta values in the denominator of each term of $\omega_{\mathbb{Q},P/B}^{(G,T)}(s)$. Hence we have

$$\hat{\zeta}_P(s) = \omega_{\mathbb{Q},P/B}^{(G,T)}(s) \prod_{h=2}^{n-1} \hat{\zeta}(h) \cdot \hat{\zeta}(s+n).$$

More precisely, we obtain

$$\hat{\zeta}_P(s) = \sum_{h=1}^{n} R_h(s)\,\hat{\zeta}(s+h)$$

with rational functions

$$R_h(s) = \sum_{\substack{w \in \mathfrak{W}_P \\ |(\Phi^+ \setminus \Phi_P^+) \cap w^{-1}\Phi^+| = h-1}} C_w \prod_{\alpha \in (w^{-1}\Delta) \setminus \Phi_P} \frac{1}{\langle \lambda_P, \alpha^\vee \rangle s + \operatorname{ht} \alpha^\vee - 1},$$

where $\mathfrak{W}_P = \{w \in W \mid w\Delta_P \subset \Delta \cup \Phi^-\}$ and

$$C_w = \prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_P \setminus \Delta_P)} \frac{1}{\operatorname{ht} \alpha^\vee - 1} \;\; \hat{\zeta}(2)^{|\Delta_P^+ \cap w^{-1}\Phi^+|} \prod_{\alpha \in (\Phi_P^+ \setminus \Delta_P) \cap w^{-1}\Phi^+} \frac{\hat{\zeta}(\operatorname{ht} \alpha^\vee + 1)}{\hat{\zeta}(\operatorname{ht} \alpha^\vee)}.$$

The constant c_p appearing in the functional equation (2.2) is calculated as follows:

$$c_P = 2\langle \lambda_P - \rho_P, \alpha_P^\vee \rangle = n.$$

We have $R_h(-n-s) = R_h(s)$ for every $1 \leqslant h \leqslant n$ by considering the involution $w \mapsto w_0 w w_P$ of $\mathfrak{W}_P$ according to [13]. Hence we obtain the functional equation

$$\hat{\zeta}_P(s) = \hat{\zeta}_P(-n-s).$$

by using the functional equation of $\hat{\zeta}(s)$. Define

$$\xi_P(s) = \prod_{k=0}^{n} (s+k) \cdot \hat{\zeta}_P(s).$$

Then we find that $\xi_P(s)$ is an entire function, and each

$$P_h(s) := \frac{1}{(s+h)(s+h-1)} R_h(s) \prod_{k=0}^{n}(s+k) \quad (1 \leqslant h \leqslant n)$$

is a polynomial. Set

$$\varepsilon_P(s) = \begin{cases} \displaystyle\sum_{h=n/2}^{n} P_h(s)\xi(s+h), & n \text{ is even,} \\[2ex] \displaystyle\sum_{h=(n+3)/2}^{n} P_h(s)\xi(s+h) + \frac{1}{2} P_{\frac{n+1}{2}}(s)\xi\left(s + \frac{n+1}{2}\right), & n \text{ is odd} \end{cases}$$

with $\xi(s) = s(s-1)\hat{\zeta}(s)$. Then we have

$$\xi_P(s) = \varepsilon_P(s) + (-1)^{n-1}\varepsilon_P(-n-s)$$

by the functional equation of $\hat{\zeta}_P(s)$. This is the formula $(\star)$ of section 3.

By definition, we have

$$\varepsilon_P(s) = \begin{cases} P_2(s)\xi(s+2), \qquad n=2, \\[2mm] P_n(s)\xi(s+n)\left(1 + \sum_{h=n/2}^{n-1} \frac{P_h(s)}{P_n(s)}\frac{\xi(s+h)}{\xi(s+n)}\right), \\[2mm] \qquad n>2 \text{ is even}, \\[2mm] P_n(s)\xi(s+n)\left(1 + \sum_{h=(n+3)/2}^{n-1} \frac{P_h(s)}{P_n(s)}\frac{\xi(s+h)}{\xi(s+n)} + \frac{1}{2}\frac{P_{\frac{n+1}{2}}(s)}{P_n(s)}\frac{\xi\left(s+\frac{n+1}{2}\right)}{\xi(s+n)}\right), \\[2mm] \qquad n \text{ is odd}. \end{cases}$$

It is well known that $\xi(s)$ has no zeros in the right half plane $\Re(s) \geqslant 1$. Thus $\xi(s+n)$ has no zeros in the right half plane $\Re(s) \geqslant -n/2$. Furthermore $|\xi(s+h)/\xi(s+n)| < 1$ for $\Re(s) > -(h+n-1)/2$ (see Lemma 9.1 below). Therefore, if

$$\deg P_n(s) \geqslant \deg P_h(s) + 1 \quad \text{for every } 1 < h < n,$$

we can conclude that the number of zeros of $\varepsilon_P(s)$ in the right half plane $\Re(s) \geqslant -n/2$. This is **2** (i) of section 3. It is not hard to find that $\deg P_h(s) \leqslant n-3$ for every $1 < h < n$ ($n \geqslant 3$) (see Lemma 10.1). Therefore assuming $\deg P_n(s) = n-2$, we have the above inequality, but it is not trivial to know the degree of $P_n(s)$ for general $n \geqslant 2$ even if it may be the actual situation. By definition of $P_n(s)$, it is equivalent to the nonvanishing of the $\mathbb{Q}$-linear combination of products of special values of $\hat{\zeta}(s)$:

$$\sum_{\substack{w \in \mathfrak{W}_P \\ |(\Phi^+ \setminus \Phi_P^+) \cap w^{-1}\Phi^+| = n-1}} C_w \prod_{\substack{\alpha \in (w^{-1}\Delta)\setminus\Phi_P \\ |(w^{-1}\Delta)\setminus\Phi_P| = 1}} \frac{1}{\langle \lambda_P, \alpha^\vee \rangle} \neq 0,$$

and it is highly nontrivial. Fortunately, this problem can be solved by using the volume formula of [23, section 4.7] (see the proof of Lemma 10.3 in section 10). In contrast with **2** (i), **2** (ii) is provided easily by using the Stirling formula for the gamma function.

5. The first step of the proof of Theorem 1.1

5.1. **A modification to entire functions.** The zeta function $\hat{\zeta}_{\Delta,p}^\Phi(s)$ is meromorphic on $\mathbb{C}$ and has finitely many poles. In order to prove Theorem 1.1, we consider the entire function $\xi_{\Delta,p}^\Phi(s)$ which is a polynomial multiple of $\hat{\zeta}_{\Delta,p}^\Phi(s)$.

At first, we recall formula (2.8) of [13]:

$$\omega_{\Delta,p}^\Phi(s) = \sum_{w \in \mathfrak{W}_p} \prod_{\alpha \in (w^{-1}\Delta)\setminus\Delta_p} \frac{1}{\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee - 1}$$
$$\times \prod_{\alpha \in \Phi_w \setminus \Delta_p} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee) \prod_{\alpha \in (-\Phi_w)} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee)^{-1}.$$

Here we understand that each product equals 1 if its index set is empty.

Definition 5.1 *We define the product of zeta functions*

$$F_p(s) := \prod_{\alpha \in \Phi^-} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee),$$

and the meromorphic function

$$Z_p(s) := F_p(s)\,\omega_{\Delta,p}^\Phi(s).$$

Then we have

$$Z_p(s) = \sum_{w \in \mathfrak{W}_p} f_{p,w}(s) \, g_{p,w}(s),$$

with rational functions $f_{p,w}$ and products of zeta functions $g_{p,w}$:

$$f_{p,w}(s) = \prod_{\alpha \in (w^{-1}\Delta) \backslash \Delta_p} \frac{1}{\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee - 1},$$

$$g_{p,w}(s) = \prod_{\alpha \in (w^{-1}\Phi^-) \backslash \Delta_p} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee),$$

since $\Phi^- \backslash \Phi_w = \Phi^- \cap w^{-1}\Phi^-$ and $(\Phi_w \backslash \Delta_p) \cup (\Phi^- \backslash \Phi_w) = (w^{-1}\Phi^-) \backslash \Delta_p$.

We modify the above formula of $g_{p,w}$ so that all coefficients $\langle \lambda_p, \alpha^\vee \rangle$ of s in $g_{p,w}$ will be nonnegative integers. We see that $g_{p,w}(s)$ is

$$\prod_{\alpha \in (w^{-1}\Phi^-) \cap \Phi^-} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee) \prod_{\alpha \in ((w^{-1}\Phi^-) \cap \Phi^+) \backslash \Delta_p} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee)$$

$$= \prod_{\alpha \in (w^{-1}\Phi^-) \cap \Phi^-} \hat{\zeta}(1 - \langle \lambda_p, \alpha^\vee \rangle s - \mathrm{ht}\, \alpha^\vee) \prod_{\alpha \in ((w^{-1}\Phi^-) \cap \Phi^+) \backslash \Delta_p} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee)$$

$$= \prod_{\alpha \in (w^{-1}\Phi^-) \cap \Phi^-} \hat{\zeta}(\langle \lambda_p, -\alpha^\vee \rangle s + \mathrm{ht}\, (-\alpha^\vee) + 1) \prod_{\alpha \in ((w^{-1}\Phi^-) \cap \Phi^+) \backslash \Delta_p} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee)$$

$$= \prod_{\alpha \in (w^{-1}\Phi^+) \cap \Phi^+} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee + 1) \prod_{\alpha \in ((w^{-1}\Phi^-) \cap \Phi^+) \backslash \Delta_p} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee)$$

for each $w \in \mathfrak{W}_p$ by using the functional equation $\hat{\zeta}(s) = \hat{\zeta}(1 - s)$. Then all coefficients $\langle \lambda_p, \alpha^\vee \rangle$ in the last line are nonnegative integers:

$$g_{p,w}(s) = \prod_{\alpha \in \Phi_w \backslash \Delta_p} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee) \prod_{\alpha \in \Phi^+ \backslash \Phi_w} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee + 1) \qquad (5.1)$$

for each $w \in \mathfrak{W}_p$. Note that $\Phi^+ \backslash \Phi_w = \Phi^+ \cap w^{-1}\Phi^+$. Define

$$\delta_{\alpha,w} = \begin{cases} 1 & \text{if } \alpha \in w^{-1}\Phi^+, \\ 0 & \text{if } \alpha \in w^{-1}\Phi^-. \end{cases} \qquad (5.2)$$

Then formula (5.1) is written as

$$g_{p,w}(s) = \hat{\zeta}(2)^{|\Delta_p \cap w^{-1}\Phi^+|} \prod_{\alpha \in \Phi^+ \backslash \Delta_p} \hat{\zeta}(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee + \delta_{\alpha,w}).$$

We define entire functions $\tilde{X}_{p,w}(s)$ by replacing $\hat{\zeta}(s)$ of $g_{p,w}$ by $\xi(s)$:

$$\tilde{X}_{p,w}(s) = \xi(2)^{|\Delta_p \cap w^{-1}\Phi^+|} \prod_{\alpha \in \Phi^+ \backslash \Delta_p} \xi(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee + \delta_{\alpha,w}). \qquad (5.3)$$

Obviously, we have

$$\tilde{X}_{p,w}(s) = 2^{|\Delta_p \cap w^{-1}\Phi^+|} g_{p,w}(s)$$

$$\times \prod_{\alpha \in \Phi^+ \backslash \Delta_p} \Big(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee + \delta_{\alpha,w}\Big)\Big(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\, \alpha^\vee + \delta_{\alpha,w} - 1\Big).$$

Define the polynomial $Q_p(s)$ by

$$Q_p(s) = \prod_{w \in \mathfrak{W}_p} \left[2^{|\Delta_p \cap w^{-1}\Phi^+|} \prod_{\alpha \in (w^{-1}\Delta) \setminus \Delta_p} \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht} \alpha^\vee - 1 \right) \right.$$

$$\left. \times \prod_{\alpha \in \Phi^+ \setminus \Delta_p} \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht} \alpha^\vee + \delta_{\alpha,w} \right) \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht} \alpha^\vee + \delta_{\alpha,w} - 1 \right) \right]$$

Definition 5.2 *Under the above notation, we define*

$$X_p(s) := Q_p(s) Z_p(s).$$

By definitions $X_p(s)$ is an entire function having the form

$$X_p(s) = \sum_{w \in \mathfrak{W}_p} \tilde{Q}_{p,w}(s) \tilde{X}_{p,w}(s),$$

where $\tilde{Q}_{p,w}(s)$ are polynomials given by

$$\tilde{Q}_{p,w}(s) = \prod_{w \neq v \in \mathfrak{W}_p} \left[2^{|\Delta_p \cap v^{-1}\Phi^+|} \prod_{\alpha \in (v^{-1}\Delta) \setminus \Delta_p} \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht} \alpha^\vee - 1 \right) \right.$$

$$\left. \times \prod_{\alpha \in \Phi^+ \setminus \Delta_p} \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht} \alpha^\vee + \delta_{\alpha,v} \right) \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht} \alpha^\vee + \delta_{\alpha,v} - 1 \right) \right].$$

Remark Note that the polynomial $Q_p(s)$ is not minimal in a sense that the polynomial of the lowest degree such that $Q(s)Z_p(s)$ has no poles. In fact $\tilde{Q}_{p,w}(s)$ $(w \in \mathfrak{W}_p)$ has a lot of common divisors as well as $\tilde{X}_{p,w}(s)$.

Now we modify $\tilde{X}_{p,w}(s)$ and $\tilde{Q}_{p,w}(s)$ a little. Define

$$C_{p,w} = \xi(2)^{|\Delta_p \cap w^{-1}\Phi^+|} \prod_{\alpha \in \Phi_p^+ \setminus \Delta_p} \xi(\operatorname{ht} \alpha^\vee + \delta_{\alpha,w}), \tag{5.4}$$

$$X_{p,w}(s) = \prod_{\alpha \in \Phi^+ \setminus \Phi_p^+} \xi(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht} \alpha^\vee + \delta_{\alpha,w})$$

and

$$Q_{p,w}(s) = C_{p,w} \tilde{Q}_{p,w}(s).$$

Then we have

$$\tilde{X}_{p,w}(s) = C_{p,w} X_{p,w}(s)$$

and

$$X_p(s) = \sum_{w \in \mathfrak{W}_p} Q_{p,w}(s) X_{p,w}(s).$$

Lemma 5.3 *Let c_p be the number defined in (2.2). Let w_0 (resp. w_p) be the longest element of W (resp. W_p). Then the functional equations*

$$\tilde{X}_{p,w}(-c_p - s) = \tilde{X}_{p,w_0 w w_p}(s),$$

$$\tilde{Q}_{p,w}(-c_p - s) = \epsilon_p \tilde{Q}_{p,w_0 w w_p}(s),$$

$$Q_p(-c_p - s) = \epsilon_p Q_p(s),$$

hold for every $w \in \mathfrak{W}_p$ for a suitable choice of sign $\epsilon_p \in \{\pm 1\}$ depending only on p. In particular we have the functional equation

$$X_p(-c_p - s) = \epsilon_p X_p(s).$$

Proof. This is a consequence of Lemma 5.3 of [13].   □

5.2. **Decomposition of $X_p(s)$.** The aim of this part is a construction of an entire function $E_p(s)$ satisfying

$$X_p(s) = E_p(s) + \epsilon_p E_p(-c_p - s),$$

where ϵ_p is the sign of Lemma 5.3.

Definition 5.4 *For $w \in \mathfrak{W}_p$, we define the number $l_p(w)$ by*

$$l_p(w) = \sum_{\alpha \in \Phi^+ \setminus \Phi_p^+} (1 - \delta_{\alpha,w}) = |(\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-|.$$

Remark Compare $l_p(w)$ with the length $l(w) = |\Phi^+ \cap w^{-1}\Phi^-|$ of $w \in W$.

Lemma 5.5 *Let w_p be the longest element of W_p. Then we have*

$$l_p(w) + l_p(w_0 w w_p) = |\Phi^+ \setminus \Phi_p^+|$$

for every $w \in \mathfrak{W}_p$.

Proof. We have

$$|(\Phi^+ \setminus \Phi_p^+) \cap (w_0 w w_p)^{-1}\Phi^-| = |w_p(\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^+| = |(\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^+|.$$

Hence we obtain

$$l_p(w) + l_p(w_0 w w_p) = |(\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-| + |(\Phi^+ \setminus \Phi_p^+) \cap (w_0 w w_p)^{-1}\Phi^-|$$
$$= |(\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-| + |(\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^+| = |\Phi^+ \setminus \Phi_p^+|$$

for every $w \in \mathfrak{W}_p$. $\square$

Definition 5.6 *We divide the subset $\mathfrak{W}_p$ of W into three parts:*

$$\mathfrak{W}_p = \mathfrak{W}_p^+ \text{ II } \mathfrak{W}_p^- \text{ II } \mathfrak{W}_p^0,$$

where II means the disjoint union, and

$$\mathfrak{W}_p^+ := \{w \in \mathfrak{W}_p \mid l_p(w) < |\Phi^+ \setminus \Phi_p^+|/2\},$$
$$\mathfrak{W}_p^- := \{w \in \mathfrak{W}_p \mid l_p(w) > |\Phi^+ \setminus \Phi_p^+|/2\},$$
$$\mathfrak{W}_p^0 := \{w \in \mathfrak{W}_p \mid l_p(w) = |\Phi^+ \setminus \Phi_p^+|/2\}.$$

Note that $\mathfrak{W}_p^0 = \emptyset$ is possible, and

$$w_0 \mathfrak{W}_p^- w_p = \mathfrak{W}_p^+, \qquad w_0 \mathfrak{W}_p^0 w_p = \mathfrak{W}_p^0.$$

by Lemma 5.5.

Definition 5.7 *Define*

$$E_p(s) = \sum_{w \in \mathfrak{W}_p^+} Q_{p,w}(s) X_{p,w}(s) + \frac{1}{2} \sum_{w \in \mathfrak{W}_p^0} Q_{p,w}(s) X_{p,w}(s).$$

Here we understand that the second sum equals zero if the subset $\mathfrak{W}_p^0$ is empty.

Proposition 5.8 *We have*

$$X_p(s) = E_p(s) + \epsilon_p E_p(-c_p - s),$$

where ϵ_p is the sign of Lemma 5.3.

Proof. This is obvious by the above definitions and lemmas. $\square$

The decomposition of Proposition 5.8 is useful for our proof of Theorem 1.1 (the weak Riemann hypothesis). However we do not know whether this choice of $E_p(s)$ is best possible for the Riemann hypothesis for $\hat{\zeta}_P(s)$.

5.3. **Reduction of $X_p(s)$.** By definition of $Q_{p,w}(s)$ and $X_{p,w}(s)$, component terms $Q_{p,w}(s)X_{p,w}(s)$ of $X_p(s)$ have a lot of "common factors". Now we define "the greatest common divisor" of these terms.

Definition 5.9 *Define*

$$D_p(s) = \prod_{k=1}^{\infty} \prod_{h=2}^{\infty} \xi(ks+h)^{N_p(k,h-1)-M_p(k,h)},$$

$$R_p(s) = \text{g.c.d}\,\{Q_{p,w}(s) \mid w \in \mathfrak{W}_p\},$$

where "g.c.d" means the monic polynomial of the maximal degree which divides all polynomials $Q_{p,w}(s)$ in the polynomial ring $\mathbb{C}[s]$.

Remark The above definition of $D_p(s)$ is different from the one of [13], since we use the notation $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ in this article.

Definition 5.10 *Define*

$$\xi_p(s) = \frac{X_p(s)}{R_p(s)D_p(s)}, \qquad \varepsilon_p(s) = \frac{E_p(s)}{R_p(s)D_p(s)}.$$

We have

$$\xi_p(s) = \varepsilon_p(s) \pm \varepsilon_p(-c_p - s).$$

The entire function $\xi_p(s)$ equals the zeta function $\hat{\zeta}_p(s)$ times a polynomial. Hence Theorem 1.1 is equivalent to the statement that all but finitely many zeros of $\xi_p(s)$ lie on the line $\Re(s) = -c_p/2$.

6. The second step of the proof Theorem 1.1

Roughly the second step is a consequence of the result that $X_{p,\text{id}}(s)$ is dominant in $\Re(s) \geqslant -c_p/2$ and $X_{p,\text{id}}(s)/D_p(s)$ has no zeros in $\Re(s) \geqslant -c_p/2$, since

$$E_p(s) = \Big[\sum_{w \in \mathfrak{W}_p^{\ddagger}} Q_w(s)\Big] \cdot X_{p,\text{id}}(s) \cdot \Big[1 + \sum_{\substack{w \in \mathfrak{W}_p \setminus \mathfrak{W}_p^{\ddagger} \\ 2l_p(w) \leqslant |\Phi^+ \setminus \Phi_p^+|}} R_w(s)\frac{X_{p,w}(s)}{X_{p,\text{id}}(s)}\Big],$$

where $\mathfrak{W}_p^{\ddagger}$ is a subset of $\mathfrak{W}_p$ defined below and $R_w(s)$ are some rational functions. The analytic behavior of $E_p(s)$ in the right half-plane $\Re(s) > -c_p/2$ is less difficult than that of the left.

6.1. **Behavior of $E_p(s)$ in a right half-plane.** In this part we construct the dominant term of $\varepsilon_p(s)$ in a right half-pane via $E_p(s)$ of Definition 5.7.

Lemma 6.1 *Let $\{v_1 = \text{id}, v_2, \cdots, v_l\}$ be the left minimal coset representative of W_p:*

$$W = \bigsqcup_{j=1}^{l} v_j W_p.$$

Then $l_p(w)$ depends only on the coset $v_j W_p$.

Proof. For $w \in v_j W_p$, we have $l_p(w) = |(\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-| = |w(\Phi^+ \setminus \Phi_p^+) \cap \Phi^-| = |v_j(\Phi^+ \setminus \Phi_p^+) \cap \Phi^-| = |(\Phi^+ \setminus \Phi_p^+) \cap v_j^{-1}\Phi^-| = l_p(v_j)$. $\qquad\square$

Lemma 6.2 *Let w_0 be the longest element of W. We have*

$$l_p(\text{id}) = 0, \qquad l_P(w_0) = |\Phi^+ \setminus \Phi_p^+|.$$

In other words the minimum value and the maximum value of $l_p(w)$ are attained by the identity element id *and the longest element w_0 respectively.*

Proof. We have $l_p(\mathrm{id}) = |(\Phi^+ \setminus \Phi_p^+) \cap \Phi^-| = 0$ and $l_p(w_0) = |(\Phi^+ \setminus \Phi_p^+) \cap \Phi^+| = |\Phi^+ \setminus \Phi_p^+|$ by definition of $l_p(w)$. $\qquad\square$

Lemma 6.3 *Define*

$$\mathfrak{W}_p^{\ddagger} = \{w \in \mathfrak{W}_p \mid l_p(w) = 0\}$$

Then $\mathrm{id} \in \mathfrak{W}_p^{\ddagger}$ *and*

$$\mathfrak{W}_p^{\ddagger} = \{w \in \mathfrak{W}_p \mid (\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^+ = (\Phi^+ \setminus \Phi_p^+)\}.$$

In particular $X_{p,w}(s) = X_{p,\mathrm{id}}(s)$ *for every* $w \in \mathfrak{W}_p^{\ddagger}$.

Proof. This lemma immediately follows from Definition 5.4 and (5.3). $\qquad\square$

By Lemmas 6.1, 6.2 and 6.3

$$\mathfrak{W}_p^{\ddagger} = W_p \amalg v_2 W_p \amalg \cdots v_m W_p,$$

for some $1 \leqslant m < l \ (= |W/W_p|)$.

Definition 6.4 (the dominant term in a right half-plane) *We define*

$$X_p^{\ddagger}(s) := X_{p,\mathrm{id}}(s), \qquad Q_p^{\ddagger}(s) := \sum_{w \in \mathfrak{W}_p^{\ddagger}} Q_{p,w}(s).$$

By definition

$$X_p^{\ddagger}(s) = X_{p,\mathrm{id}}(s) = \prod_{\alpha \in \Phi^+ \setminus \Phi_p^+} \xi(\langle \lambda_p, \alpha^{\vee} \rangle s + \mathrm{ht}\,\alpha^{\vee} + 1) = \prod_{k=1}^{\infty} \prod_{h=2}^{\infty} \xi(ks + h)^{N_p(k, h-1)}.$$

The following two propositions assert that

$$Q_p^{\ddagger}(s)\, X_p^{\ddagger}(s) = \Big(\sum_{w \in \mathfrak{W}_p^{\ddagger}} Q_{p,w}(s) \Big) \prod_{\alpha \in \Phi^+ \setminus \Phi_p^+} \xi(\langle \lambda_p, \alpha^{\vee} \rangle s + \mathrm{ht}\,\alpha^{\vee} + 1)$$

is the dominant term of $E_p(s)$ in a right half-plane.

Proposition 6.5 *We have*

$$\deg Q_p^{\ddagger}(s) \geqslant \deg Q_{p,w}(s) + 1$$

for every $w \in (\mathfrak{W}_p^+ \cup \mathfrak{W}_p^0) \setminus \mathfrak{W}_p^{\ddagger}$.

Proposition 6.6 *We have*

$$\left| \frac{X_{p,w}(s)}{X_p^{\ddagger}(s)} \right| < 1 \quad for \quad \Re(s) > -\frac{c_p}{2}$$

and

$$\left| \frac{X_{p,w}(s)}{X_p^{\ddagger}(s)} \right| \leqslant 1 \quad for \quad \Re(s) = -\frac{c_p}{2}$$

for every $w \in \mathfrak{W}_p \setminus \mathfrak{W}_p^{\ddagger}$.

The dominant term $X_p^{\ddagger}(s)$ has the following property.

Proposition 6.7 *Let* $D_p(s)$ *be the function defined in Definition 5.9. Then*

$$\frac{X_p^{\ddagger}(s)}{D_p(s)} \neq 0 \quad for \quad \Re(s) \geqslant -\frac{c_p}{2}.$$

Furthermore, there exists a positive function $\delta(t)$ *defined on the real line satisfying* $\delta(t)\log|t| \to \infty$ $(|t| \to \infty)$ *such that* $X_p^{\ddagger}(s)/D_p(s)$ *has no zero in* $\Re(s) \geqslant -c_p/2 - \delta(t)$.

Remark By a known result [22, p.135] obtained by Vinogradov's method, we could take $\delta(t)$ such that

$$\delta(t) \ll \frac{1}{(\log t)^{2/3}(\log \log t)^{1/3}}$$

as $t \to \infty$.

We prove Proposition 6.5 in section 10, and Propositions 6.6 and 6.7 in section 9. These three propositions derive the following result.

Proposition 6.8 *The number of zeros of $\varepsilon_p(s)$ lying in right half-plane $\Re(s) \geqslant -c_p/2$ is finitely many at most. Furthermore, there exists a positive function $\delta(t)$ on the real line satisfying $\delta(t) \log|t| \to \infty$ ($|t| \to \infty$) such that the number of zeros of $\varepsilon_p(s)$ in $\Re(s) \geqslant -c_p/2 - \delta(t)$ is finitely many at most.*

Proof. We have

$$E_p(s) = Q_p^{\ddagger}(s) X_p^{\ddagger}(s)[1 + r_p(s)] \tag{6.1}$$

with

$$r_p(s) = \sum_{w \in \mathfrak{W}_p^+ \setminus \mathfrak{W}_p^{\ddagger}} \frac{Q_{p,w}(s)}{Q_p^{\ddagger}(s)} \cdot \frac{X_{p,w}(s)}{X_p^{\ddagger}(s)} + \frac{1}{2} \sum_{w \in \mathfrak{W}_p^0} \frac{Q_{p,w}(s)}{Q_p^{\ddagger}(s)} \cdot \frac{X_{p,w}(s)}{X_p^{\ddagger}(s)}$$

Define

$$\mathfrak{D}_p = \left\{ s \in \mathbb{C} \ \Big| \ \sum_{w \in \mathfrak{W}_p^+ \setminus \mathfrak{W}_p^{\ddagger}} \left| \frac{Q_{p,w}(s)}{Q_p^{\ddagger}(s)} \right| + \frac{1}{2} \sum_{w \in \mathfrak{W}_p^0} \left| \frac{Q_{p,w}(s)}{Q_p^{\ddagger}(s)} \right| \geqslant 1 \right\}.$$

Then $\mathfrak{D}_p$ is a bounded region in $\mathbb{C}$ by Proposition 6.5. Therefore we have

$$|r_p(s)| < 1 \quad \text{for} \quad \Re(s) \geqslant -\frac{c_p}{2} \text{ with } s \notin \mathfrak{D}_p, \tag{6.2}$$

by Proposition 6.6. Hence Proposition 6.7 and (6.2) implies Proposition 6.8. $\qquad\square$

The final proposition of the second step is about the zero-free region of $E_p(s)$ on a left half-plane.

Proposition 6.9 *There exists a positive real number κ such that $E_p(s)$ has no zeros in the region $\Re(s) \leqslant -\kappa \log(|\Im(s)| + 10)$.*

This will be proved in section 11 by using the Stirling formula. Combining Propositions 6.8 and 6.9, we find that all but finitely many zeros of $\varepsilon_p(s)$ lie in the region

$$\left\{ s = \sigma + it \in \mathbb{C} \ \Big| -\kappa \log(|t| + 10) < \sigma < -\frac{c_p}{2} \right\}.$$

7. THE THIRD STEP OF THE PROOF THEOREM 1.1

The final step of the proof of (the front half of) Theorem 1.1 consists of three parts. The first one is about the number of zeros of $\varepsilon_p(s)$ in a given region. The second one is the Hadamard factorization formula of $\varepsilon_p(s)$. The third one is an application of a result of de Bruijn [3, p.215] (see also Lemma 3.1 of [12]) which was established by the first author in [12].

Proposition 7.1 *Let $T > 1$, and $\sigma > c_p/2$. Denote by $N(T; \sigma)$ the number of zeros of $\varepsilon_p(s)$ in the region*

$$-\sigma < \Re(s) < -c_p/2 - \delta(t), \quad 0 < \Im(s) < T.$$

Then there exist a positive number $\sigma_{\mathrm{L}} > 0$ such that

$$N(T; \sigma_{\mathrm{L}}) = C_1 T \log T + C_2 T + O(\log T)$$

for some positive real number $C_1 > 0$ and real number C_2, and

$$N(T; +\infty) = C_1 T \log T + C_3 T + O(\log^2 T)$$

for some real number C_3.

Thus we have $N(T; +\infty) - N(T; \sigma_L) = O(T)$. Also, we can justify this by applying and modifying the method in [22, p.230]. This will be proved in section 12.

Proposition 7.2 *Define*

$$W_p(z) = \varepsilon_p(-c_p/2 + iz).$$

Then it has the product formula

$$W_p(z) = \omega\, e^{\alpha z} V(z) W_1(z) W_2(z),$$

where ω is a nonzero real number, α is a real number, $V(z)$ is a polynomial having no zeros in $\Im(z) > 0$ except for purely imaginary zeros,

$$W_1(z) = \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\rho_n} \right) \left(1 + \frac{z}{\bar{\rho}_n} \right) \right],$$

$$W_2(z) = \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\eta_n} \right) \left(1 + \frac{z}{\bar{\eta}_n} \right) \right]$$

with $\Re(\rho_n) > 0$, $\Re(\eta_n) > 0$ and $0 < \delta(t) < \Im(\rho_n) < \sigma_L + 1 < \Im(\eta_n) < \kappa \log(\Re(\eta_n) + 10)$ for every $n \geqslant 1$. Here $\delta(t)$ is the function of Proposition 6.8, κ is the positive number of Proposition 6.9, and σ_L is the positive number of Proposition 7.1. The products W_1 and W_2 converge uniformly on every compact subset in $\mathbb{C}$.

This will be proved in section 13 by using Proposition 7.1.

Proposition 7.3 (Proposition 3.1 of [12]) *Let $W(z)$ be a function in $\mathbb{C}$. Suppose that $W(z)$ has the product factorization*

$$W(z) = h(z)\, e^{\alpha z} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\rho_n} \right) \left(1 + \frac{z}{\bar{\rho}_n} \right) \right],$$

where $h(z)$ is a nonzero polynomial having N many zeros counted with multiplicity in the lower half-plane, $\alpha \in \mathbb{R}$, $\Im(\rho_n) > 0$ $(n = 1, 2, \cdots)$, and the product converges uniformly in any compact subset of $\mathbb{C}$. Then, $W(z) + \overline{W(\bar{z})}$ and $W(z) - \overline{W(\bar{z})}$ has at most N pair of conjugate complex zeros counted with multiplicity.

Now we achieve the following goal which is an immediate consequence of Propositions 7.2 and 7.3. Note that the realness of the exponent α of Proposition 7.2 is crucial.

Theorem 7.4 (Weak Riemann Hypothesis for ξ_p) *There exists a bounded region $\mathfrak{B}_p$ such that all zeros of $\xi_p(s)$ outside $\mathfrak{B}_p$ lie on the line $\Re(s) = -c_p/2$.*

By studying of the behavior of the argument of $\varepsilon_p(-c_p/2 + it)$ $(t > 0)$, we obtain the following additional result.

Theorem 7.5 (Simple zeros of ξ_p) *There exists a bounded region $\mathfrak{B}'_p (\supset \mathfrak{B}_p)$ such that all zeros of $\xi_p(s)$ outside $\mathfrak{B}'_p$ lie on the line $\Re(s) = -c_p/2$ and simple.*

This will be proved in section 14. Theorems 7.4 and 7.5 imply the main result Theorem 1.1 because of Chevalley's fundamental theorem.

8. PRELIMINARIES FOR PROOF OF PROPOSITIONS 6.6 AND 6.7

In this section, we prepare several lemmas for the proof of Propositions 6.6 and 6.7. Indeed, Lemmas 8.1, 8.4 and 8.15 will play an important role for it. The condition (2.1) in Definition 2.1 is essential for Lemma 8.1.

For integers k and l, we define

$$\Sigma_p(k) = \{\alpha \in \Phi \mid \langle \lambda_p, \alpha^\vee \rangle = k\},$$
$$\Sigma_p(k, h) = \{\alpha \in \Phi \mid \langle \lambda_p, \alpha^\vee \rangle = k, \ \mathrm{ht}\, \alpha^\vee = h\}.$$

They are not empty for finitely many (k, h). The set of positive roots Φ^+ is decomposed into the disjoint union

$$\Phi^+ \setminus \Phi_p^+ = \bigsqcup_{k=1}^{\infty} \Sigma_p(k), \quad \Sigma_p(k) = \bigsqcup_{h=1}^{\infty} \Sigma_p(k, h).$$

By definition, $N_p(k, h)$ is the cardinality of $\Sigma_p(k, h)$.

Lemma 8.1 *Let $w \in \mathfrak{W}_p$. Let α be a positive root. Suppose that $\alpha \in (\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-$ and $\alpha + \alpha_j \in \Phi^+ \setminus \Phi_p^+$ for some $\alpha_j \in \Delta_p$. Then $\alpha + \alpha_j \in (\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-$.*

Note. Besides $\alpha \in (\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-$ and $\alpha + \alpha_p \in \Phi^+ \setminus \Phi_p^+$ does not imply $\alpha + \alpha_p \in (\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-$ in general.

Proof. It suffices to show that $w(\alpha + \alpha_j) \in \Phi^-$ under the assumption. By the assumption and the definition of $\mathfrak{W}_p$, we have $w\alpha \in \Phi^-$ and $w\alpha_j \in \Delta \cup \Phi^-$. If $w\alpha_j \in \Phi^-$, we have $w(\alpha + \alpha_j) \in \Phi^-$, since $\alpha + \alpha_j$ is a root. If $w\alpha_j \in \Delta$, we also have $w\alpha + w\alpha_j \in \Phi^-$. In fact, it is impossible that $w\alpha_j \in \Delta$ and $w\alpha + w\alpha_j \in \Phi^+$, since $w\alpha \in \Phi^-$. In each case, we have $w(\alpha + \alpha_j) \in \Phi^-$. $\qquad\square$

Lemma 8.2 *Let α be a positive root in $\Phi^+ \setminus \Phi_p^+$. Suppose that $\langle \rho_p, \alpha^\vee \rangle < 0$. Then there exists $\alpha_j \in \Delta_p$ such that $\alpha^\vee + \alpha_j^\vee \in (\Phi^+ \setminus \Phi_p^+)^\vee$. In particular, there exists $\alpha_j \in \Delta_p$ such that $\alpha^\vee + \alpha_j^\vee \in ((\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-)^\vee$, if $\langle \rho_p, \alpha^\vee \rangle < 0$ for $\alpha \in (\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-$.*

Proof. By $\langle \rho_p, \alpha^\vee \rangle < 0$, there exists $\alpha_j \in \Delta_p$ such that $\langle \alpha_j, \alpha^\vee \rangle < 0$, since

$$2\rho_p = \sum_{\beta \in \Phi_p^+} \beta = \sum_{\substack{j=1 \\ j \neq p}}^{r} n_j \alpha_j \quad (n_j \in \mathbb{Z}_{>0}).$$

Thus $\langle \alpha_j^\vee, \alpha^\vee \rangle < 0$ by $\alpha_j^\vee = 2\alpha_j / \langle \alpha_j, \alpha_j \rangle$. Hence $\alpha^\vee + \alpha_j^\vee$ is a positive root in $\Phi^\vee$, ([7, §9]). Moreover $\alpha^\vee + \alpha_j^\vee$ is a positive root in $(\Phi^+ \setminus \Phi_p^+)^\vee$, since $\alpha^\vee \in (\Phi^+ \setminus \Phi_p^+)^\vee$ and $\alpha_p^\vee \in \Phi_p^+$. The second statement follows from the first statement and Lemma 8.1. $\qquad\square$

Lemma 8.3 *Let $k \geqslant 1$. Then*
   (1) *$N_p(k, h) = N_p(k, kc_p - h)$ for every $h \geqslant 1$, and*
   (2) *$N_p(k, h) \leqslant N_p(k, h + 1)$ if $2h + 1 \leqslant kc_p$.*

Proof. See Proposition 1 of [16]. See also Lemma 4.3 (1) of [13] for (1). $\qquad\square$

Lemma 8.4 *Let k and h be positive integers. Write*

$$\Sigma_p(k, h) = \{\beta_1, \cdots, \beta_N\} \quad (N = N_p(k, h))$$

if $\Sigma_p(k, h)$ is not empty. Suppose that $k \geqslant 1$ and $2h + 1 \leqslant kc_p$. Then there exists simple roots $\alpha_{j_1}, \cdots, \alpha_{j_N}$ (not necessary distinct) such that
   - *$\alpha_{j_n} \in \Delta_p$ $(1 \leqslant n \leqslant N)$,*
   - *$\beta_n^\vee + \alpha_{j_n}^\vee \in (\Phi^+ \setminus \Phi_p^+)^\vee$ $(1 \leqslant n \leqslant N)$, and*
   - *$\beta_m^\vee + \alpha_{j_m}^\vee \neq \beta_n^\vee + \alpha_{j_n}^\vee$ $(m \neq n)$.*

Proof. See the proof of Lemma 8.15. $\qquad\square$

Lemma 8.5 *Let $k \geqslant 1$ and $w \in \mathfrak{W}_p$. Then $N_{p,w}(k, h) \leqslant N_{p,w}(k, h + 1)$ if $2h + 1 \leqslant kc_p$. In particular, $M_p(k, h) = 0$ if $2h - 1 \leqslant kc_p$*

Proof. This is a consequence of Lemmas 8.1 and 8.4. $\qquad\square$

Lemma 8.6 *Let $k \geqslant 1$. Then we have*

$$M_p(k, h) = \begin{cases} 0 & \text{if } 2h - 1 \leqslant kc_p, \\ N_p(k, h - 1) - N_p(k, h) & \text{if } 2h - 1 > kc_p. \end{cases}$$

Proof. We put

$$D_p^{(1)}(s) = \prod_{k=1}^{\infty} \prod_{h=2}^{\infty} \xi(ks+h)^{N_p(k,h-1)-M_p(k,h)}$$

(cf. Definition 5.9). Then we have

$$D_p^{(1)}(s) = \prod_{k=1}^{\infty} \prod_{h \leqslant (kc_p+1)/2} \xi(ks+h)^{N_p(k,h-1)}$$
$$\times \prod_{h > (kc_p+1)/2} \xi(ks+h)^{N_p(k,h-1)-M_p(k,h)}$$

by Lemma 8.5. On the other hand, we have

$$D_p^{(1)}(-c_p - s) = \prod_{k=1}^{\infty} \prod_{h < (kc_p+1)/2} \xi(ks+h)^{N_p(k,h)-M_p(k,kc_p-h+1)}$$
$$\times \prod_{h \geqslant (kc_p+1)/2} \xi(ks+h)^{N_p(k,h)}$$

by using Lemma 4.3 (1) of [13]. Because of $D_p^{(1)}(-c_p - s) = D_p^{(1)}(s)$ by Lemma 5.5 of [13], we have $M_p(k,h) = N_p(k,h-1) - N_p(k,h)$ for $h > (kc_p + 1)/2$. □

Corollary 8.7 *In the definition of $M_p(k,h)$, the longest element w_0 attains the maximum*

$$\max_{w \in \mathfrak{W}_p} \left(N_{p,w}(k,h-1) - N_{p,w}(k,h) \right).$$

Proof. This is a consequence of Lemmas 8.3 and 8.6, since $N_p(k,h) = N_{p,w_0}(k,h)$. □

Let $\widetilde{\alpha}$ be the highest root of Φ^+. Define integers $k_1, \cdots, k_r$ by

$$\widetilde{\alpha} = \sum_{i=1}^{r} k_i \alpha_i \quad (k_i > 0). \tag{8.1}$$

For a positive integer k with $\Sigma_p(k) \neq \emptyset$, a lowest root $\alpha_p^-(k)$ for $\Sigma_p(k)$ is the root such that $\beta - \alpha_p^-(k)$ is a (possibly empty) sum of simple roots for every $\beta \in \Sigma_p(k)$. A highest root $\alpha_p^+(k)$ for $\Sigma_p(k)$ is defined by a similarly way. Lowest roots and highest roots always exist and are unique if $\Sigma_p(k)$ is not empty ([4, Proposition 1.4.2]).

Lemma 8.8 (Lemma 1.4.5 of [4]) *Let k be a positive integer. Then $\Sigma_p(k)$ is not empty if and only if $1 \leqslant k \leqslant k_p$, where k_p is the number defined in (8.1).*

Lemma 8.9 (Lemma 1.4.6 of [4]) *Suppose that $1 \leqslant k \leqslant k_p$. Let w_p be the longest element of W_p, i.e., $w_p \Delta_p = -\Delta_p$. Then we have $\alpha_p^-(1) = \alpha_p$ and $w_p \alpha_p^-(k) = \alpha_p^+(k)$.*

For a positive integer k with $\Sigma_p(k) \neq \emptyset$, we write

$$\alpha_p^{\pm}(k)^{\vee} = k\alpha_p^{\vee} + \gamma_p^{\pm}(k)^{\vee} \quad (\gamma_p^{\pm}(k)^{\vee} \in Q^+(\Delta_p^{\vee})),$$

where $Q^+(\Delta_p^{\vee}) = \sum_{j=1}^{r} \mathbb{Z}_{\geqslant 0} \alpha_j^{\vee}$. Then the heights $\operatorname{ht} \gamma_p^+(k)^{\vee}$ and $\operatorname{ht} \gamma_p^-(k)^{\vee}$ are nonnegative, since they must be written as a combination of simple roots $\alpha_j^{\vee} \in \Delta_p^{\vee}$ with nonnegative integer coefficients. We define

$$\hbar_k^{\pm} = \operatorname{ht} \gamma_p^{\pm}(k)^{\vee} = \langle \rho, \gamma_p^{\pm}(k)^{\vee} \rangle \tag{8.2}$$

for $1 \leqslant k \leqslant k_p$. For an element $\alpha \in \Sigma_p(h)$, the height $\operatorname{ht}(\alpha - k\alpha_p)^{\vee}$ is called the $(p\text{-})$level of α. In this terminology, $\hbar_k^+$ (resp. $\hbar_k^-$) is the highest (resp. lowest) level of $\Sigma_p(k)$. Note that $\gamma_p^-(1) = 0$, since $\alpha_p^-(1) = \alpha_p$.

Lemma 8.10 *For every $1 \leqslant k_1 < k_2 \leqslant k_p$, we have $\hbar_{k_1}^+ \leqslant \hbar_{k_2}^+$.*

Proof. We have $\alpha_p^+(k_2)^\vee - \alpha_p^+(k_1)^\vee \in \Phi^\vee$ by Corollary 1.2 of [10] and

$$\alpha_p^+(k_2)^\vee - \alpha_p^+(k_1)^\vee = (k_2 - k_1)\alpha_p^\vee + (\gamma_p^+(k_2)^\vee - \gamma_p^+(k_1)^\vee).$$

Therefore $\gamma_p^+(k_2) - \gamma_p^+(k_1) \in Q^+(\Delta_p)$, since $k_2 - k_1 > 0$ and $\gamma_p^+(k) \in Q^+(\Delta_p)$. This implies the result by definition of $\hbar_k^+$. $\qquad\square$

Lemma 8.11 *Let w_p be the longest element of W_p. Then we have*

$$\gamma_p^-(k) - w_p\gamma_p^+(k) = k\gamma_p^+(1).$$

In particular, $w_p\gamma_p^+(1) = -\gamma_p^+(1)$ and $\gamma_p^+(k) - w_p\gamma_p^-(k) = k\,\gamma_p^+(1)$.

Proof. We have $w_p\alpha_p^-(k)^\vee = \alpha_p^+(k)^\vee = k\alpha_p^\vee + \gamma_p^+(k)^\vee$. Therefore

$$\begin{aligned}
\alpha_p^-(k)^\vee &= w_p(w_p\alpha_p^-(k)^\vee) = kw_p\alpha_p^\vee + w_p\gamma_p^+(k)^\vee \\
&= k\alpha_p^+(1)^\vee + w_p\gamma_p^+(k)^\vee = k\alpha_p^\vee + k\gamma_p^+(1)^\vee + w_p\gamma_p^+(k)^\vee
\end{aligned}$$

Hence $k\alpha_p^\vee + \gamma_p^-(k)^\vee = k\alpha_p^\vee + k\gamma_p^+(1)^\vee + w_p\gamma_p^+(k)^\vee$. This implies the assertion. $\qquad\square$

Lemma 8.12 *Let c_p be the number of (2.2). Then we have*

$$c_p = 2 + \hbar_1^+ = 1 + \operatorname{ht}\alpha_p^+(1)^\vee = 1 + \operatorname{ht}(w_p\alpha_p^\vee),$$

where w_p is the longest element of W_p.

Proof. We have

$$\begin{aligned}
c_p &= 2\langle\lambda_p - \rho_p, \alpha_p^\vee\rangle = 2 - \langle 2\rho_p, \alpha_p^\vee\rangle = 2 - \langle\rho - w_p\rho, \alpha_p^\vee\rangle \\
&= 1 + \langle w_p\rho, \alpha_p^\vee\rangle = 1 + \langle\rho, w_p\alpha_p^\vee\rangle = 1 + \langle\rho, \alpha_p^\vee + \gamma_p^+(1)^\vee\rangle \\
&= 2 + \langle\rho, \gamma_p^+(1)^\vee\rangle = 2 + \hbar_1^+.
\end{aligned}$$

Then the second (resp. the third) equality of Lemma 8.12 follows from definition of $\hbar_1^+$ (resp. Lemma 8.9). $\qquad\square$

Lemma 8.13 *Let $k \geqslant 1$. We have $2h = kc_p + m$ if $2\langle\rho_p, \alpha^\vee\rangle = m \in \mathbb{Z}$ for $\alpha \in \Sigma_p(k, h)$. In particular if $\alpha \in \Sigma_p(k, h)$ for $h < kc_p/2$ (resp. $h > kc_p/2$), $\langle\rho_p, \alpha^\vee\rangle < 0$ (resp. > 0).*

Proof. Because of $c_p\lambda_p = 2\rho - 2\rho_p$ from the proof of Lemma 4.1 of [13], we have

$$2\operatorname{ht}\alpha^\vee - \langle\lambda_p, \alpha^\vee\rangle c_p = 2\operatorname{ht}\alpha^\vee - \langle 2\rho - 2\rho_p, \alpha^\vee\rangle = 2\langle\rho_p, \alpha^\vee\rangle.$$

This implies the assertion by definition of $\Sigma_p(k, h)$. $\qquad\square$

Lemma 8.14 *Let $1 \leqslant k \leqslant k_p$ and let $\hbar_k^\pm$ be numbers in (8.2). We have*

$$\hbar_k^+ + \hbar_k^- = k\,\hbar_1^+.$$

Therefore we have

$$k + 1 \leqslant \operatorname{ht}\alpha^\vee \leqslant kc_p - k - 1 \quad (k \geqslant 2) \quad and \quad 1 \leqslant \operatorname{ht}\alpha^\vee \leqslant c_p - 1 \quad (k = 1)$$

for $\alpha \in \Sigma_p(k)$ if $c_p \geqslant 3$, and $\hbar_k^+ \leqslant h_{k+1}^+ \leqslant (k+1)\hbar_1^+ - 1$.

Remark If $c_p = 2$, $\alpha_p^+(1) = \alpha_p^-(1) = \alpha_p$ by Lemma 8.12. Hence Φ is of type A_1, and the case $k \geqslant 2$ does not appear.

Proof. We note that

$$\operatorname{ht}(w_p\gamma^\vee) = -\operatorname{ht}\gamma^\vee$$

for every $\gamma^\vee \in Q^+(\Delta_p^\vee)$. In fact, we have

$$\operatorname{ht}\gamma^\vee = \langle\rho_p, \gamma^\vee\rangle = \langle w_p\rho_p, w_p\gamma^\vee\rangle = -\langle\rho_p, w_p\gamma^\vee\rangle = -\operatorname{ht}(w_p\gamma^\vee)$$

for $\gamma^\vee \in Q^+(\Delta_p^\vee)$, since $\langle \rho_p, \alpha_i^\vee \rangle = 1$ for $i \neq p$ from the proof of Lemma 4.1 in [13]. Therefore we have

$$
\begin{aligned}
k + \hbar_k^- &= \mathrm{ht}\,(\alpha_p^-(k))^\vee = \mathrm{ht}\,(w_p \alpha_p^+(k)^\vee) = \mathrm{ht}\,(w_p(k\alpha_p^\vee + \gamma_p^+(k)^\vee)) \\
&= k\,\mathrm{ht}\,(w_p \alpha_p^\vee) + \mathrm{ht}\,(w_p \gamma_p^+(k)^\vee) = k\,\mathrm{ht}\,(w_p \alpha_p^\vee) - \mathrm{ht}\,(\gamma_p^+(k))^\vee \\
&= k(1 + \hbar_1^+) - \hbar_k^+.
\end{aligned}
$$

Here we used Lemmas 8.11 and 8.12. This implies Lemma 8.14. $\qquad\square$

Lemma 8.15 *Let $1 \leqslant k \leqslant k_p$. Then $\Sigma_p(k)$ is divided into the disjoint union*

$$
\Sigma_p(k) = \bigsqcup_{m=1}^{M_k} L_m(k)
$$

satisfying the following conditions:

(1) $\alpha_p^\pm(k) \in L_1(k)$ *and* $|L_1(k)| = \mathrm{ht}\,\alpha_p^+(k)^\vee - \mathrm{ht}\,\alpha_p^-(k)^\vee + 1$,
(2) $|L_1(k)| > |L_2(k)|, |L_3(k)|, \ldots, |L_{M_k}(k)| \geqslant 1$,
(3) *we have*

$$
L_m(k)^\vee = \{\beta_m(k)^\vee\} \cup \{\beta_m(k)^\vee + \sum_{j=1}^J \alpha_{m,j}(k)^\vee \mid 1 \leqslant J \leqslant |L_m(k)| - 1\}
$$

for some $\beta_m(k) \in \Sigma_p(k)$ $(1 \leqslant m \leqslant M)$ with $\beta_1(k) = \alpha_p^-(k)$, and for some $\alpha_{m,j}(k) \in \Delta_p$ $(1 \leqslant m \leqslant M_k, 1 \leqslant j \leqslant |L_m(k)| - 1)$,
(4) $k + \hbar_k^- = \mathrm{ht}\,\beta_1(k)^\vee < \mathrm{ht}\,\beta_2(k)^\vee \leqslant \mathrm{ht}\,\beta_3(k)^\vee \leqslant \cdots \leqslant \mathrm{ht}\,\beta_M(k)^\vee \leqslant kc_p/2$,
(5) $\mathrm{ht}\,\widetilde{\beta}_m(k)^\vee = kc_p - \mathrm{ht}\,\beta_m^\vee$ $(1 \leqslant m \leqslant M_k)$, *where*

$$
\widetilde{\beta}_m(k)^\vee = \beta_m^\vee + \sum_{j=1}^{|L_m(k)|-1} \alpha_{m,j}(k)^\vee.
$$

Proof. At first we note that the problem is reduced to the cases of irreducible root systems with $\Sigma_p(1)$ by using results of section 1.4 of [4]. Then we see that the assertions of Lemmas 8.4 and 8.15 for $\Sigma_p(1)$ hold by constructing explicitly the required disjoint union directly. See Appendix 1. $\qquad\square$

9. PROOF OF PROPOSITIONS 6.6 AND 6.7

9.1. **Proof of Proposition 6.6.**

Lemma 9.1 *Let a, b be real numbers satisfying $a \leqslant b$. Then*

$$
\left| \frac{\xi(s+a)}{\xi(s+b+1)} \right| < 1 \quad for \quad \Re(s) > -\frac{a+b}{2}
$$

and

$$
\left| \frac{\xi(s+a)}{\xi(s+b+1)} \right| = 1 \quad for \quad \Re(s) = -\frac{a+b}{2}.
$$

Proof. This is equivalent to $|\xi(s+(a-b-1)/2)| < |\xi(s-(a-b-1)/2)|$ for $\Re(s) > 1/2$ by shifting $s + (a+b+1)/2$ to s, and it holds by applying Theorem 4 of [14] to $F(s) = \xi(s)$ and $c = (a-b-1)/2$. The second statement immediately follows from the functional equation for $\xi(s)$. $\qquad\square$

Now we prove Proposition 6.6. We recall

$$
X_p^\dagger(s) = X_{p,\mathrm{id}}(s) = \prod_{\alpha \in \Phi_+ \backslash \Phi_p^+} \xi(\langle \lambda_p, \alpha^\vee \rangle s + \mathrm{ht}\,\alpha^\vee + 1). \tag{9.1}
$$

by Lemma 6.3. From (5.3) and (9.1), we have

$$\frac{X_{p,w}(s)}{X_p^{\ddagger}(s)} = \prod_{\alpha \in (\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-} \frac{\xi(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht} \alpha^\vee)}{\xi(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht} \alpha^\vee + 1)}. \tag{9.2}$$

Using the notation of Lemma 8.15, we define

$$L_m(k)_w = L_m(k) \cap w^{-1}\Phi^-$$

and put $\Lambda_w(k) = \{m \mid 1 \leqslant m \leqslant M_k, L_m(k)_w \neq \emptyset\}$. In addition, we define

$$h_m(k) = \max\{\operatorname{ht} \alpha^\vee \mid \alpha \in L_m(k)\}, \quad h_{m,w}(k) = \min\{\operatorname{ht} \alpha^\vee \mid \alpha \in L_m(k)_w\}.$$

for $m \in \Lambda_w(k)$. Note that $h_m(k)$ is attained by one of $\alpha \in L_m(k)_w$ by Lemma 8.2. Then the right-hand side of (9.2) equals

$$\prod_{k=1}^{k_p} \prod_{m \in \Lambda_w(k)} \prod_{h_m = h_{m,w}(k)}^{h_m(k)} \frac{\xi(ks + h_m)}{\xi(ks + h_m + 1)} = \prod_{k=1}^{k_p} \prod_{m \in \Lambda_w(k)} \frac{\xi(ks + h_{m,w}(k))}{\xi(ks + h_m(k) + 1)}$$

by Lemma 8.2 and Lemma 8.15 (3). Now, by Lemma 9.1, we have

$$\left| \frac{\xi(ks + h_{m,w}(k))}{\xi(ks + h_m(k) + 1)} \right| < 1 \quad \text{for} \quad \Re(s) > -\frac{h_{m,w}(k) + h_m(k)}{2k}.$$

for every $m \in \Lambda_w(k)$. We have

$$\frac{h_{m,w}(k) + h_m(k)}{2k} \geqslant \frac{c_p}{2},$$

for every $m \in \Lambda_w(k)$, since $h_{m,w}(k) \geqslant \operatorname{ht} \beta_m(k)^\vee = kc_p - h_m(k)$ by Lemma 8.15 (5). Also, we obtain

$$\left| \frac{\xi(ks + h_{m,w}(k))}{\xi(ks + h_m(k) + 1)} \right| < 1 \quad \text{for} \quad \Re(s) = -\frac{c_p}{2}.$$

unless $h_{m,w}(k) + h_m(k) = kc_p$. If $h_{m,w}(k) + h_m(k) = kc_p$,

$$\left| \frac{\xi(ks + h_{m,w}(k))}{\xi(ks + h_m(k) + 1)} \right| = 1 \quad \text{for} \quad \Re(s) = -\frac{c_p}{2}.$$

Now we complete the proof of Proposition 6.6 $\square$

Thus, we see that Proposition 6.6 follows from Lemma 8.15 whose the essential part is due to Lemmas 8.1 and explicit descriptions of root systems.

9.2. Proof of Proposition 6.7. We have

$$D_p(s) = \prod_{k=1}^{\infty} \prod_{h \leqslant (kc_p+1)/2} \xi(ks + h)^{N_p(k,h-1)} \prod_{h > (kc_p+1)/2} \xi(ks + h)^{N_p(k,h)}.$$

by Lemma 8.6. Hence we have

$$\frac{X_p^{\ddagger}(s)}{D_p(s)} = \prod_{k=1}^{\infty} \prod_{h > (kc_p+1)/2} \xi(ks + h)^{N_p(k,h-1)-N_p(k,h)}. \tag{9.3}$$

Note that $N_p(k, h-1) - N_p(k, h) \geqslant 0$ by Lemma 8.3. (This non-negativity should be hold by the construction of $M_p(k, h)$.) It is well-known that $\xi(s) \neq 0$ for $\Re(s) \geqslant 1$. Therefore, $\xi(ks + h) \neq 0$ for $\Re(s) \geqslant (1 - h)/k$. On the other hand, we have $(1 - h)/k \leqslant -c_p/2$, since kc_p and h are both integers. If kc_p is odd, $h > (kc_p + 1)/2$ implies $(1 - h)/k \leqslant -c_p/2 - 1/(2k) < -c_p/2$. Hence $X_p^{\ddagger}(s)/D_p(s) \neq 0$ for $\Re(s) \geqslant -c_p/2$. If kc_p is even, $h > (kc_p + 1)/2$ implies $(1 - h)/k \leqslant -c_p/2$. It also derives $X_p^{\ddagger}(s)/D_p(s) \neq 0$ for $\Re(s) \geqslant -c_p/2$. We complete the proof of Proposition 6.7. $\square$

Thus, Proposition 6.7 is a simple consequence of Lemma 8.6. As well as Proposition 6.6, the essential part of Lemma 8.6 is due to Lemmas 8.1 and 8.4.

10. Proof of Proposition 6.5

The most essential part of the proof of Proposition 6.5 is the volume formula (**Theorem**) of [23, section 4.7] which will be used in the proof of Lemma 10.3 below.

Lemma 10.1 *Let* $w \in \mathfrak{W}_p$. *If* $|w^{-1}\Delta \setminus \Phi_p| = 1$, *the set* $\Phi^+ \setminus \Phi_p^+$ *equals to either* $(\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^+$ *or* $(\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-$.

Proof. We show that $\Phi^+ \cap w(\Phi^+ \setminus \Phi_p^+) = w(\Phi^+ \setminus \Phi_p^+)$ or $\Phi^- \cap w(\Phi^+ \setminus \Phi_p^+) = w(\Phi^+ \setminus \Phi_p^+)$ if $|\Delta \setminus w\Phi_p| = 1$. Suppose $|\Delta \setminus w\Phi_p| = 1$, and denote by α_w the only one simple root in $\Delta \setminus w\Phi_p$. Then α_w belongs to $\Phi^+ \cap w(\Phi^+ \setminus \Phi_p^+)$ or $\Phi^+ \cap w(\Phi^- \setminus \Phi_p^-)$, and hence α_w belongs to $\Phi^+ \cap w(\Phi^+ \setminus \Phi_p^+)$ or $-\alpha_w$ belongs to $\Phi^- \cap w(\Phi^+ \setminus \Phi_p^+)$.

Put $\Delta_w^+ = \Delta \cap w\Phi_p^+$ and $\Delta_w^- = \Delta \cap w\Phi_p^-$ so that

$$\Delta = \Delta_w^+ \cup \Delta_w^- \cup \{\alpha_w\}$$

(disjoint). Then we claim $\Delta_w^+ = \Delta \cap w\Delta_p$. In fact $\Delta_w^+ \supset \Delta \cap w\Delta_p$ is obvious. An arbitrary $\alpha \in \Delta_w^+$ has the form

$$\alpha = \sum_{j \neq p} n_j w\alpha_j \quad (n_j \geqslant 0, w\alpha_j \in \Phi^- \cup \Delta),$$

since $w\Delta_p \subset \Phi^- \cup \Delta$. The right-hand side decomposes into two parts according to $w\alpha_j \in \Phi^-$ or $w\alpha_j \in \Delta$:

$$\alpha = \sum_{\substack{j \neq p \\ w\alpha_j \in \Phi^-}} n_j w\alpha_j + \sum_{\substack{j \neq p \\ w\alpha_j \in \Delta}} n_j w\alpha_j.$$

We note that the left-hand side is a simple root. Therefore, α should be one of $\Delta \cap w\Delta_p$, and hence $\Delta_w^+ = \Delta \cap w\Delta_p$.

Now we prove the assertion of the lemma by using a different way according to two cases $w\alpha_p \in \Phi^+$ or $w\alpha_p \in \Phi^-$.

First we deal with the case $w\alpha_p \in \Phi^+$. Assume $\alpha_w \in \Phi^+ \cap w(\Phi^+ \setminus \Phi_p^+)$. Then we will have a contradiction unless $\Phi^- \cap w(\Phi^+ \setminus \Phi_p^+) = \emptyset$. If $\Phi^- \cap w(\Phi^+ \setminus \Phi_p^+) \neq \emptyset$, there exists at least one simple root α_k such that

$$-\alpha_k \in \Phi^- \cap w(\Phi^+ \setminus \Phi_p^+). \tag{10.1}$$

Actually we have

$$\sum_i a_i w^{-1}(-\alpha_i) = w^{-1}(-\sum_i a_i \alpha_i) = b_p \alpha_p + \sum_{j \neq p} b_j \alpha_j \quad (b_p > 0, \, a_i, b_j \geqslant 0)$$

for $-\sum_i a_i \alpha_i \in \Phi^- \cap w(\Phi^+ \setminus \Phi_p^+)$. Hence we have

$$w^{-1}(-\alpha_k) = b_p' \alpha_p + \sum_{j \neq p} b_j' \alpha_j \quad (b_p' > 0, \, , b_j' \geqslant 0)$$

for at least one α_k, since $w^{-1}(-\alpha_i) \in \Phi$.

The simple root α_k belongs to one of Δ_w^+, Δ_w^- or $\{\alpha_w\}$. If $\alpha_k \in \Delta_w^+$, we have $\alpha_k \in \Delta \cap w\Delta_p \subset \Phi^+ \cap w\Phi^+$. This contradicts (10.1). If $\alpha_k \in \Delta_w^-$, we have $\alpha_k \in \Phi^+ \cap w\Phi_p^-$. This contradicts (10.1). If $\alpha_k = \alpha_w$, we have $\alpha_k = \alpha_w \in \Phi^+ \cap w(\Phi^+ \setminus \Phi_p^+)$ by assumption. This also contradicts (10.1). Hence $\Phi^- \cap w(\Phi^+ \setminus \Phi_p^+) = \emptyset$ which implies $\Phi^+ \cap w(\Phi^+ \setminus \Phi_p^+) = w(\Phi^+ \setminus \Phi_p^+)$.

On the other hand, we have $\Phi^- \cap w(\Phi^+ \setminus \Phi_p^+) = w(\Phi^+ \setminus \Phi_p^+)$ if we assume that $-\alpha_w$ belongs to $\Phi^- \cap w(\Phi^+ \setminus \Phi_p^+)$ by a way similar to the above.

Finally we deal with the case $w\alpha_p \in \Phi^-$. We show that $(\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^- = (\Phi^+ \setminus \Phi_p^+)$ by induction on k for $\Sigma_p(k)$ if $w\alpha_p \in \Phi^-$ (i.e. $\alpha_p \in (\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-$). By Lemma 8.1, we have $\Sigma_p(1) \subset (\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-$ if $w\alpha_p \in \Phi^-$. Suppose that $\sum_p(k-1) \subset$

$(\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-$ for $1 < k \leqslant k_p$. Let $\alpha_p^-(k)$ be the lowest element of $\Sigma_p(k)$. Then there exists $\beta \in \Sigma_p(k-1)$ such that

$$\alpha_p^-(k)^\vee = \beta^\vee + \alpha_p^\vee,$$

since $(\alpha_p^-(k)^\vee - \alpha_j^\vee)^\vee$ is not a root for any $j \neq p$ by the lowest property of $\alpha_p^-(k)^\vee$. We have $w\alpha_p \in \Phi^-$ by the first assumption and $w\beta \in \Phi^-$ by the assumption of induction. Therefore $w\alpha_p^-(k) = (w\beta^\vee + w\alpha_p^\vee)^\vee \in \Phi^-$, i.e. $\alpha_p^-(k) \in (\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-$. Hence $\Sigma_p(k) \subset (\Phi^+ \setminus \Phi_p^+) \cap w^{-1}\Phi^-$ by Lemma 8.1. $\qquad\square$

Lemma 10.2 *We have*

$$\{w \in \mathfrak{W}_p^\ddagger \mid |(w^{-1}\Delta) \setminus \Phi_p| = 1\} = \{w \in W_p \mid \Delta_p \subset w^{-1}(\Delta_p \cup \Phi_p^-)\}.$$

Proof. At first we prove the the left-hand side is contained in the right-hand side. We have

$$\{w \in \mathfrak{W}_p^\ddagger \mid |(w^{-1}\Delta) \setminus \Phi_p| = 1\} = \{w \in \mathfrak{W}_p \mid l_p(w) = 0, \ \Delta_p \subset w^{-1}(\Delta \cup \Phi^-)\}.$$

by definition of $\mathfrak{W}_p^\ddagger$ in Lemma 6.3. Recall that $l_p(w)$ depends only on the coset wW_p (Lemma 6.1). Let W^p be the set of minimal coset representatives of W/W_p. If $v^p \in W^p$, it implies $v^p\Phi_p^+ \subset \Phi^+$. Thus $l_p(w) = 0$ is equivalent to $v^p\Phi^+ = \Phi^+$, and also $v^p = \mathrm{id}$. Hence $\mathfrak{W}_p^\ddagger \subset W_p$. On the other hand, if $w \in W_p$, $\Delta_p \cap w^{-1}(\Delta \cup \Phi^-) = \Delta_p \cap w^{-1}(\Delta_p \cup \Phi_p^-)$, and $|(w^{-1}\Delta) \setminus \Phi_p| = 1$ is equivalent to $(w^{-1}\Delta) \setminus \Phi_p = \{w^{-1}\alpha_p\}$. Hence we obtain the desired consequence. The opposite inclusion is obtained by tracing back the above argument. $\qquad\square$

Lemma 10.3 *Let $C_{p,w}$ be real numbers defined in (5.4). Define real numbers $D_{p,w}$ by*

$$
\frac{1}{D_{p,w}} = 2^{|\Delta_p \cap w^{-1}\Phi^+|} \prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_p \setminus \Delta_p)} (\mathrm{ht}\,\alpha^\vee - 1)
$$
$$
\times \prod_{\alpha \in \Phi_p^+ \setminus \Delta_p} (\mathrm{ht}\,\alpha^\vee + \delta_{\alpha,w})(\mathrm{ht}\,\alpha^\vee + \delta_{\alpha,w} - 1), \tag{10.2}
$$

where $\delta_{\alpha,w}$ is defined in (5.2). Then

$$\sum_{\substack{w \in \mathfrak{W}_p^\ddagger \\ |(w^{-1}\Delta) \setminus \Phi_p| = 1}} \frac{1}{\langle \lambda_p, \alpha_w^\vee \rangle} C_{p,w} D_{p,w} \neq 0, \tag{10.3}$$

where α_w is the only one element of $(w^{-1}\Delta) \setminus \Phi_p$.

Proof. We first prove

$$\operatorname*{Res}_{\lambda=\rho} \omega_\Delta^\Phi(\lambda)$$

$$= \sum_{\substack{w \in W \\ \Delta \subset w^{-1}(\Delta \cup \Phi^-)}} \hat{\zeta}(2)^{-|\Delta \cap w^{-1}\Phi^-|} \prod_{\alpha \in (w^{-1}\Delta) \setminus \Delta} \frac{1}{\mathrm{ht}\,\alpha^\vee - 1} \prod_{\alpha \in (\Phi^+ \setminus \Delta) \cap w^{-1}\Phi^-} \frac{\hat{\zeta}(\mathrm{ht}\,\alpha^\vee)}{\hat{\zeta}(\mathrm{ht}\,\alpha^\vee + 1)} \tag{10.4}$$

holds for an arbitrary reduced root system Φ (which is not necessary irreducible) and its fundamental system Δ. If Φ is not irreducible and its irreducible decomposition is $\Phi = \Phi_1 \amalg \cdots \amalg \Phi_m$, we find that $\operatorname*{Res}_{\lambda=\rho} \omega_\Delta^\Phi(\lambda) = \prod_{i=1}^m \operatorname*{Res}_{\lambda=\rho_i} \omega_{\Delta_i}^{\Phi_i}(\lambda)$, $\Delta_i = \Delta \cap \Phi_i$, $\rho_i = \frac{1}{2}\sum_{\alpha \in \Phi^+ \cap \Phi_i} \alpha$ and (10.4) holds for each component, since the Weyl group $W(\Phi)$ of Φ decomposes into

$$W(\Phi) = W(\Phi_1) \times \cdots \times W(\Phi_m).$$

We apply (10.4) to Φ_p, Δ_p and ρ_p later.

We have

$$\operatorname{Res}_{\lambda=\rho} \omega_\Delta^\Phi(\lambda) = \operatorname{Res}_{s=0} \omega_{\Delta,p}^\Phi(s)$$

by (2.3). Therefore, it is enough to show that $\operatorname{Res}_{s=0} \omega_{\Delta,p}^\Phi(s)$ equals to the right-hand side in (10.4). By (2.8) in [13] we have

$$\omega_{\Delta,p}^\Phi(s) = \sum_{w \in \mathfrak{W}_p} \hat\zeta(2)^{-|\Delta_p \cap w^{-1}\Phi^-|} \prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_p \backslash \Delta_p)} \frac{1}{\operatorname{ht}\alpha^\vee - 1} \prod_{\alpha \in (\Phi_p^+ \backslash \Delta_p) \cap w^{-1}\Phi^-} \frac{\hat\zeta(\operatorname{ht}\alpha^\vee)}{\hat\zeta(\operatorname{ht}\alpha^\vee + 1)}$$

$$\times \prod_{\alpha \in (w^{-1}\Delta) \backslash \Phi_p} \frac{1}{\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee - 1} \prod_{\alpha \in (\Phi^+ \backslash \Phi_p^+) \cap w^{-1}\Phi^-} \frac{\hat\zeta(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee)}{\hat\zeta(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee + 1)}.$$

Here we note that

$$(1) \quad \Delta \cap ((w^{-1}\Delta) \backslash \Phi_p) = \{\alpha_p\} \cap (w^{-1}\Delta) = \begin{cases} \{\alpha_p\}, & \text{if } \alpha_p \in w^{-1}\Delta, \\ \emptyset, & \text{otherwise,} \end{cases}$$

and

$$(2) \quad \Delta \cap ((\Phi^+ \backslash \Phi_p^+) \cap w^{-1}\Phi^-) = \begin{cases} \{\alpha_p\}, & \text{if } \alpha_p \in w^{-1}\Phi^-, \\ \emptyset, & \text{otherwise.} \end{cases}$$

In addition, (1) and (2) do not occur simultaneously. Using these facts together with the fact that $\langle \lambda_p, \alpha_p^\vee \rangle = 1$, we obtain that $\operatorname{Res}_{s=0} \omega_{\Delta,p}^\Phi(s)$ is

$$\sum_{\substack{w \in \mathfrak{W}_p \\ \alpha_p \in w^{-1}\Delta}} \hat\zeta(2)^{-|\Delta_p \cap w^{-1}\Phi^-|} \prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_p \backslash \Delta_p)} \frac{1}{\operatorname{ht}\alpha^\vee - 1} \prod_{\alpha \in (\Phi_p^+ \backslash \Delta_p) \cap w^{-1}\Phi^-} \frac{\hat\zeta(\operatorname{ht}\alpha^\vee)}{\hat\zeta(\operatorname{ht}\alpha^\vee + 1)}$$

$$\times \frac{1}{\langle \lambda_p, \alpha_p^\vee \rangle} \prod_{\alpha \in (w^{-1}\Delta) \backslash (\Phi_p \cup \{\alpha_p\})} \frac{1}{\operatorname{ht}\alpha^\vee - 1} \prod_{\alpha \in (\Phi^+ \backslash \Phi_p^+) \cap w^{-1}\Phi^-} \frac{\hat\zeta(\operatorname{ht}\alpha^\vee)}{\hat\zeta(\operatorname{ht}\alpha^\vee + 1)}$$

$$+ \sum_{\substack{w \in \mathfrak{W}_p \\ \alpha_p \in w^{-1}\Phi^-}} \frac{\hat\zeta(2)^{-|\Delta_p \cap w^{-1}\Phi^-|}}{\hat\zeta(2) \langle \lambda_p, \alpha_p^\vee \rangle} \prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_p \backslash \Delta_p)} \frac{1}{\operatorname{ht}\alpha^\vee - 1} \prod_{\alpha \in (\Phi_p^+ \backslash \Delta_p) \cap w^{-1}\Phi^-} \frac{\hat\zeta(\operatorname{ht}\alpha^\vee)}{\hat\zeta(\operatorname{ht}\alpha^\vee + 1)}$$

$$\times \prod_{\alpha \in (w^{-1}\Delta) \backslash \Phi_p} \frac{1}{\operatorname{ht}\alpha^\vee - 1} \prod_{\alpha \in (\Phi^+ \backslash \Phi_p^+ \cup \{\alpha_p\}) \cap w^{-1}\Phi^-} \frac{\hat\zeta(\operatorname{ht}\alpha^\vee)}{\hat\zeta(\operatorname{ht}\alpha^\vee + 1)}$$

$$= \sum_{\substack{w \in \mathfrak{W}_p \\ \alpha_p \in w^{-1}\Delta}} \hat\zeta(2)^{-|\Delta \cap w^{-1}\Phi^-|} \prod_{\alpha \in (w^{-1}\Delta) \backslash \Delta} \frac{1}{\operatorname{ht}\alpha^\vee - 1} \prod_{\alpha \in (\Phi^+ \backslash \Delta) \cap w^{-1}\Phi^-} \frac{\hat\zeta(\operatorname{ht}\alpha^\vee)}{\hat\zeta(\operatorname{ht}\alpha^\vee + 1)}$$

$$+ \sum_{\substack{w \in \mathfrak{W}_p \\ \alpha_p \in w^{-1}\Phi^-}} \hat\zeta(2)^{-|\Delta \cap w^{-1}\Phi^-|} \prod_{\alpha \in (w^{-1}\Delta) \backslash \Delta} \frac{1}{\operatorname{ht}\alpha^\vee - 1} \prod_{\alpha \in (\Phi^+ \backslash \Delta) \cap w^{-1}\Phi^-} \frac{\hat\zeta(\operatorname{ht}\alpha^\vee)}{\hat\zeta(\operatorname{ht}\alpha^\vee + 1)}.$$

Hence we obtain (10.4).

We recall $C_{p,w}$ in (5.4). We claim

$$\sum_{\substack{w \in \mathfrak{W}_p^\ddagger \\ |(w^{-1}\Delta) \backslash \Phi_p| = 1}} \frac{1}{\langle \lambda_p, \alpha_w^\vee \rangle} C_{p,w} D_{p,w} = \prod_{\alpha \in \Phi_p^+} \hat\zeta(\operatorname{ht}\alpha^\vee + 1) \cdot \operatorname{Res}_{\lambda=\rho_p} \omega_{\Delta_p}^{\Phi_p}(\lambda). \tag{10.5}$$

We justify this as follows. We have

$$C_{p,w}D_{p,w} = \hat{\zeta}(2)^{|\Delta_p \cap w^{-1}\Phi^+|} \prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_p \setminus \Delta_p)} \frac{1}{\operatorname{ht}\alpha^\vee - 1} \prod_{\alpha \in \Phi_p^+ \setminus \Delta_p} \hat{\zeta}(\operatorname{ht}\alpha^\vee + \delta_{\alpha,w}).$$

by definitions of $C_{p,w}$ and $D_{p,w}$. Therefore

$$\sum_{\substack{w \in \mathfrak{W}_p^\ddagger \\ |(w^{-1}\Delta) \setminus \Phi_p| = 1}} \frac{1}{\langle \lambda_p, \alpha_w^\vee \rangle} C_{p,w}D_{p,w}$$

$$= \sum_{\substack{w \in \mathfrak{W}_p^\ddagger \\ |(w^{-1}\Delta) \setminus \Phi_p| = 1}} \frac{\hat{\zeta}(2)^{|\Delta_p \cap w^{-1}\Delta|}}{\langle \lambda_p, \alpha_w^\vee \rangle} \prod_{\alpha \in (w^{-1}\Delta) \cap (\Phi_p \setminus \Delta_p)} \frac{1}{\operatorname{ht}\alpha^\vee - 1}$$

$$\times \prod_{\alpha \in (\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi^+} \hat{\zeta}(\operatorname{ht}\alpha^\vee + 1) \prod_{\alpha \in (\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi^-} \hat{\zeta}(\operatorname{ht}\alpha^\vee),$$

since $|\Delta_p \cap w^{-1}\Phi^+| = |\Delta_p \cap w^{-1}\Delta|$ if $w \in \mathfrak{W}_p$. If $w \in W_p$, we have

$$(w^{-1}\Delta) \cap (\Phi_p \setminus \Delta_p) = (w^{-1}\Delta_p) \setminus \Delta_p; \qquad \Delta_p \cap w^{-1}\Delta = \Delta_p \cap w^{-1}\Delta_p.$$

Moreover, if $w \in W_p$, we have

$$(\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi^+ = (\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi_p^+$$

$$(\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi^- = (\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi_p^-,$$

since

$$(\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi^+ = \left((\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi_p^+\right) \cup \left((\Phi_p^+ \setminus \Delta_p) \cap w^{-1}(\Phi^+ \setminus \Phi_p^+)\right)$$

$$(\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi^- = \left((\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi_p^-\right) \cup \left((\Phi_p^+ \setminus \Delta_p) \cap w^{-1}(\Phi^- \setminus \Phi_p^-)\right),$$

and

$$(\Phi_p^+ \setminus \Delta_p) \cap w^{-1}(\Phi^+ \setminus \Phi_p^+) = \emptyset, \qquad (\Phi_p^+ \setminus \Delta_p) \cap w^{-1}(\Phi^- \setminus \Phi_p^-) = \emptyset$$

for $w \in W_p$. Hence, by Lemma 10.2 and the fact that $\langle \lambda_p, w^{-1}\alpha_p^\vee \rangle = \langle w\lambda_p, \alpha_p^\vee \rangle = 1$ for $w \in W_p$, we obtain

$$\sum_{\substack{w \in \mathfrak{W}_p^\ddagger \\ |(w^{-1}\Delta) \setminus \Phi_p| = 1}} \frac{1}{\langle \lambda_p, \alpha_w^\vee \rangle} C_{p,w}D_{p,w}$$

$$= \sum_{\substack{w \in W_p \\ \Delta_p \subset w^{-1}(\Delta_p \cup \Phi_p^-)}} \frac{\hat{\zeta}(2)^{|\Delta_p \cap w^{-1}\Delta_p|}}{\langle \lambda_p, w^{-1}\alpha_p^\vee \rangle} \prod_{\alpha \in (w^{-1}\Delta_p) \setminus \Delta_p} \frac{1}{\operatorname{ht}\alpha^\vee - 1}$$

$$\times \prod_{\alpha \in (\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi_p^+} \hat{\zeta}(\operatorname{ht}\alpha^\vee + 1) \prod_{\alpha \in (\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi_p^-} \hat{\zeta}(\operatorname{ht}\alpha^\vee)$$

$$= \sum_{\substack{w \in W_p \\ \Delta_p \subset w^{-1}(\Delta_p \cup \Phi_p^-)}} \hat{\zeta}(2)^{|\Delta_p \cap w^{-1}\Delta_p|} \prod_{\alpha \in (w^{-1}\Delta_p) \setminus \Delta_p} \frac{1}{\operatorname{ht}\alpha^\vee - 1}$$

$$\times \prod_{\alpha \in (\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi_p^+} \hat{\zeta}(\operatorname{ht}\alpha^\vee + 1) \prod_{\alpha \in (\Phi_p^+ \setminus \Delta_p) \cap w^{-1}\Phi_p^-} \hat{\zeta}(\operatorname{ht}\alpha^\vee).$$

With this formula, we obtain (10.5) by applying (10.4) to Φ_p, Δ_p and ρ_p.

By the theorem of [23, section 4.7], the right-hand side of (10.5) is positive, in particular, it is not equal to zero. In fact, it should be a product of special values of the Riemann zeta function and volumes of several (truncated) domains corresponding to irreducible components of Φ_p. Hence (10.3) follows. We complete the proof of Lemma 10.3. $\qquad\square$

Proof of Proposition 6.5. Using real numbers $C_{p,w}$ and $D_{p,w}$ defined in (5.4) and (10.2), respectively, we have

$$Q_{p,w}(s) = C_{p,w} \prod_{\substack{v \in \mathfrak{W}_p \\ v \neq w}} \left[2^{|\Delta_p \cap v^{-1}\Phi^+|} \prod_{\alpha \in (v^{-1}\Delta)\setminus\Delta_p} \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee - 1 \right) \right.$$

$$\left. \times \prod_{\alpha \in \Phi^+\setminus\Delta_p} \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee + \delta_{\alpha,v} \right) \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee + \delta_{\alpha,v} - 1 \right) \right]$$

$$= C_{p,w} \prod_{\substack{v \in \mathfrak{W}_p \\ v \neq w}} \left[D_{p,v}^{-1} \prod_{\alpha \in (v^{-1}\Delta)\setminus\Phi_p} \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee - 1 \right) \right.$$

$$\left. \times \prod_{\alpha \in \Phi^+\setminus\Phi_p^+} \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee + \delta_{\alpha,v} \right) \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee + \delta_{\alpha,v} - 1 \right) \right]$$

$$= Q_p^*(s) \left[C_{p,w} D_{p,w} \prod_{\alpha \in (w^{-1}\Delta)\setminus\Phi_p} \frac{1}{\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee - 1} \right.$$

$$\left. \times \prod_{\alpha \in \Phi^+\setminus\Phi_p^+} \frac{1}{(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee + \delta_{\alpha,w})(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee + \delta_{\alpha,w} - 1)} \right],$$

where $Q_p^*(s)$ is the polynomial

$$Q_p^*(s) = \prod_{v \in \mathfrak{W}_p} \left[D_{p,v}^{-1} \prod_{\alpha \in (v^{-1}\Delta)\setminus\Phi_p} \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee - 1 \right) \right.$$

$$\left. \times \prod_{\alpha \in \Phi^+\setminus\Phi_p^+} \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee + \delta_{\alpha,v} \right) \left(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee + \delta_{\alpha,v} - 1 \right) \right].$$

In particular, we have

$$Q_p^\ddagger(s) = \sum_{v \in \mathfrak{W}_p^\ddagger} Q_{p,v}(s)$$

$$= Q_p^*(s) \sum_{v \in \mathfrak{W}_p^\ddagger} \left[C_{p,v} D_{p,v} \prod_{\alpha \in (v^{-1}\Delta)\setminus\Phi_p} \frac{1}{\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee - 1} \right.$$

$$\left. \times \prod_{\alpha \in \Phi^+\setminus\Phi_p^+} \frac{1}{(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee + \delta_{\alpha,v})(\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee + \delta_{\alpha,v} - 1)} \right].$$

Hence we have

$$\frac{Q_{p,w}(s)}{Q_p^\ddagger(s)} = \frac{C_{p,w} D_{p,w} \displaystyle\prod_{\alpha \in (w^{-1}\Delta)\setminus\Phi_p} \frac{1}{\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee - 1}}{\displaystyle\sum_{v \in \mathfrak{W}_p^\ddagger} \left[C_{p,v} D_{p,v} \displaystyle\prod_{\alpha \in (v^{-1}\Delta)\setminus\Phi_p} \frac{1}{\langle \lambda_p, \alpha^\vee \rangle s + \operatorname{ht}\alpha^\vee - 1} \left(1 + o(1) \right) \right]}$$

as $|s| \to \infty$. Therefore $|Q_{p,w}(s)/Q_p^\ddagger(s)| = o(1)$ as $|s| \to \infty$ for every $w \in (\mathfrak{W}_p^+ \cup \mathfrak{W}_p^0) \setminus \mathfrak{W}_p^\ddagger$ by Lemmas 10.1 and 10.3. We complete the proof of Proposition 6.5. $\qquad\square$

11. PROOF OF PROPOSITION 6.9

Write $s = \sigma + it$ $(\sigma, t \in \mathbb{R})$. By Lemma 6.3, we have

$$E_p(s) = X_p^{\ddagger}(s)[Q_p^{\ddagger}(s) + V_p(s)]$$

with

$$V_p(s) = \sum_{w \in \mathfrak{W}_p^+ \setminus \mathfrak{W}_p^{\ddagger}} Q_{p,w}(s) \cdot \frac{X_{p,w}(s)}{X_p^{\ddagger}(s)} + \frac{1}{2} \sum_{w \in \mathfrak{W}_p^0} Q_{p,w}(s) \cdot \frac{X_{p,w}(s)}{X_p^{\ddagger}(s)}.$$

Thus, it suffices to prove $Q_p^{\ddagger}(s) + V_p(s) \neq 0$ in a left half plane, since $X_p^{\ddagger}(s)$ is a finite product of zeta functions.

From the proof of Proposition 6.6, we have

$$\frac{X_{p,w}(s)}{X_p^{\ddagger}(s)} = \prod_{k=1}^{k_p} \prod_{m \in \Lambda_w(k)} \frac{\xi(ks + h_{m,w}(k))}{\xi(ks + h_m(k) + 1)}.$$

Using the functional equation $\xi(s) = \xi(1-s)$, we have

$$\frac{\xi(ks+a)}{\xi(ks+b+1)} = \frac{\xi((1-ks)-a)}{\xi((1-ks)-b-1)}$$

$$= \pi^{\frac{a-b-1}{2}} \frac{(ks+a)(ks+a-1)}{(ks+b)(ks+b+1)} \frac{\Gamma((1-ks)/2 - a/2)}{\Gamma((1-ks)/2 - (b+1)/2)} \frac{\zeta((1-ks)-a)}{\zeta((1-ks)-b-1)}.$$

Because of $\zeta(s)^{-1} = \sum_{n=1}^{\infty} \mu(n) n^{-s}$ as $\sigma \to \infty$, we have

$$\frac{\zeta((1-ks)-a)}{\zeta((1-ks)-b-1)} = 1 + \sum_{n=2}^{\infty} \frac{\mathfrak{C}_n(k; a, b)}{n^{-s}}.$$

Using the Stirling formula

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}}\, e^{-z} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \cdots + \frac{1}{c_n z^n} + O(|z|^{-n-1})\right)$$

for $|\arg z| < \pi - \epsilon$, we have

$$\frac{\Gamma(z+\lambda)}{\Gamma(z)} = z^{\lambda}\left(1 + \frac{a_1(\lambda)}{z} + \cdots + \frac{a_n(\lambda)}{z^n} + O(|z|^{-n-1})\right)$$

as $|z| \to \infty$ with $|\arg z| < \pi - \epsilon$, where $a_j(x) \in \mathbb{R}[x]$ are polynomials of x and the implied constant depends on λ and ϵ. Using the above facts, we obtain

$$\frac{\xi(ks+a)}{\xi(ks+b+1)} = (2\pi/k)^{\frac{a-b-1}{2}} (-s)^{-\frac{a-b-1}{2}} \left(1 + \frac{c_1(a,b)}{s} + \cdots + \frac{c_n(a,b)}{s^n} + O(|s|^{-n-1})\right)$$

$$\times \left(1 + \sum_{\nu=2}^{\infty} \frac{\mathfrak{C}_\nu(k; a, b)}{\nu^{-s}}\right)$$

as $\sigma \to -\infty$ for any fixed $n \geqslant 0$, where $c_j(a,b)$ are real numbers depending on real numbers a, b. Applying the result to each term of $X_{p,w}(s)/X_p^{\ddagger}(s)$, we obtain

$$\frac{X_{p,w}(s)}{X_p^{\ddagger}(s)} = K_{p,w}(-s)^{\frac{1}{2}A_{p,w}(k)}\left(1 + \frac{c_1(w)}{s} + \cdots + \frac{c_n(w)}{s^n} + O(|s|^{-n-1})\right)\left(1 + \sum_{\nu=2}^{\infty} \frac{\mathfrak{C}_\nu(w)}{\nu^{-s}}\right)$$

for some real numbers $c_1(w), \cdots, c_n(w)$, where

$$A_{p,w} = \sum_{k=1}^{k_p} \sum_{m \in \Lambda_w(k)} (h_m(k) - h_{m,w}(k) + 1), \qquad (11.1)$$

$$K_{p,w} = (2\pi)^{-\frac{1}{2}A_{p,w}} \exp\left(\frac{1}{2}\sum_{k=1}^{k_p} \log k \sum_{m \in \Lambda_w(k)} (h_m(k) - h_{m,w}(k) + 1)\right) \quad (> 0)$$

Note that $A_{p,w}$ are positive integers, since $h_m(k) - h_{m,w}(k) \geqslant 0$ for every $1 \leqslant k \leqslant k_p$ and $m \in \Lambda_w(k)$ by definition. Therefore, for sufficiently large n, we have

$$V_p(s) = \sum_{w \in (\mathfrak{W}_p^+ \cup \mathfrak{W}_p^0) \setminus \mathfrak{W}_p^{\ddagger}} K'_{p,w}(-s)^{\frac{1}{2}A_{p,w}} Q_{p,w}(s)$$

$$\times \left(1 + \frac{c_1(w)}{s} + \cdots + \frac{c_n(w)}{s^n} + O(|s|^{-n-1})\right) \left(1 + \sum_{\nu=2}^{\infty} \frac{\mathfrak{C}_\nu(w)}{\nu^{-s}}\right)$$

as $\sigma \to -\infty$, where $K'_{p,w} = K_{p,w}$ for $w \in \mathfrak{W}_p^+ \setminus \mathfrak{W}_p^{\ddagger}$ and $K'_{p,w} = K_{p,w}/2$ for $w \in \mathfrak{W}_p^0$.

We write $Q_p^{\ddagger}(s) + V_p(s)$ as

$$Q_p^{\ddagger}(s) + V_p(s) = \sum_{\mu=1}^{\infty} \frac{1}{\mu^{-s}} \mathcal{Q}_\mu(s^{1/2}) \left(1 + \sum_{k=1}^{n} \frac{c_k(\mu)}{s^{k/2}} + O(|s|^{-(n+1)/2})\right),$$

where $\mathcal{Q}_\mu(s)$ $(m = 1, 2, 3, \cdots)$ are polynomials with $M =: \max_{\mu \geqslant 1} \deg \mathcal{Q}_\mu < \infty$. Let μ_0 be the smallest positive integer such that $\deg \mathcal{Q}_{m_0} = M$. Then

$$Q_p^{\ddagger}(s) + V_p(s) = \frac{1}{\mu_0^{-s}} Q_{\mu_0}(s^{1/2}) \left[\left(1 + \sum_{k=1}^{n} \frac{c_k(\mu_0)}{s^{k/2}} + O(|s|^{-(n+1)/2})\right)\right.$$

$$+ \sum_{\mu=1}^{\mu_0-1} \frac{1}{(\mu/\mu_0)^{-s}} \frac{\mathcal{Q}_\mu(s^{1/2})}{\mathcal{Q}_{\mu_0}(s^{1/2})} \left(1 + \sum_{k=1}^{n} \frac{c_k(\mu)}{s^{k/2}} + O(|s|^{-(n+1)/2})\right)$$

$$\left. + \sum_{\mu=\mu_0+1}^{\infty} \frac{1}{(\mu/\mu_0)^{-s}} \frac{\mathcal{Q}_\mu(s^{1/2})}{\mathcal{Q}_{\mu_0}(s^{1/2})} \left(1 + \sum_{k=1}^{n} \frac{c_k(\mu)}{s^{k/2}} + O(|s|^{-(n+1)/2})\right)\right].$$

We can take $\sigma_0 > 0$ such that

$$\sum_{\mu=\mu_0+1}^{\infty} \frac{1}{(\mu/\mu_0)^{-\sigma}} \left|\frac{\mathcal{Q}_\mu(s^{1/2})}{\mathcal{Q}_{\mu_0}(s^{1/2})}\right| \left|1 + \sum_{k=1}^{n} \frac{c_k(\mu)}{s^{k/2}} + O(|s|^{-(n+1)/2})\right| < \frac{1}{2}.$$

holds for $\Re(s) < -\sigma_0$ and $|\Im(s)| > 1$. On the other hand

$$\sum_{\mu=1}^{\mu_0-1} \frac{1}{(\mu/\mu_0)^{-\sigma}} \left|\frac{\mathcal{Q}_\mu(s^{1/2})}{\mathcal{Q}_{\mu_0}(s^{1/2})}\right| \left|1 + \sum_{k=1}^{n} \frac{c_k(\mu)}{s^{k/2}} + O(|s|^{-(n+1)/2})\right| = O(\mu_0^{-\sigma} |s|^{-1/2})$$

as $|\Im(s)| \to \infty$ in $\Re(s) < -\sigma_0$ by the choice of μ_0. Hence we have

$$Q_p^{\ddagger}(s) + V_p(s) = \frac{\mathcal{Q}_{\mu_0}(s^{1/2})}{\mu_0^{-s}} \left[1 + \Theta(1/2) + O(\mu_0^{-\sigma} |s|^{-1/2})\right]$$

for $\Re(s) < -\sigma_0$ and $|\Im(s)| > R$, where R is a large positive number and $\Theta(1/2)$ means a function whose absolute value is bounded by $1/2$. Therefore

$$\varepsilon_p(s) = \frac{X_p^{\ddagger}(s)}{D_p(s)} \frac{1}{R_p(s)} \left(Q_p^{\ddagger}(s) + V_p(s)\right) = \frac{X_p^{\ddagger}(s)}{D_p(s)} \frac{Q_p^{\ddagger}(s)}{R_p(s)} \frac{\mathcal{Q}_{\mu_0}(s^{1/2})}{Q_p^{\ddagger}(s)} \mu_0^s [1 + g(s)],$$

and $|g(s)| < 1$ as $|s| \to \infty$ with $\sigma < \kappa \log(|t| + 10)$. This formula implies Proposition 6.9 by making $\kappa > 0$ large if necessary. $\qquad\square$

12. Proof of Proposition 7.1

We need the following lemma.

Lemma 12.1 *Let $\sigma_0 > 0$, $T > 10$. Let $0 \leqslant \alpha < \beta < \sigma_0$, where α and β can be depending on T. Let $f(s)$ be an analytic function, real for real, regular for $\sigma \geqslant \alpha$, except at finitely many poles on the real line; let*

$$|\Re f(\sigma + it)| \geqslant m > 0$$

and

$$|f(\sigma' + it')| \leqslant M_{\sigma,t} \quad (\sigma' \geqslant \sigma, \ 1 \leqslant t' \leqslant t).$$

Then, if T is not the ordinate of a zero of $f(s)$

$$|\arg f(\sigma + iT)| \leqslant \frac{\pi}{\log(\sigma_0 - \alpha)/(\sigma_0 - \beta)} \left(\log M_{\alpha, T+2} + \log \frac{1}{m} \right) + \frac{3}{2}\pi$$

for $\sigma \geqslant \beta$.

Proof. We readily prove this by the similar method as in [22, p. 213]. $\qquad\square$

Proof of Proposition 7.1. It suffices to show the proposition for large T, since we shall make T large if necessary. From the proof of Proposition 6.9, we have

$$Q_p^\ddagger(s) + V_p(s) = \mathcal{Q}(s^{1/2}) \, \mu_0^s \, (1 + g(s))$$

for some positive integer μ_0 and a polynomial $\mathcal{Q}(s) = \mathcal{Q}_{\mu_0}(s)$ such that

$$|g(-\sigma_{\mathrm{L}} + it)| < 1 \quad \text{as} \quad |t| \to \infty \tag{12.1}$$

for some fixed $\sigma_{\mathrm{L}} > 0$, and

$$|g(s)| < 1 \quad \text{as} \quad |s| \to \infty \quad \text{with} \quad \Re(s) < -\kappa \log(|\Im(s)| + 10). \tag{12.2}$$

Thus, from this, we have

$$\varepsilon_p(s) = \frac{X_p^\ddagger(s)}{D_p(s)} \frac{Q_p^\ddagger(s)}{R_p(s)} \cdot \frac{\mathcal{Q}(s^{1/2})}{Q_p^\ddagger(s)} \mu_0^s (1 + g(s)) = \frac{X_p^\ddagger(s)}{D_p(s)} \frac{Q_p^\ddagger(s)}{R_p(s)} \cdot (1 + r_p(s))$$

where $r_p(s)$ is of (6.1). Recall the bound (6.2) for $r_p(s)$. Note that $Q_p^\ddagger(s)/R_p(s)$ is a polynomial by Definition 5.9 of $R_p(s)$.

For $N(T; \sigma_{\mathrm{L}})$, we consider the rectangle R_T with vertices at $-\sigma_{\mathrm{L}} + ci$, $\sigma_{\mathrm{R}} + ci$, $\sigma_{\mathrm{R}} + iT$, $-\sigma_{\mathrm{L}} + iT$, where σ_{R} is a positive real number and $c > 0$ is a positive constant. We take σ_{R} and c large positive real numbers such that $\varepsilon_p(s)$ has no zeros on the line from $-\sigma_{\mathrm{L}} + ci$ to $\sigma_{\mathrm{R}} + ci$ and $|g(-\sigma_{\mathrm{L}} + it)|$, $|r_p(\sigma_{\mathrm{R}} + it)| < 1$ for $|t| \geqslant c$. Also, we can assume that $\varepsilon_P(s)$ has no zeros on the rectangle.

We apply the standard method of the counting of zeros to the above rectangle with (6.2) and (12.1) (see [22, p. 212]). We have

$$N(T; \sigma_{\mathrm{L}}) = \frac{1}{2\pi i} \int_{R_T} d \log \varepsilon_p(s)$$

$$= \frac{1}{2\pi} \Delta_1 \arg \varepsilon_p(s) + \frac{1}{2\pi} \Delta_2 \arg \varepsilon_p(s) + \frac{1}{2\pi} \Delta_3 \arg \varepsilon_p(s) + \frac{1}{2\pi} \Delta_4 \arg \varepsilon_p(s),$$

where Δ_1, Δ_2, Δ_3, Δ_4 denote the variations from $\sigma_{\mathrm{R}} + ci$ to $\sigma_{\mathrm{R}} + iT$, from $\sigma_{\mathrm{R}} + iT$ to $-\sigma_{\mathrm{L}} + iT$, from $-\sigma_{\mathrm{L}} + iT$ to $-\sigma_{\mathrm{L}} + ic$, from $-\sigma_{\mathrm{L}} + ic$ to $\sigma_{\mathrm{R}} + ci$, respectively. Clearly, $\Delta_4 \arg \varepsilon_p(s) = O(1)$. Using (6.2) and (12.1), we can readily compute $\Delta_1 \arg \varepsilon_p(s)$ and $\Delta_3 \arg \varepsilon_p(s)$. For $\Delta_2 \arg \varepsilon_p(s)$, referring to (9.3), we define

$$\Gamma^*(s) = \prod_{k=1}^\infty \prod_{h > (kc_p+1)/2} \gamma(ks + h)^{N_p(k, h-1) - N_p(k, h)};$$

$$L(s) = \frac{Q_p^\ddagger(s)}{R_p(s)} \prod_{k=1}^\infty \prod_{h > (kc_p+1)/2} \zeta(ks + h)^{N_p(k, h-1) - N_p(k, h)};$$

$$L^*(s) = L(s)\big(1 + r_p(s)\big);$$

where $\gamma(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)$. Then

$$\varepsilon_p(s) = \Gamma^*(s)L^*(s)$$

by definition, and

$$\Delta_2 \arg \varepsilon_p(s) = \Delta_2 \arg \Gamma^*(s) + \Delta_2 \arg L^*(s).$$

The valuation $\Delta_2 \arg \Gamma^*(s)$ is computed easily by the Stirling formula. On the other hand, we have

$$
\begin{aligned}
L(s)r_p(s) = \sum_{w\in\mathfrak{W}_p^+ \cup \mathfrak{W}_p^0 \backslash \mathfrak{W}_p^\ddagger} \frac{\tilde{Q}_{p,w}(s)}{R_p(s)} \prod_{\alpha\in(\Phi^+\backslash\Phi_p^+)\cap w^{-1}\Phi^-} \frac{\gamma(\langle\lambda_p,\alpha^\vee\rangle s + \operatorname{ht}\alpha^\vee)}{\gamma(\langle\lambda_p,\alpha^\vee\rangle s + \operatorname{ht}\alpha^\vee + 1)} \\
\times \prod_{\alpha\in(\Phi^+\backslash\Phi_p^+)\cap w^{-1}\Phi^-} \zeta(\langle\lambda_p,\alpha^\vee\rangle s + \operatorname{ht}\alpha^\vee),
\end{aligned}
\tag{12.3}
$$

where $\tilde{Q}_{p,w}(s)/R_p(s)$ are polynomials by definition of $R_p(s)$. Therefore we see that

$$L^*(s) \ll T^M \qquad (-\sigma_{\mathrm{L}} \leqslant \Re(s) \leqslant \sigma_{\mathrm{R}})$$

for some positive M if T is sufficiently large ([22, Chap.V]). Hence we have

$$\Delta_2 \arg L^*(s) = O(\log T).$$

by Lemma 12.1. Thus, we get the formula for $N(T;\sigma_{\mathrm{L}})$.

For $N(T;+\infty)$, it suffices to consider $N(T;-\kappa\log T)$ by Proposition 6.9. We form the rectangle $-\kappa\log T + c_1 i$, $\sigma_{\mathrm{R}} + c_1 i$, $\sigma_{\mathrm{R}} + iT$, $-\kappa\log T + iT$, where $c_1 > 0$ is taken such that $\varepsilon_p(s)$ has no zeros on the boundary of this rectangle. We follow the method as above. We similarly have

$$N(T;\infty) = \Delta_1^* \arg \varepsilon_p(s) + \Delta_2^* \arg \varepsilon_p(s) + \Delta_3^* \arg \varepsilon_p(s) + \Delta_4^* \arg \varepsilon_p(s)$$

and $\Delta_2^* \arg \varepsilon_p(s) = \Delta_2^* \arg \Gamma^*(s) + \Delta_2^* \arg L^*(s)$, where where Δ_1^*, Δ_2^*, Δ_3^*, Δ_4^* denote the variations from $\sigma_{\mathrm{R}} + c_1 i$ to $\sigma_{\mathrm{R}} + iT$, from $\sigma_{\mathrm{R}} + iT$ to $-\kappa\log T + iT$, from $-\kappa\log T + iT$ to $-\kappa\log T + ic_1$, from $-\sigma_{\mathrm{L}} + ic_1$ to $\sigma_{\mathrm{R}} + ic_1$, respectively. Using (6.2) and (12.2), we can readily compute $\Delta_1^* \arg \varepsilon_p(s)$, $\Delta_3^* \arg \varepsilon_p(s)$ and $\Delta_4^* \arg \Gamma^*(s)$. We see that

$$L^*(s) \ll T^{\kappa^* \log T} \qquad (\sigma = -\kappa\log T \leqslant \Re(s) \leqslant \sigma_{\mathrm{R}}),$$

by (12.3) (and [22, Chap.V]), where $\kappa^* > \kappa$ is a constant depending on κ. Thus we obtain

$$\Delta_2^* \arg L^*(s) = O(\log^2 T)$$

by applying Lemma 12.1 to $\alpha = -2\kappa\log T$, $\beta = -\kappa\log T$ and $\sigma_0 = \sigma_{\mathrm{R}}$. It is not hard to see that $\Delta_4^* \arg L^*(s) = O(\log T)$ by Lemma 12.1. Thus we obtain the formula for $N(T;+\infty)$. $\qquad\square$

13. Proof of Proposition 7.2

The function $\varepsilon_p(s)$ is a linear combination of products of several zeta functions $\xi(ks+h)$ with coefficients of rational functions. Therefore $W_p(s)$ is an entire function of the order at most one, since $\xi(s)$ is an entire function of order one (and maximal type). Note that if ρ is a zero of $W_p(z)$, $-\bar{\rho}$ is so, since $\varepsilon_p(s)$ is real for real s. Thus, Propositions 6.8 and 6.9 imply the product factorization of $W_p(s)$ of the form

$$W_p(z) = \omega\, e^{\beta z}\, V(z)\, w_1(z)w_2(z)$$

with

$$w_1(z) = \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\rho_n}\right)\left(1 + \frac{z}{\bar{\rho}_n}\right)\right] \exp\left(\frac{z}{\rho_n} - \frac{z}{\bar{\rho}_n}\right),$$

$$w_2(z) = \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\eta_n}\right)\left(1 + \frac{z}{\bar{\eta}_n}\right)\right] \exp\left(\frac{z}{\eta_n} - \frac{z}{\bar{\eta}_n}\right),$$

where ω is a nonzero real number, β is a complex number, $V(z)$ is a nonzero polynomial having no zeros in $\Im(z) > 0$ except for purely imaginary zeros, $\Re(\rho_n) > 0$, $\Re(\eta_n) > 0$, $\Im(\eta_n) \geqslant 0$, and $0 < \delta(t) < \Im(\rho_n) < \sigma_L + 1 < \Im(\eta_n) < \kappa \log(\Re(\eta_n) + 10)$ for every $n \geqslant 1$. Here $\delta(t)$ is the function of Proposition 6.8, and κ is the positive number of Proposition 6.9, and σ_L is the positive number of Proposition 7.1. The products of the right-hand sides converge uniformly on every compact subset in $\mathbb{C}$. Write $\rho_n = a_n + ib_n$ $(b_n > 0)$, and $\eta_n = c_n + id_n$ $(d_n > 0)$. Then

$$\sum_{n=1}^{\infty} \left| \frac{1}{\rho_n} - \frac{1}{\bar{\rho}_n} \right| \leqslant 2 \sum_{n=1}^{\infty} \frac{b_n}{|\rho_n|^2} \leqslant 2(\sigma_L + 1) \sum_{n=1}^{\infty} \frac{1}{|\rho_n|^2},$$

and

$$\sum_{n=1}^{\infty} \left| \frac{1}{\eta_n} - \frac{1}{\bar{\eta}_n} \right| \leqslant 2 \sum_{n=1}^{\infty} \frac{d_n}{|\eta_n|^2} \leqslant 2\kappa \sum_{n=1}^{\infty} \frac{\log(c_n + 10)}{|\eta_n|^2} \ll_\epsilon \sum_{n=1}^{\infty} \frac{1}{|\eta_n|^{2-\epsilon}}.$$

Because the sum on the right-hand side is finite, we can take factors $\sum_{n=1}^{\infty} \exp\left(\frac{z}{\rho_n} - \frac{z}{\bar{\rho}_n} \right)$ and $\sum_{n=1}^{\infty} \exp\left(\frac{z}{\eta_n} - \frac{z}{\bar{\eta}_n} \right)$ out of the infinite product. Hence we obtain the desired product formula except for the requirement for the exponent

$$\alpha := \beta + \sum_{n=1}^{\infty} \exp\left(\frac{1}{\rho_n} - \frac{1}{\bar{\rho}_n} \right) + \sum_{n=1}^{\infty} \exp\left(\frac{1}{\eta_n} - \frac{1}{\bar{\eta}_n} \right)$$

of the new exponential factor.

Now we prove $\Im(\alpha) = 0$ to complete the proof of Proposition 7.2. We have

$$\log \left| \frac{W_p(-iy)}{W_p(iy)} \right| = 2y \cdot \Im(\alpha) + \log \left| \frac{V(-iy)}{V(iy)} \right| + \sum_{n=1}^{\infty} \log \left| \frac{iy + \rho_n}{iy - \rho_n} \right|^2 + \sum_{n=1}^{\infty} \log \left| \frac{iy + \eta_n}{iy - \eta_n} \right|^2$$

$$= 2y \cdot \Im(\alpha) + O(\log y) \quad (y \to +\infty).$$

In fact $\log |V(-iy)/V(iy)| = o(1)$,

$$\sum_{n=1}^{\infty} \log \frac{a_n^2 + (y + b_n)^2}{a_n^2 + (y - b_n)^2} \leqslant \sum_{n=1}^{\infty} \frac{4yb_n}{a_n^2 + (y - b_n)^2} = O\left(\sum_{\rho=1}^{\infty} \frac{y \log \rho}{\rho^2 + y^2} \right) = O(\log y)$$

as $y \to +\infty$ by $0 < b_n < \sigma_L + 1$ and Proposition 7.1, and

$$\sum_{n=1}^{\infty} \log \frac{c_n^2 + (y + d_n)^2}{c_n^2 + (y - d_n)^2} \leqslant \sum_{n=1}^{\infty} \frac{4yd_n}{c_n^2 + (y - d_n)^2} = O\left(\sum_{\rho=1}^{\infty} \frac{y \log \rho}{(\rho^2 + y^2)^{1-\epsilon}} \right) = O(\log y)$$

as $y \to +\infty$ by $0 < d_n < \kappa \log(c_n + 10)$ and Proposition 7.1. Hence, in order to prove $\Im(\alpha) = 0$, it suffices to show

$$\frac{\varepsilon_p(-c_p/2 + y)}{\varepsilon_p(-c_p/2 - y)} = Ay^m(1 + o(1)) \quad (y \to +\infty)$$

for some $A \neq 0$ and $m \geqslant 0$. Because of $D_p(-c_p - s) = D_p(s)$ and $R_p(-c_p - s) = R_p(s)$,

$$\frac{\varepsilon_p(-c_p/2 + y)}{\varepsilon_p(-c_p/2 - y)} = \frac{E_p(-c_p/2 + y)}{E(-c_p/2 - y)} \frac{R_p(-c_p - y)D_p(-c_p/2 - y)}{R_p(-c_p + y)D_p(-c_p/2 + y)} = \frac{E_p(-c_p/2 + y)}{E(-c_p/2 - y)}.$$

We need to show

$$\frac{E_p(-c_p/2 - y)}{E_p(-c_p/2 + y)} = \frac{\sum_w Q'_{p,w}(-c_p/2 - y)X_{p,w}(-c_p/2 - y)}{\sum_w Q'_{p,w}(-c_p/2 + y)X_{p,w}(-c_p/2 + y)} = A^{-1}y^{-m}(1 + o(1))$$

as $y \to +\infty$ for some $A \neq 0$ and $m \geqslant 0$, where the summation $\sum_w$ taken over all $w \in \mathfrak{W}_p^+ \cup \mathfrak{W}_p^0$, $Q'_{p,w}(s) = Q_{p,w}(s)$ for $w \in \mathfrak{W}_p^+$ and $Q'_{p,w}(s) = Q_{p,w}(s)/2$ for $w \in \mathfrak{W}_p^0$.

By Propositions 6.5 and 6.6, we have

$$E_p(-c_p/2 + y) = Q_p^{\ddagger}(-c_p/2 + y)X_p^{\ddagger}(-c_p/2 + y)(1 + o(1)) \quad (y \to +\infty).$$

Therefore it suffices to show that

$$\frac{E_p(-c_p/2-y)}{Q_p^{\ddagger}(-c_p/2+y)X_p^{\ddagger}(-c_p/2+y)} = \sum_w \frac{Q'_{p,w}(-c_p/2-y)}{Q_p^{\ddagger}(-c_p/2+y)}\frac{X_{p,w}(-c_p/2-y)}{X_p^{\ddagger}(-c_p/2+y)} \qquad (13.1)$$
$$= A^{-1}y^{-m}(1+o(1))$$

as $y \to +\infty$ for some $A \neq 0$ and $m > 0$. We have

$$\frac{Q'_{p,w}(-c_p/2-y)}{Q_p^{\ddagger}(-c_p/2+y)} = \delta'_w + \frac{a_1(w)}{y} + \cdots + \frac{a_n(w)}{y^n} + O(y^{-n-1}) \quad (y \to +\infty),$$

for some real numbers $a_1(w), \cdots, c_n(w)$, where $\delta_w = 1$ if $w \in \mathfrak{W}_p^{\ddagger}$ and $|w^{-1}\Delta \setminus \Phi_p| = 1$, and $\delta_w = 0$ otherwise. We write

$$\frac{X_{p,w}(-c_p/2-y)}{X_p^{\ddagger}(-c_p/2+y)} = \frac{X_{p,w}(-c_p/2+y)}{X_p^{\ddagger}(-c_p/2+y)}\frac{X_{p,w}(-c_p/2-y)}{X_{p,w}(-c_p/2+y)}.$$

Here we find that

$$\frac{X_{p,w}(s)}{X_p^{\ddagger}(s)} = B_{p,w}\,s^{-\frac{A_{p,w}}{2}}\left(1+\frac{b_1(w)}{s}+\cdots+\frac{b_n(w)}{s^n}+O\big(|s|^{-n-1}\big)\right)(1+O(2^{-\sigma}))$$

as $\sigma \to +\infty$ for some real numbers $B_{p,w} > 0$, $b_1(w), \cdots, b_n(w)$, and positive integers $A_{p,w}$ of (11.1) by a way similar to the proof of Proposition 6.9. On the other hand,

$$\frac{\xi(k(-c_p-s)+h+\delta)}{\xi(ks+h+\delta)} = \frac{\xi(ks+kc_p-h-\delta+1)}{\xi(ks+h+\delta)}$$
$$= (2\pi/k)^{-\frac{kc_p-2h-2\delta+1}{2}}s^{\frac{kc_p-2h-2\delta+1}{2}}\left(1+\frac{c_1(k,h)}{s}+\cdots+\frac{c_n(k,h)}{s^n}+O\big(|s|^{-n-1}\big)\right)(1+O(2^{-\sigma}))$$

as $\sigma \to +\infty$. Therefore we obtain

$$\frac{X_{p,w}(-c_p-s)}{X_{p,w}(s)} = \prod_{\alpha\in\Phi^+\setminus\Phi_p^+}\frac{\xi(\langle\lambda_p,\alpha^\vee\rangle(-c_p-s)++\mathrm{ht}\,\alpha^\vee+\delta_{\alpha,w})}{\xi(\langle\lambda_p,\alpha^\vee\rangle s+\mathrm{ht}\,\alpha^\vee+\delta_{\alpha,w})}$$
$$= \prod_{k=1}^{k_p}\prod_{h=1}^{kc_p+\hbar_k^+}\left(\frac{\xi(k(-c_p-s)+h)}{\xi(ks+h)}\right)^{N_{p,w}(k,h)}\left(\frac{\xi(k(-c_p-s)+h+1)}{\xi(ks+h+1)}\right)^{N_p(k,h)-N_{p,w}(k,h)}$$
$$= B'_{p,w}\,s^{-\frac{A'_{p,w}}{2}}\left(1+\frac{b'_1(w)}{s}+\cdots+\frac{b'_n(w)}{s^n}+O\big(|s|^{-n-1}\big)\right)(1+O(2^{-\sigma}))$$

as $\sigma \to +\infty$ for some real numbers $B'_{p,w} > 0$ and integers $A'_{p,w}$ given by

$$-A'_{p,w} = \sum_{k=1}^{k_p}\sum_{h=k+\hbar_k^-}^{k+\hbar_k^+}\left(N_{p,w}(k,h)(kc_p-2h+1)+(N_p(k,h)-N_{p,w}(k,h))(kc_p-2h-1)\right).$$

Now we calculate $A'_{p,w}$. We have

$$-A'_{p,w} = \sum_{k=1}^{k_p}\sum_{h=k+\hbar_k^-}^{k+\hbar_k^+}\left(N_p(k,h)(kc_p-2h)+2N_{p,w}(k,h)-N_p(k,h)\right)$$
$$= \sum_{k=1}^{k_p}\sum_{h=k+\hbar_k^-}^{k+\hbar_k^+}N_p(k,h)(kc_p-2h) - \sum_{k=1}^{k_p}\sum_{h=k+\hbar_k^-}^{kc_p+\hbar_k^+}\left(N_p(k,h)-2N_{p,w}(k,h)\right).$$

The first sum on the right-hand side is zero, since $N_p(k,h) = N_p(k, kc_p - h)$ and $\hbar_k^- + \hbar_k^+ = k(c_p - 2)$ by Lemmas 8.3 and 8.14. Therefore, we get

$$
\begin{aligned}
A'_{p,w} &= \sum_{k=1}^{k_p} \sum_{h=k+\hbar_k^-}^{kc_p+\hbar_k^+} \Big(N_p(k,h) - 2N_{p,w}(k,h) \Big) \\
&= \sum_{k=1}^{k_p} \sum_{h=k+\hbar_k^-}^{kc_p+\hbar_k^+} 2\Big(N_p(k,h) - N_{p,w}(k,h) \Big) - \sum_{k=1}^{k_p} \sum_{h=k+\hbar_k^-}^{kc_p+\hbar_k^+} N_p(k,h) \\
&= 2\left(|\Phi^+ \setminus \Phi_p^+| - l_p(w) \right) - |\Phi^+ \setminus \Phi_p^+| = |\Phi^+ \setminus \Phi_p^+| - 2\,l_p(w).
\end{aligned}
$$

Hence $A'_{p,w} > 0$ for $w \in \mathfrak{W}_p^+$ and $A'_{p,w} = 0$ for $w \in \mathfrak{W}_p^0$.

Combining the above facts, we obtain

$$
\begin{aligned}
&\frac{E_p(-c_p/2 - y)}{Q_p^\ddagger(-c_p/2 + y) X_p^\ddagger(-c_p/2 + y)} \\
&= \sum_{w \in \mathfrak{W}_p^+ \cup \mathfrak{W}_p^0} B_{p,w}^* \, y^{-\frac{A_{p,w} + A'_{p,w}}{2}} \left(\delta'_w + \frac{b_1^*(w)}{y} + \cdots + \frac{b_n^*(w)}{y^n} + O\big(y^{-n-1}\big) \right) (1 + O(2^{-y}))
\end{aligned}
$$

as $y \to +\infty$ for some positive integers $B_{p,w}^*$ and real numbers $b_1^*(w), \cdots, b_n^*(w)$. Now we obtain (13.1), and complete the proof of Proposition 7.2. $\qquad\square$

14. PROOF OF THEOREM 7.5

In order to prove that all but finitely many zeros of $\xi_p(s)$ are simple, it suffices to show that $\theta(t) = \arg \varepsilon_p(-c_p/2 + it)$ is strictly increasing as $t \to +\infty$, since we already know that all but finitely many zeros of $\xi_p(s)$ lie on the line $\Re(s) = -c_p/2$, and

$$
\xi_p(-c_p/2 + it) = |\varepsilon_p(-c_p/2 + it)| \left(e^{i\theta(t)} \pm e^{-i\theta(t)} \right)
$$

holds for a suitable choice of sign. We have

$$
E_p(s) = Q_p^\ddagger(s) X_p^\ddagger(s)(1 + o(1))
$$

as $|s| \to \infty$ on $\Re(s) = -c_p/2$ by Propositions 6.5 and 6.6, and hence

$$
\varepsilon_p(s) = \frac{Q_p^\ddagger(s)}{R_p(s)} \frac{X_p^\ddagger(s)}{D_p(s)}(1 + o(1))
$$

as $|s| \to \infty$ on $\Re(s) = -c_p/2$. Note that $Q_p^\ddagger(s)/R_p(s)$ and $X_p^\ddagger(s)/D_p(s)$ are entire by Definitions 5.9 and 5.10. Using the above asymptotic formula and (9.3) of $X_p^\ddagger(s)/D_p(s)$, we can see that

$$
\frac{d \arg \varepsilon_p(-c_p/2 + it)}{dt} = \kappa \log t \, (1 + o(1)) \quad (t \to +\infty)
$$

for some positive constant κ. Here the factor $\log t$ comes from the gamma factors of $X_p^\ddagger(s)/D_p(s)$ (see (9.3)), and we need to use the fact that

$$
\frac{\zeta'}{\zeta}(1 + it) = O\left(\frac{\log t}{\log \log t} \right) = o(\log t) \quad (t \to +\infty)
$$

(see Theorem 5.7 of [22], for example), if kc_p is even and $N_p(k, h-1) - N_p(k,h) > 0$ for some $1 \leqslant k \leqslant k_p$ with $h = 1 + kc_p/2$. In any case $\theta(t) = \arg \varepsilon_p(-c_p/2 + it)$ is strictly increasing as $t \to \infty$. Hence we obtain the desired result. $\qquad\square$

The worst case of the proof could occur. In fact the zeta function of A_1 is

$$
\hat{\zeta}_1^{A_1}(s) = \frac{\hat{\zeta}(s+2)}{s} - \frac{\hat{\zeta}(s+1)}{s+2}.
$$

In this case, we have the factor $\zeta(1+it)$ in $\varepsilon_1(s) = \xi(s+2)$ on $\Re(s) = -1$ ($c_p = 2$).

15. Appendix 1: Decomposition of $\Sigma_p(1)$

We use the same numbering as in [9, p.53]. In the following, we abbreviate $L_j^\vee(1)$ of Lemma 8.15 as $L_j^\vee$, and $\alpha^\vee = \sum_{i=1}^r a_i \alpha_i^\vee$ as the sequence $a_1 \cdots a_r$ with $a_p = 1$. We will give explicit forms of the sets $L_j^\vee$ according to the type of the root system $\Phi^\vee$.

15.1. A_n case.
Due to the symmetry of the root system, it is sufficient to consider the case $1 \leqslant p \leqslant [n/2+1]$. Let $\alpha_{i,j}^\vee = \sum_{k=i}^j \alpha_k^\vee$. Then for $1 \leqslant j \leqslant p$,

$$L_j^\vee = \{\alpha_{j,n}^\vee, \alpha_{j,n-1}^\vee, \ldots, \alpha_{j,p+j-1}^\vee, \alpha_{j+1,p+j-1}^\vee, \ldots, \alpha_{p,p+j-1}^\vee\}.$$

For example, in the A_6 case, we have

- $p = 1$ case:
  $L_1^\vee$: 111111 111110 111100 111000 110000 100000
- $p = 2$ case:
  $L_1^\vee$: 111111 111110 111100 111000 110000 010000
  $L_2^\vee$: 011111 011110 011100 011000
- $p = 3$ case:
  $L_1^\vee$: 111111 111110 111100 111000 011000 001000
  $L_2^\vee$: 011111 011110 011100 001100
  $L_3^\vee$: 001111 001110
- $p = 4$ case:
  $L_1^\vee$: 111111 111110 111100 011100 001100 000100
  $L_2^\vee$: 011111 011110 001110 000110
  $L_3^\vee$: 001111 000111
- $p = 5$ case:
  $L_1^\vee$: 111111 111110 011110 001110 000110 000010
  $L_2^\vee$: 011111 001111 000111 000011
- $p = 6$ case:
  $L_1^\vee$: 111111 011111 001111 000111 000011 000001

15.2. B_n case.
Let $\alpha_{i,j}^\vee = \sum_{l=i}^j \alpha_l^\vee$ and $\beta_{i,j,k}^\vee = \sum_{l=i}^j \alpha_l^\vee + 2\sum_{l=j+1}^k \alpha_l^\vee$. Then for $1 \leqslant j \leqslant \min\{p, n-p+1\}$,

$$L_j^\vee = \{\beta_{j,p,n}^\vee, \beta_{j,p+1,n}^\vee, \ldots, \beta_{j,n-1,n}^\vee, \alpha_{j,n}^\vee, \alpha_{j,n-1}^\vee, \ldots, \alpha_{j,p+j-1}^\vee, \alpha_{j+1,p+j-1}^\vee, \ldots, \alpha_{p,p+j-1}^\vee\}.$$

Further if $n-p+1 < p \leqslant n$, then for $n-p+1 < j \leqslant \min\{p, 2n-2p+1\}$,

$$L_j^\vee = \{\beta_{j,p,n}^\vee, \beta_{j,p+1,n}^\vee, \ldots, \beta_{j,2n-p-j+1,n}^\vee, \beta_{j+1,2n-p-j+1,n}^\vee, \ldots, \beta_{p,2n-p-j+1,n}^\vee\}.$$

For example, in the B_6 case, we have

- $p = 1$ case:
  $L_1^\vee$: 122222 112222 111222 111122 111112 111111 111110 111100 111000 110000 100000
- $p = 2$ case:
  $L_1^\vee$: 112222 111222 111122 111112 111111 111110 111100 111000 110000 010000
  $L_2^\vee$: 012222 011222 011122 011112 011111 011110 011100 011000
- $p = 3$ case:
  $L_1^\vee$: 111222 111122 111112 111111 111110 111100 111000 011000 001000
  $L_2^\vee$: 011222 011122 011112 011111 011110 011100 001100
  $L_3^\vee$: 001222 001122 001112 001111 001110
- $p = 4$ case:
  $L_1^\vee$: 111122 111112 111111 111110 111100 011100 001100 000100
  $L_2^\vee$: 011122 011112 011111 011110 001110 000110

$L_3^\vee$: 001122 001112 001111 000111

$L_4^\vee$: 000122 000112

- $p = 5$ case:

  $L_1^\vee$: 111112 111111 111110 011110 001110 000110 000010

  $L_2^\vee$: 011112 011111 001111 000111 000011

  $L_3^\vee$: 001112 000112 000012

- $p = 6$ case:

  $L_1^\vee$: 111111 011111 001111 000111 000011 000001

15.3. C_n case. Let $\alpha_{i,j}^\vee = \sum_{l=i}^j \alpha_l^\vee$ and $\beta_{i,j}^\vee = \sum_{l=i}^j \alpha_l^\vee + 2\sum_{l=j+1}^{n-1} \alpha_l^\vee + \alpha_n^\vee$. Then for $1 \leqslant j \leqslant \min\{p, n-p+1\}$,

$$L_j^\vee = \{\beta_{j,p}^\vee, \beta_{j,p+1}^\vee, \ldots, \beta_{j,n-2}^\vee, \alpha_{j,n}^\vee, \alpha_{j,n-1}^\vee, \ldots, \alpha_{j,p+j-1}^\vee, \alpha_{j+1,p+j-1}^\vee, \ldots, \alpha_{p,p+j-1}^\vee\}.$$

Further if $n - p + 1 < p \leqslant n - 1$, then for $n - p + 1 < j \leqslant \min\{p, 2n - 2p\}$,

$$L_j^\vee = \{\beta_{j,p}^\vee, \beta_{j,p+1}^\vee, \ldots, \beta_{j,2n-j-p}^\vee, \beta_{j+1,2n-j-p}^\vee, \ldots, \beta_{p,2n-j-p}^\vee\}.$$

In the case $p = n$, for $1 \leqslant j \leqslant \frac{n+1}{2}$,

$$L_j^\vee = \{\beta_{j,j-1}^\vee, \beta_{j,j}^\vee, \beta_{j,j+1}^\vee, \ldots, \beta_{j,n-j}^\vee, \beta_{j+1,n-j}^\vee, \ldots, \beta_{n-j+1,n-j}^\vee\}.$$

For example, in the C_6 case, we have

- $p = 1$ case:

  $L_1^\vee$: 122221 112221 111221 111121 111111 111110 111100 111000 110000 100000

- $p = 2$ case:

  $L_1^\vee$: 112221 111221 111121 111111 111110 111100 111000 110000 010000

  $L_2^\vee$: 012221 011221 011121 011111 011110 011100 011000

- $p = 3$ case:

  $L_1^\vee$: 111221 111121 111111 111110 111100 111000 011000 001000

  $L_2^\vee$: 011221 011121 011111 011110 011100 001100

  $L_3^\vee$: 001221 001121 001111 001110

- $p = 4$ case:

  $L_1^\vee$: 111121 111111 111110 111100 011100 001100 000100

  $L_2^\vee$: 011121 011111 011110 001110 000110

  $L_3^\vee$: 001121 001111 000111

  $L_4^\vee$: 000121

- $p = 5$ case:

  $L_1^\vee$: 111111 111110 011110 001110 000110 000010

  $L_2^\vee$: 011111 001111 000111 000011

- $p = 6$ case:

  $L_1^\vee$: 222221 122221 112221 111221 111121 111111 011111 001111 000111 000011 000001

  $L_2^\vee$: 022221 012221 011221 011121 001121 000121 000021

  $L_3^\vee$: 002221 001221 000221

15.4. D_n case. Let $\alpha_{i,j}^\vee = \sum_{l=i}^j \alpha_l^\vee$, $\beta_{i,j}^\vee = \sum_{l=i}^j \alpha_l^\vee + 2\sum_{l=j+1}^{n-2} \alpha_l^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee$ and $\gamma_i^\vee = \sum_{l=i}^{n-2} \alpha_l^\vee + \alpha_n^\vee$. Then in the cases $1 \leqslant p \leqslant n - 2$, for $1 \leqslant j \leqslant \min\{p, n-p+1\}$,

$$L_j^\vee = \{\beta_{j,p}^\vee, \beta_{j,p+1}^\vee, \ldots, \beta_{j,n-2}^\vee, \alpha_{j,n-1}^\vee, \ldots, \alpha_{j,p+j-1}^\vee, \alpha_{j+1,p+j-1}^\vee, \ldots, \alpha_{p,p+j-1}^\vee\}.$$

Further if $n - p + 1 < p \leqslant n - 2$, then for $n - p + 1 < j \leqslant \min\{p, 2n - 2p - 1\}$,

$$L_j^\vee = \{\beta_{j,p}^\vee, \beta_{j,p+1}^\vee, \ldots, \beta_{j,2n-j-p-1}^\vee, \beta_{j+1,2n-j-p-1}^\vee, \ldots, \beta_{p,2n-j-p-1}^\vee\}.$$

In addition, for $j = \min\{p, 2n - 2p - 1\} + 1$,

$$L_j^\vee = \{\gamma_1^\vee, \gamma_2^\vee, \ldots, \gamma_p^\vee\}.$$

In the case $p = n - 1$, for $j = 1$,

$$L_1^\vee = \{\beta_{1,1}^\vee, \beta_{1,2}^\vee, \ldots, \beta_{1,n-2}^\vee, \alpha_{1,n-1}^\vee, \alpha_{2,n-1}^\vee, \ldots, \alpha_{n-1,n-1}^\vee\},$$

and for $2 \leqslant j \leqslant \frac{n}{2}$,

$$L_j^\vee = \{\beta_{j,j}^\vee, \beta_{j,j+1}^\vee, \ldots, \beta_{j,n-j}^\vee, \beta_{j+1,n-j}^\vee, \ldots, \beta_{n-j,n-j}^\vee\},$$

In the case $p = n$, for $1 \leqslant j \leqslant \frac{n}{2}$, $L_j^\vee$ is the same as in the case $p = n - 1$ with the roles of α_{n-1} and α_n exchanged.

For example, in the D_6 case, we have

- $p = 1$ case:
  $L_1^\vee$: 122211 112211 111211 111111 111110 111100 111000 110000 100000
  $L_2^\vee$: 111101
- $p = 2$ case:
  $L_1^\vee$: 112211 111211 111111 111110 111100 111000 110000 010000
  $L_2^\vee$: 012211 011211 011111 011110 011100 011000
  $L_3^\vee$: 111101 011101
- $p = 3$ case:
  $L_1^\vee$: 111211 111111 111110 111100 111000 011000 001000
  $L_2^\vee$: 011211 011111 011110 011100 001100
  $L_3^\vee$: 001211 001111 001110
  $L_4^\vee$: 111101 011101 001101
- $p = 4$ case:
  $L_1^\vee$: 111111 111110 111100 011100 001100 000100
  $L_2^\vee$: 011111 011110 001110 000110
  $L_3^\vee$: 001111 000111
  $L_4^\vee$: 111101 011101 001101 000101
- $p = 5$ case:
  $L_1^\vee$: 122211 112211 111211 111111 111110 011110 001110 000110 000010
  $L_2^\vee$: 012211 011211 011111 001111 000111
  $L_3^\vee$: 001211
- $p = 6$ case:
  $L_1^\vee$: 122211 112211 111211 111111 111101 011101 001101 000101 000001
  $L_2^\vee$: 012211 011211 011111 001111 000111
  $L_3^\vee$: 001211

15.5. E_6 case.

- $p = 1$ case:
  $L_1^\vee$: 123212 123211 122211 122111 122101 112101 111101 111001 111000 110000 100000
  $L_2^\vee$: 112211 112111 111111 111110 111100
- $p = 2$ case:
  $L_1^\vee$: 112211 112111 111111 111110 111100 111000 110000 010000
  $L_2^\vee$: 012211 012111 012101 011101 011100 011000
  $L_3^\vee$: 112101 111101 111001 011001
  $L_4^\vee$: 011111 011110
- $p = 3$ case:
  $L_1^\vee$: 111111 111110 111100 111000 011000 001000
  $L_2^\vee$: 111101 111001 011001 001001
  $L_3^\vee$: 011111 011110 011100 001100
  $L_4^\vee$: 011101 001101
  $L_5^\vee$: 001111 001110
- $p = 4$ case:
  $L_1^\vee$: 122111 122101 112101 111101 111100 011100 001100 000100

$L_2^\vee$: 112111 111111 111110 011110 001110 000110
$L_3^\vee$: 012111 012101 011101 001101
$L_4^\vee$: 011111 001111

- $p = 5$ case:
  $L_1^\vee$: 123212 123211 122211 122111 112111 111111 111110 011110 001110 000110 000010
  $L_2^\vee$: 112211 012211 012111 011111 001111
- $p = 6$ case:
  $L_1^\vee$: 123211 122211 122111 122101 112101 111101 111001 011001 001001 000001
  $L_2^\vee$: 112211 112111 111111 011111 001111 001101
  $L_3^\vee$: 012211 012111 012101 011101

15.6. E_7 **case.**

- $p = 1$ case:
  $L_1^\vee$: 1343212 1243212 1233212 1233211 1232211 1222211 1122211 1122111 1121111 1111111 1111110 1111100 1111000 1110000 1100000 1000000
  $L_2^\vee$: 1232212 1232112 1232102 1232101 1222101 1122101 1121101 1111101 1111001 1110001
  $L_3^\vee$: 1232111 1222111 1221111 1221101 1221001 1121001
- $p = 2$ case:
  $L_1^\vee$: 1122211 1122111 1121111 1111111 1111110 1111100 1111000 1110000 1100000 0100000
  $L_2^\vee$: 0122211 0122111 0121111 0111111 0111110 0111100 0111000 0110000
  $L_3^\vee$: 1122101 0122101 0121101 0111101 0111001 0110001
  $L_4^\vee$: 1121101 1111101 1111001 1110001
  $L_5^\vee$: 1121001 0121001
- $p = 3$ case:
  $L_1^\vee$: 1111111 1111110 1111100 1111000 1110000 0110000 0010000
  $L_2^\vee$: 1111101 1111001 1110001 0110001 0010001
  $L_3^\vee$: 0111111 0111110 0111100 0111000 0011000
  $L_4^\vee$: 0111101 0111001 0011001
  $L_5^\vee$: 0011111 0011110 0011100
  $L_6^\vee$: 0011101
- $p = 4$ case:
  $L_1^\vee$: 1221111 1221101 1221001 1121001 1111001 1111000 0111000 0011000 0001000
  $L_2^\vee$: 1121111 1121101 1111101 0111101 0111100 0011100 0001100
  $L_3^\vee$: 1111111 1111110 0111110 0011110 0001110
  $L_4^\vee$: 0121111 0121101 0121001 0111001 0011001
  $L_5^\vee$: 0111111 0011111 0011101
  $L_6^\vee$: 1111100
- $p = 5$ case:
  $L_1^\vee$: 1232112 1232102 1232101 1222101 1221101 1121101 1111101 1111100 0111100 0011100 0001100 0000100
  $L_2^\vee$: 1232111 1222111 1221111 1121111 1111111 1111110 0111110 0011110 0001110 0000110
  $L_3^\vee$: 1122111 1122101 0122101 0121101 0111101 0011101
  $L_4^\vee$: 0122111 0121111 0111111 0011111
- $p = 6$ case:
  $L_1^\vee$: 2343212 1343212 1243212 1233212 1232212 1232112 1232111 1222111 1221111 1121111 1111111 1111110 0111110 0011110 0001110 0000110 0000010
  $L_2^\vee$: 1233211 1232211 1222211 1122211 1122111 0122111 0121111 0111111 0011111
  $L_3^\vee$: 0122211

- $p = 7$ case:

  $L_1^\vee$: 1233211 1232211 1232111 1232101 1222101 1122101 1121101 1111101 1111001 1110001 0110001 0010001 0000001

  $L_2^\vee$: 1222211 1222111 1221111 1221101 1221001 1121001 0121001 0111001 0011001

  $L_3^\vee$: 1122211 1122111 1121111 1111111 0111111 0011111 0011101

  $L_4^\vee$: 0122211 0122111 0122101 0121101 0111101

  $L_5^\vee$: 0121111

15.7. E_8 **case.**

- $p = 1$ case:

  $L_1^\vee$: 13456423 12456423 12356423 12346423 12345423 12345422 12345322 12344322 12334322 12334312 12234312 12234212 12233212 12223212 12223211 12222211 12222111 12222101 11222101 11122101 11112101 11111101 11111001 11111000 11110000 11100000 11000000 10000000

  $L_2^\vee$: 12345323 12345313 12345312 12344312 12344212 12334212 12333212 12333211 12233211 11233211 11223211 11222211 11222111 11122111 11112111 11111111 11111110 11111100

  $L_3^\vee$: 12234322 11234322 11234312 11234212 11233212 11223212 11123212 11123211 11122211 11112211

- $p = 2$ case:

  $L_1^\vee$: 11234322 11234312 11234212 11233212 11223212 11223211 11222211 11222111 11222101 11122101 11112101 11111101 11111001 11111000 11110000 11100000 11000000 01000000

  $L_2^\vee$: 01234322 01234312 01234212 01233212 01223212 01123212 01123211 01122211 01112211 01112111 01111111 01111110 01111100 01111000 01110000 01100000

  $L_3^\vee$: 11233211 01233211 01223211 01222211 01222111 01222101 01122101 01112101 01111101 01111001

  $L_4^\vee$: 11123212 11123211 11122211 11112211 11112111 11111111 11111110 11111100

  $L_5^\vee$: 11122111 01122111

- $p = 3$ case:

  $L_1^\vee$: 11123212 11123211 11122211 11122111 11122101 11112101 11111101 11111001 11111000 11110000 11100000 01100000 00100000

  $L_2^\vee$: 01123212 00123212 00123211 00122211 00112211 00112111 00111111 00111101 00111001 00111000 00110000

  $L_3^\vee$: 01123211 01122211 01122111 01122101 01112101 01111101 01111001 01111000 01110000

  $L_4^\vee$: 11112211 11112111 11111111 11111110 11111100 01111100 00111100

  $L_5^\vee$: 01112211 01112111 01111111 01111110 00111110

  $L_6^\vee$: 00122111 00122101 00112101

- $p = 4$ case:

  $L_1^\vee$: 11112211 11112111 11112101 11111101 11111001 11111000 11110000 01110000 00110000 00010000

  $L_2^\vee$: 01112211 00112211 00012211 00012111 00012101 00011101 00011001 00011000

  $L_3^\vee$: 11111111 11111110 11111100 01111100 01111000 00111000

  $L_4^\vee$: 01112111 01111111 01111110 00111110 00111100 00011100

  $L_5^\vee$: 01112101 00112101 00111101 00111001

  $L_6^\vee$: 00112111 00111111 00011111 00011110

  $L_7^\vee$: 01111101 01111001

- $p = 5$ case:

  $L_1^\vee$: 11111111 11111110 11111100 11111000 01111000 00111000 00011000 00001000

  $L_2^\vee$: 11111101 11111001 01111001 00111001 00011001 00001001

  $L_3^\vee$: 01111111 01111110 01111100 00111100 00011100 00001100

  $L_4^\vee$: 00111111 00111110 00011110 00001110

$L_5^\vee$: 01111101 00111101 00011101 00001101
$L_6^\vee$: 00011111 00001111

- $p = 6$ case:
  $L_1^\vee$: 12222111 12222101 11222101 11122101 11112101 11111101 11111100 01111100 00111100 00011100 00001100 00000100
  $L_2^\vee$: 11222111 11122111 11112111 11111111 11111110 01111110 00111110 00011110 00001110 00000110
  $L_3^\vee$: 01222111 01222101 01122101 00122101 00112101 00012101 00011101 00001101
  $L_4^\vee$: 01122111 00122111 00112111 00012111 00011111 00001111
  $L_5^\vee$: 01112111 01112101 01111101 00111101
  $L_6^\vee$: 01111111 00111111

- $p = 7$ case:
  $L_1^\vee$: 12345313 12345312 12344312 12344212 12334212 12333212 12333211 12233211 12223211 12222211 12222111 11222111 11122111 11112111 11111111 11111110 01111110 00111110 00011110 00001110 00000110 00000010
  $L_2^\vee$: 12334312 12234312 12234212 12233212 12223212 11223212 11123212 11123211 11122211 11112211 01112211 01112111 01111111 00111111 00011111 00001111
  $L_3^\vee$: 11234312 11234212 11233212 11233211 11223211 11222211 01222211 01222111 01122111 00122111 00112111 00012111
  $L_4^\vee$: 01234312 01234212 01233212 01233211 01223211 01123211 01122211 00122211 00112211 00012211
  $L_5^\vee$: 01223212 01123212 00123212 00123211

- $p = 8$ case:
  $L_1^\vee$: 12333211 12233211 12223211 12222211 12222111 12222101 11222101 11122101 11112101 11111101 11111001 01111001 00111001 00011001 00001001 00000001
  $L_2^\vee$: 11233211 01233211 01223211 01123211 00123211 00122211 00112211 00012211 00012111 00011111 00001111 00001101
  $L_3^\vee$: 11223211 11222211 11222111 11122111 11112111 11111111 01111111 00111111 00111101 00011101
  $L_4^\vee$: 11123211 11122211 11112211 01112211 01112111 00112111 00112101 00012101
  $L_5^\vee$: 01222211 01222111 01222101 01122101 01112101 01111101
  $L_6^\vee$: 01122211 01122111 00122111 00122101

15.8. F_4 **case.**

- $p = 1$ case:
  $L_1^\vee$: 1342 1242 1232 1231 1221 1220 1120 1110 1100 1000
  $L_2^\vee$: 1222 1122 1121 1111
- $p = 2$ case:
  $L_1^\vee$: 1122 1121 1120 1110 1100 0100
  $L_2^\vee$: 0122 0121 0120 0110
  $L_3^\vee$: 1111 0111
- $p = 3$ case:
  $L_1^\vee$: 1111 1110 0110 0010
  $L_2^\vee$: 0111 0011
- $p = 4$ case:
  $L_1^\vee$: 1231 1221 1121 1111 0111 0011 0001
  $L_2^\vee$: 0121

15.9. G_2 **case.**

- $p = 1$ case:
  $L_1^\vee$: 13 12 11 10
- $p = 2$ case:
  $L_1^\vee$: 11 01

16. Appendix 2: Table of numbers c_p

We use the numbering of simple roots as in Appendix 1.

(1) A_r case ($r \geqslant 1$): $c_p = r + 1$ (independent of p).
(2) B_r case ($r \geqslant 2$): $c_p = 2r - p$ ($p \neq r$), $c_r = 2r$.
(3) C_r case ($r \geqslant 3$): $c_p = 2r - p + 1$.
(4) D_r case ($r \geqslant 4$): $c_p = 2r - p - 1$ ($p \neq r - 1, r$), $c_{r-1} = c_r = 2r - 2$.
(5) E_6 case:

p	1	2	3	4	5	6
c_p	12	9	7	9	12	11

(6) E_7 case:

p	1	2	3	4	5	6	7
c_p	17	11	8	10	13	18	14

(7) E_8 case:

p	1	2	3	4	5	6	7	8
c_p	29	19	14	11	9	13	23	17

(8) F_4 case: $c_1 = 11$, $c_2 = 7$, $c_3 = 5$, $c_4 = 8$.
(9) G_2 case: $c_1 = 5$, $c_2 = 3$.

17. Appendix 3: Homogeneous vector bundles

We present basic facts on homogeneous vector bundles according to [18]. Let G be a semisimple complex Lie group of type Φ with a maximal torus T. Let B be a Borel subgroup containing T, and let P be a maximal parabolic subgroup of G corresponding a simple root α_p in the fundamental system Δ attached to B.

Then $X_p = G/P$ is a connected compact complex manifold with the Picard number one, and the homogeneous line bundles on X_p are in one-to-one correspondence with the set of weights $\Lambda_p = \mathbb{Z}\lambda_p$ and $\operatorname{Pic}(X_p) \simeq \Lambda_p$, where λ_p is the fundamental weight corresponding to α_p.

Let L be a holomorphic line bundle on X_p, and let $c_1(L) \in H^2(X_p, \mathbb{R})$ be the first Chern class of L. The first Chern class of a line bundle L is identified with its associated weight $\lambda \in \Lambda_p \simeq \operatorname{Pic}(X_p)$ as follows:

$$c_1(L) = \frac{i}{2\pi} \sum_{\alpha \in \Phi^+ \setminus \Phi_p^+} \langle \lambda, \alpha^\vee \rangle \, dx_\alpha \wedge d\bar{x}_\alpha,$$

and hence we shall just write $c_1(L) = \lambda$. Let T_{X_p} be the tangent bundle of X_p. Then we find that

$$c_1(T_{X_p}) = \sum_{\alpha \in \Phi^+ \setminus \Phi_p^+} \alpha = c_p \lambda_p,$$

where c_p is the number of (2.2).

The index $\mathfrak{i}_p$ of X_p is defined by the identity $-K_{X_p} = \mathfrak{i}_p L_{\lambda_p}$ for the canonical bundle K_{X_p} and the (ample) line bundle L_{λ_p} associated with the fundamental weight. For the first Chern class of K_{X_p}, we have

$$c_1(K_{X_p}) = - \sum_{\alpha \in \Phi^+ \setminus \Phi_p^+} \alpha = -c_1(T_{X_p}) = -c_p \lambda_p.$$

This derives the identity

$$-K_{X_p} = c_p L_{\lambda_p} \quad (\mathfrak{i}_p = c_p).$$

Now the functional equation of the zeta function $\hat{\zeta}_p(s)$ associated with (Φ, Δ, p) can be written as

$$\hat{\zeta}_p(\langle c_1(K_{X_p}), \alpha_p^\vee \rangle - s) = \hat{\zeta}_p(s).$$

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