MINIMAL UNKNOTTING SEQUENCES OF REIDEMEISTER MOVES CONTAINING UNMATCHED RII MOVES

CHUICHIRO HAYASHI, MIWA HAYASHI, MINORI SAWADA AND SAYAKA YAMADA

Abstract. Arnold introduced invariants J^+, J^- and St for generic planar curves. It is known that both $J^+/2 + St$ and $J^-/2 + St$ are invariants for generic spherical curves. Applying these invariants to underlying curves of knot diagrams, we can obtain lower bounds for the number of Reidemeister moves for uknotting. $J^-/2 + St$ works well for unmatched RII moves. However, it works only by halves for RI moves. Let w denote the writhe for a knot diagram. We show that $J^-/2 + St \pm w/2$ works well also for RI moves, and demonstrate that it gives a precise estimation for a certain knot diagram of the unknot with the underlying curve $r = 2 + \cos(n\theta/(n + 1))$, $(0 \leq \theta \leq 2(n + 1)\pi)$.

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1. Introduction

In this paper, all the knots are assumed to be oriented. A *Reidemeister move* is a local move of a knot diagram as in Figure 1. An RI (resp. II) move creates or deletes a monogon face (resp. a bigon face). An RII move is called *matched* or *unmatched* with respect to the orientation of the knot as shown in Figure 2. An RIII move is performed on a 3-gon face, deleting it and creating a new one. Any such move does not change the knot type. As Alexander and Briggs [2] and Reidemeister [13] showed, for any pair of diagrams D_1, D_2 which represent the same knot type, there is a finite sequence of Reidemeister moves which deforms D_1 to D_2.

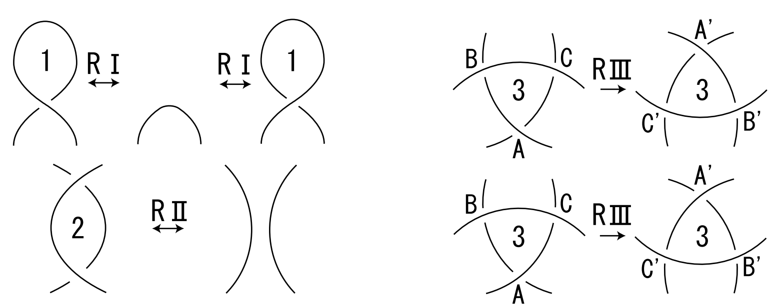

Figure 1.

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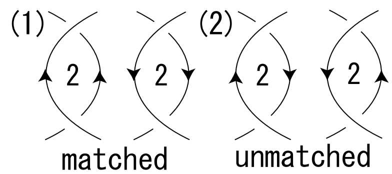

FIGURE 2.

Necessity of Reidemeister moves of type II and III is studied in [11], [10] and [5]. In [6], the knot diagram invariant cowrithe is introduced, and it gives a lower bound for the number of matched RII and RIII moves. In [4], Carter, Elhamdadi, Saito and Satoh gave a lower bound for the number of RIII moves by using extended n-colorings of knot diagrams in $\mathbb{R}^2$. Hass and Nowik introduced a certain knot diagram invariant by using smoothing and linking number in [8], and gave in [9] an example of an infinite sequence of diagrams of the trivial knot such that the n-th one has $7n-1$ crossings, can be unknotted by $2n^2+3n$ Reidemeister moves, and needs at least $2n^2+3n-2$ Reidemeister moves for being unknotted. Using cowrithe, it is shown in [7] that a certain sequence of Reidemeister moves bringing $D(n+1,n)$ to $D(n,n+1)$ is minimal, where $D(p,q)$ denotes the usual diagram of the (p,q)-torus knot. In the above papers [9] and [7], the sequences of Reidemeister moves do not contain unmatched RII moves. It is not easy to estimate the number of unmatched RII moves needed for unknotting. In this paper, we show that a certain unknotting sequence of Reidemeister moves containing unmatched RII moves is minimal, using the writhe and the Arnold invariants of the underlying spherical curve, the knot diagram with over-under informations at the crossings forgotten.

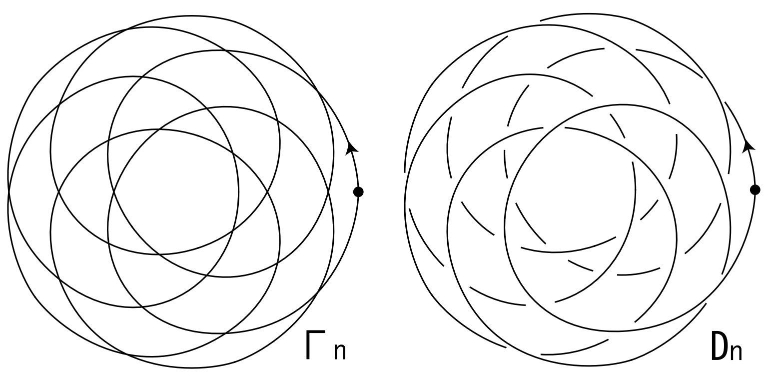

FIGURE 3.

Let n be an integer larger than or equal to 2. As the underlying spherical curve of a knot diagram, we consider Γ_n as shown in Figure 3, where $n=5$. We regard the 2-sphere S^2 as $\mathbb{R}^2 \cup \{\infty\}$. For an integer n larger than or equal to 2, Γ_n is given by the equation $r = 2+\cos(n\theta/(n+1))$, $(0 \le \theta \le 2(n+1)\pi)$ with respect to the polar coordinates (r,θ)

on the plane. The curve Γ_n has an n-gonal face at center, surrounded by a cycle of n trigonnal faces surrounded by $n-2$ cycles of n quadrilateral faces surrounded by a cycle of n trigonnal faces. The outermost region of Γ_n is an n-gonal face. We set the base point p to be $(r,\theta) = (3,0) = (3, 2(n+1)\pi)$, and give Γ_n an orientation in the direction of which θ increases. The knot diagram D_n is obtained from Γ_n by giving over-under informations at all double points so that they are ascending as below. Every crossing is composed of two subarcs of the knot. When we travel along the knot, staring at the base point and going in the direction of the orientation, we meet the first subarc and then the second one. In the diagram D_n the second subarc goes over the first one. Thus D_n represents the trivial knot.

This knot diagram D_n is also obtained from the usual diagram of the $(n+1,n)$-torus knot $T(n+1,n)$ by changing crossings so that D_n is ascending. The usual diagram of $T(n+1,n)$ is the closure of the $(n+1)$-braid $(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_n^{-1})^n$, while D_n is the closed braid of the $(n+1)$-braid below.

$$(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_n^{-1})(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_n)$$
$$(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}\sigma_n)\cdots(\sigma_1^{-1}\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}\sigma_n)$$

In this braid, the j-th strand goes over the i-th strand if $i < j$.

Theorem 1.1. *For any integer n larger than or equal to 3, the knot digram D_n of the trivial knot can be deformed to the trivial diagram with no crossing by a sequence of $n(n^2+5)/6$ Reidemeister moves, which consists of ${}_nC_1 = n$ RI moves deleting positive crossings, ${}_nC_2 = (n-1)n/2$ unmatched RII moves deleting bigons and ${}_nC_3 = (n-2)(n-1)n/6$ positive RIII moves. Moreover, any sequence of Reidemeister moves bringing D_n to the trivial diagram must contain at least $n(n^2+5)/6$ RI moves deleting positive crossings, unmatched RII move deleting bigons or positive RIII moves. Hence, the above sequence is minimal.*

To prove the above theorem, we use the knot diagram invariant $J^-/2 + St \pm w/2$, where J^- and St are the Arnold invariants for plane curves, and w is the writhe. We consider the changes of this invariant under Reidemeister moves in Section 2 after recalling the definitions of the Arnold invariants. Theorem 1.1 is proved in Section 3.

2. Knot diagram invariants

First, we recall the definition of the Arold invariants. A *plane curve* is a smooth immersion of the oriented circle S^1 to the plane $\mathbb{R}^2$. It is *generic* if it has only a finite number of multiple points, and they are transverse double points.

When a knot diagram in the plane is deformed by an RII move, a *self-tangency perestroika* occurs on the underlying plane curve. A self-tangency perestroika is called *positive* (resp. *negative*) if it creates (resp. *deletes*) a bigon face, and called *direct* (resp. *inverse*) if the corresponding RII move is matched (resp. unmatched). See Figure 4. When two knot diagrams are connected by an RIII move, then their underlying plane curves are connected

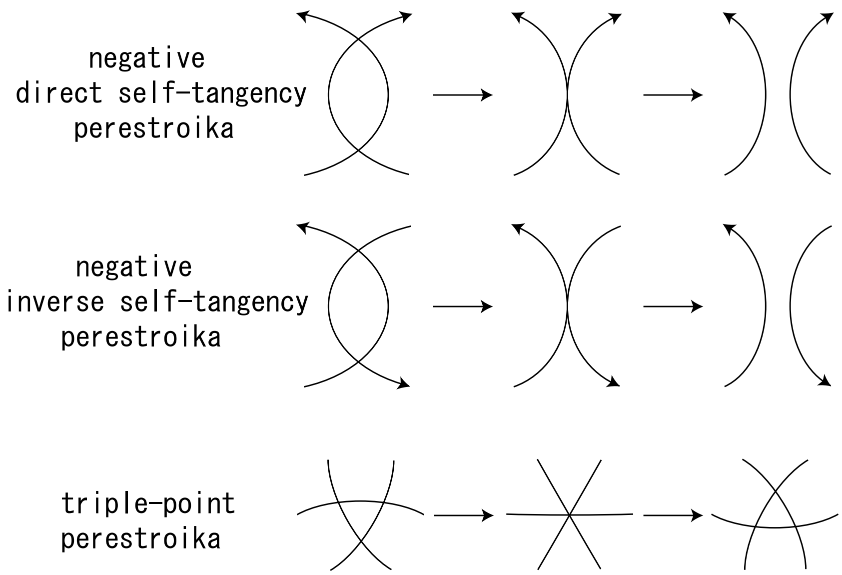

FIGURE 4.

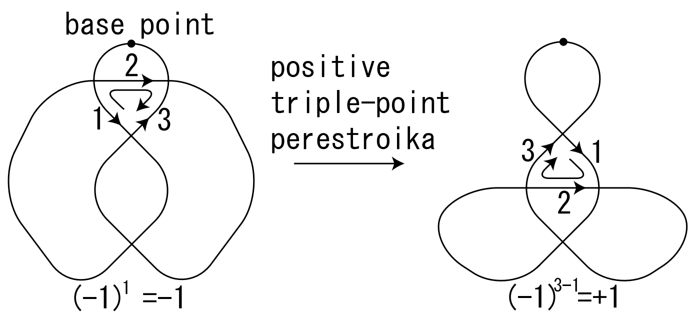

FIGURE 5.

by a *triple-point perestroika*. The *sign* of a triple-point perestroika is determined by the sign of the created triangle face of the plane curve after the perestroika. Let Δ be a triangle face of a plane curve Γ. We take a base point p on Γ so that it is disjoint from Δ. Let e_1, e_2, e_3 be the edges of Δ, where they are numbered so that they appear in this order when we go around Γ once from p to p in the direction of the orientation of Γ. We can orient the boundary circle $\partial\Delta$ so that we meet e_1, e_2 and e_3 in this order when we go around $\partial\Delta$ in the direction of its orientation. Let q be the number of the edges among e_1, e_2 and e_3 on which the orientation induced from Γ matches that from $\partial\Delta$. Then the *sign* of Δ is defined by $(-1)^q$. Note that changing the base point does not affect the sign of Δ. It can be easily seen that the triangle faces deleted and created by a triple-point perestroika have opposite signs. See Figure 5, where an example of a positive triple-point perestroika is described.

Arnold showed in [3] that there are invariants J^+, J^- and St for plane curves as below. See also [12], where Polyak gave formulae calculating J^+, J^- and St via Gauss diagrams.

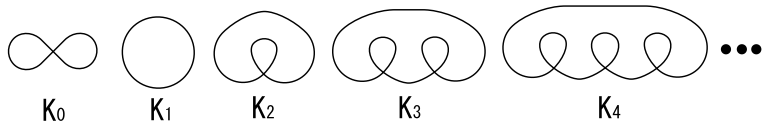

FIGURE 6.

Definition 2.1. (1) J^+, J^- and St are independent of the choice of orientation of a plane curve.

(2) J^+ does not change under an inverse self-tangency or triple-point perestroika but increases by 2 under a positive direct self-tangency perestroika.

(3) J^- does not change under a direct self-tangency or triple-point perestroika but decreases by 2 under a positive inverse self-tangency perestroika.

(4) St does not change under a self-tangency perestroika but increases by 1 under a positive triple-point perestroika.

(5) For the plane curves K_i, $i \in \mathbb{N} \cup \{0\}$ depicted in Figure 6,
$J^+(K_0) = 0,\ J^-(K_0) = -1,\ St(K_0) = 0,$
$J^+(K_{i+1}) = -2i,\ J^-(K_{i+1}) = -3i,\ St(K_{i+1}) = i$, where $i \geq 0$.

Note that K_i has Whitney index (or widing number) $+i$ or $-i$ according to a choice of orientation of K_i. (The Whitney index of a plane curve Γ is calculated as below. Smoothing (cutting and pasting) all the double points with respect to the orientation of Γ, we obtain disjoint union of oriented circles with no double points. Then the index is the number of circles oriented anti-clockwise minus the number of circles oriented clockwise.) Whitney showed in [14] that two plane curves are connected by a smooth homotopy if and only if they are of the same index. Two homotopic plane curves are connected by a sequence of self-tangency perestroikas and triple-point perestroikas.

Aicardi studied the invariant $J^+/2 + St$ in [1]. We regard the 2-sphere S^2 as $\mathbb{R}^2 \cup \{\infty\}$. Then $J^+/2 + St$ and also $J^-/2 + St$ give invariants for generic spherical curves. In fact, they does not depend on the choice of the point at infinity ∞ in $S^2 - \Gamma$, where Γ is a generic spherical curve. This fact is also implied by the formulae $J^+ = J^- + n$ and $J^+(\Gamma) + 2St(\Gamma) = -2 < B_4, G_\Gamma >$ in 4.3 in [12], where n denotes the number of double points and the term $< B_4, G_\Gamma >$ depends only on the Gass diagram G_Γ of the spherical curve Γ. We obtain $J^-(\Gamma)/2 + St(\Gamma) = - < B_4, G_\Gamma > -n/2$ from these formulae. (Note that $St(\Gamma)$ should be equal to $\frac{1}{2} < -B_2 + B_3 + B_4 > + \dfrac{n-1}{4} + \dfrac{ind(\Gamma)^2}{4}$ in Theorem 1 in [12].)

We consider changes of $J^+/2 + St$ and $J^-/2 + St$ under a cusp perestroika as shown in Figure 7, which can be lifted to an RI move on a knot diagram. The next proposition is probably well-known. It follows easily from Definition 2.1 and the above formulae in [12]. See also Proposition 3 in [8].

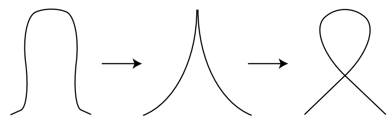

FIGURE 7.

Proposition 2.2. (well-known)

(1) *$J^+/2 + St$ does not change under a cusp or inverse self-tangency perestroika, but increases by 1 under a positive direct self-tangency perestroika or positive triple-point perestroika.*

(2) *$J^-/2 + St$ does not change under a direct self-tangency perestroika, but decreases by $1/2$ under a cusp perestroika creating a monogon, by 1 under a positive inverse self-tangency perestroika or negative triple-point perestroika.*

We can obtain lower bounds for the minimal number of Reidemeister moves connecting two knot diagrams in S^2 representing the same knot by calculating these invariants of underlying spherical curves of the knot diagrams. In fact, as Hass and Nowik showed in Section 4 in [8], the cowrithe of a knot diagram is equal to $-\{J^+/2 + St - 4c_2\}$, where c_2 is the coefficient of x^2 of the Conway polynomial of the knot. Hence the estimation of the number of Reidemeister moves by the cowrithe coincides with that by $J^+/2 + St$.

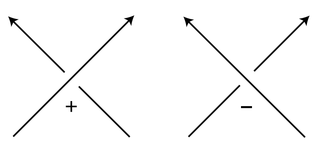

FIGURE 8.

The invariant $J^-/2 + St$ is sensitive to inverse self-tangency perestroikas, and hence to unmatched RII moves. However, it reacts by halves to cusp moves (or RI moves). Hence we consider $J^-(\bar{D})/2 + St(\bar{D}) \pm w(D)/2$, where D is a knot diagram in S^2, $\bar{D}$ the underlying spherical curve, and $w(D)$ the writhe of D. The *writhe* of a knot diagram D is the sum of signs of all the crossings of D, where the sign of a crossing is defined as shown in Figure 8.

We call an RIII move *positive* (resp. *negative*) if it causes a positive (resp. negative) triple-point perestroika on the underlying spherical curve.

The next theorem follows easily from Proposition 2.2 since the writhe does not change under an RII or RIII move and increases (resp. decreases) by 1 under an RI move creating a positive (resp. negative) crossing.

Theorem 2.3. *$J^-/2+St+w/2$ (resp. $J^-/2+St-w/2$) does not change under an RI move creating a positive (resp. negative) crossing or matched RII move, but decreases by 1 under* *an RI move creating a negative (resp. positive) crossing, unmatched RII move creating a bigon face or negative RIII move.*

The two formulae $J^+ = J^- + n$ in Section 4.3 in [12], and $x = 4c_2 - (J^+/2 + St)$ in Section 4 in [8] together imply $x + n/2 \mp w/2 = 4c_2 - (J^-/2 + St \pm w/2)$, where x is the cowrithe. Hence we obtain the next corollary.

Corollary 2.4. *$x + n/2 - w/2$ (resp. $x + n/2 + w/2$) does not change under an RI move creating a positive (resp. negative) crossing or matched RII move, but increases by 1 under an RI move creating a negative (resp. positive) crossing, unmatched RII move creating a bigon face or negative RIII move.*

Note that $n/2 + w/2$ (resp. $n/2 - w/2$) does not change under an RI move creating a negative (resp. positive) crossing, but increases by 1 under an RI move creating a positive (resp. negative) crossing.

3. Minimal sequence of Reidemeister moves

We prove Theorem 1.1 in this section. In the course of the proof, we obtain the next proposition. A knot diagram of the unknot rarely has the cowrithe with positive value. In fact, any knot diagram of the unknot with 8 or less number of crossings has negative cowrithe. Note that $c_2(D_n) = 0$ since D_n represents the unknot.

Proposition 3.1.
$J^-(\bar{D_n})/2 + St(\bar{D_n}) - w(D_n)/2 = -({}_nC_1 + {}_nC_2 + {}_nC_3) = -n(n^2+5)/6$
$J^+(\bar{D_n})/2 + St(\bar{D_n}) = -x(D_n) = -{}_nC_3 = -(n-2)(n-1)n/6$

Proof of Theorem 1.1. We first sketch the proof very roughly. The trivial knot diagram is the unit circle S^1 in $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$, and it is the union of n arcs $\gamma_1, \gamma_2, \cdots, \gamma_n$, where γ_i is given by the equation below.

$$r = 1, \ \ (2(i-1)\pi/n \leq \theta \leq 2i\pi/n)$$

We apply RI moves creating a positive crossing ${}_nC_1 = n$ times to the trivial knot diagram so that each subarc γ_i is deformed into a kink λ_i with a positive crossing and a small monogon, and so that the circle is deformed into a knot diagram with the curve K_n in Figure 6 being the underlying spherical curve. Let I be a subset of $\{1, 2, \cdots, n\}$, and $K(I)$ the knot digram obtained from the circle S^1 by replacing γ_i by λ_i for all $i \in I$. We perform ${}_nC_2 = (n-1)n/2$ unmatched RII moves creating a bigon and ${}_nC_3 = (n-2)(n-1)n/6$ negative RIII moves so that the monogons of the kinks are enlarged, that λ_j goes over λ_i if $i < j$, that $K(\{i, j\})$ is deformed to a diagram equivalent to D_2 for every pair of two distinct numbers i, j in $\{1, 2, \cdots, n\}$, and that $K(\{i, j, k\})$ is deformed to a diagram equivalent to D_3 for every triple of three distinct numbers i, j, k in $\{1, 2, \cdots, n\}$. Then the resulting knot diagram is D_n. This deformation and Theorem 2.3 show that $J^{-}(\bar{D_n})/2 + St(\bar{D_n}) - w(D_n)/2 = -{}_nC_1 - {}_nC_2 - {}_nC_3$, and the threorem follows.

Now we describe the proof of the theorem in detail. Let μ_i be the subarc of the diagram D_n given by the formula below.

$$r = 2 + \cos(n\theta/(n+1)), \quad (2(i-1)(n+1)\pi/n \leq \theta \leq 2i(n+1)\pi/n)$$

Each arc λ_i is going to be deformed to μ_i. For a subset I of $\{1, 2, \cdots, n\}$, let $D(I)$ denote the knot digram obtained from the circle S^1 by replacing γ_i by μ_i for all $i \in I$.

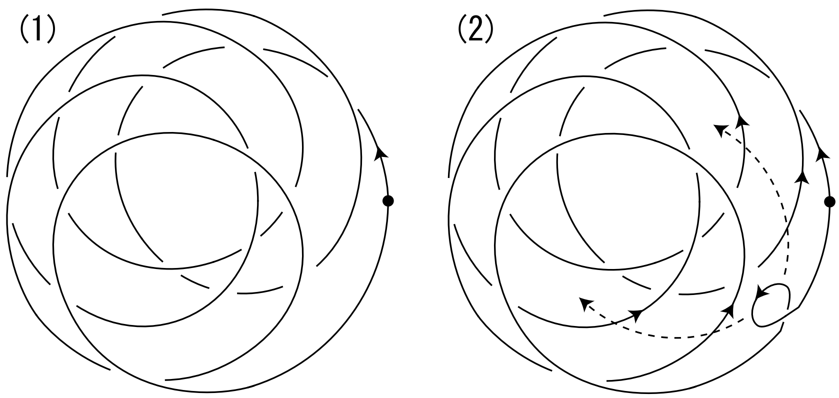

FIGURE 9.

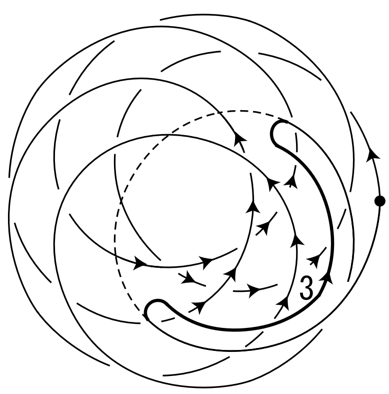

FIGURE 10.

The theorem is proved by an induction on n. In the case of D_3, the theorem can be easily confirmed. We assume that the theorem holds for D_{n-1} and consider the case of D_n. Note that the diagram $D(\{1, 2, \cdots, n-1\})$ is equivalent to D_{n-1}. See Figure 9(1). We begin with $D(\{1, 2, \cdots, n-1\})$, and deform the subarc γ_n to obtain the diagram D_n. First, we apply an RI move to γ_n to create the kink λ_n with a positive crossing. See Figure 9(2). We enlarge the monogon bounded by λ_n. Let λ_n keep on denoting the subarc of the knot

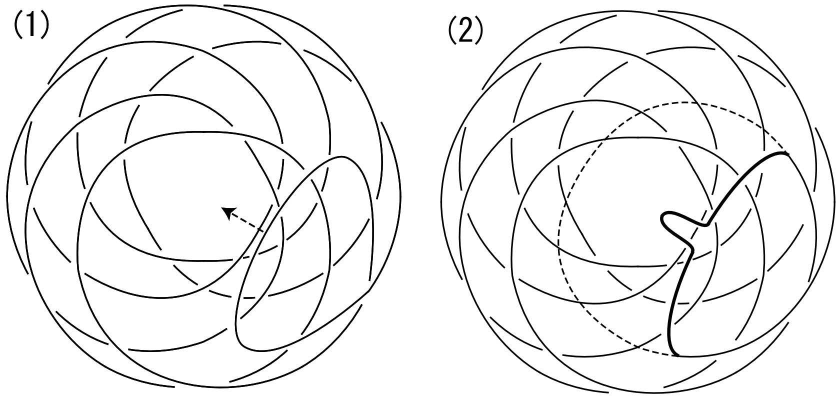

FIGURE 11.

diagram obtained from λ_n by the deformation below. We denote by $R(i)$ the RII move between the arc μ_i and λ_n, and by $R(i,j)$ the RIII move on the arcs μ_i, μ_j and λ_n. The first enlargement of the monogon bounded by λ_n is done by the sequence of RII moves $R(1), R(2), \cdots, R(k)$ and $R(n-1), R(n-2), \cdots, R(\ell)$, where $k = (n-1)/2$ and $\ell = k+1$ when n is odd, and $k = (n-2)/2$ and $\ell = k+2$ when n is even. See Figure 9(2) and Figure 10. These RII moves are performed along the arcs parallel to μ_1 and μ_{n-1} as shown in Figure 9(2). Then, as in Figure 10, we deform the arc drawed in a bold line to that in a broken line. Precisely, we first perform RIII moves $R(1, n-1)$ along subarcs of μ_1 and μ_{n-1}, $R(1, n-2)$, $R(2, n-1)$, $R(2, n-2)$ along subarcs of μ_2 and μ_{n-2}, $R(1, n-3)$, $R(3, n-1)$, $R(2, n-3)$, $R(3, n-2)$, $R(3, n-3)$ along subarcs of μ_3 and μ_{n-3}, $R(1, n-4)$, $R(4, n-1)$, $R(2, n-4)$, $R(4, n-2)$, $R(3, n-4)$, $R(4, n-3)$, $R(4, n-4)$ along subarcs of μ_4 and μ_{n-4}, $\cdots$, $R(1, n-k)$, $R(k, n-1)$, $R(2, n-k)$, $R(k, n-2)$, $\cdots$, $R(k-1, n-k)$, $R(k, n-(k-1))$, $R(k, n-k)$ along subarcs of μ_k and μ_{n-k}. See Figure 11(1).

When n is odd, we further perform RIII moves $R(k-1, k)$, $R(k-2, k)$, $\cdots$, $R(1, k)$ along a subarc of μ_k, $R(k+1, k+2)$, $R(k+1, k+3)$, $\cdots$, $R(k+1, n-1)$ along a subarc of μ_{k+1}, $R(k-2, k-1)$, $R(k-3, k-1)$, $\cdots$, $R(1, k-1)$ along a subarc of μ_{k-1}, $R(k+2, k+3)$, $R(k+2, k+4)$, $\cdots$, $R(k+2, n-1)$ along a subarc of μ_{k+2}, $\cdots$, $R(1, 2)$ along a subarc of μ_2, $R(n-2, n-1)$ along a subarc of μ_{n-2}. Thus D_n is obtained.

When n is even, we do the RII move $R(k+1)$. See Figure 11. Then, we apply RIII moves $R(k, k+1)$, $R(k-1, k+1)$, $\cdots$, $R(1, k+1)$ along a subarc of μ_{k+1}, $R(k+1, k+2)$, $R(k+1, k+3)$, $\cdots$, $R(k+1, n-1)$ along a subarc of μ_{k+1}, $R(k-1, k)$, $R(k-2, k)$, $\cdots$, $R(1, k)$ along a subarc of μ_k, $R(k+2, k+3)$, $R(k+2, k+4)$, $\cdots$, $R(k+2, n-1)$ along a subarc of μ_{k+2}, $R(k-2, k-1)$, $R(k-3, k-1)$, $\cdots$, $R(1, k-1)$ along a subarc of μ_{k-1}, $R(k+3, k+4)$, $R(k+3, k+5)$, $\cdots$, $R(k+3, n-1)$ along a subarc of μ_{k+3}, $\cdots$, $R(1, 2)$ along a subarc of μ_2, $R(n-2, n-1)$ along a subarc of μ_{n-2}. Thus we obtain D_n.

In both cases, we have performed a single RI move, ${}_{n-1}C_1 = n-1$ RII moves and ${}_{n-1}C_2 = (n-2)(n-1)/2$ RIII moves to deform γ_n to μ_n. (In fact, the RII move $R(i)$ has been performed for every integer i with $1 \leq i \leq n-1$, and the RIII move $R(i,j)$ has been performed for every pair of integer i, j with $1 \leq i < j \leq n-1$.) We do ${}_{n-1}C_m$ Reidemeister moves of the m-th type to obtain D_{n-1}. Hence the formula ${}_{n-1}C_m + {}_{n-1}C_{m-1} = {}_nC_m$ implies the theorem. □

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All authors: Department of Mathematical and Physical Sciences, Faculty of Science, Japan Women's University, 2-8-1 Mejirodai, Bunkyo-ku, Tokyo, 112-8681, Japan.

Chuichiro Hayashi: hayashic@fc.jwu.ac.jp
Miwa Hayashi: miwakura@fc.jwu.ac.jp
Minori Sawada: m0716060sm@ug.jwu.ac.jp
Sayaka Yamada: m0716131ys@ug.jwu.ac.jp