On a coloured tree with non i.i.d. random labels

Skevi Michael* and Stanislav Volkov*†

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Abstract

We obtain new results for the probabilistic model introduced in [3] and [4] which involves a d-ary regular tree. All vertices are coloured in one of d distinct colours so that d children of each vertex all have different colours. Fix d^2 strictly positive random variables. For any two connected vertices of the tree assign to the edge between them *a label* which has the same distribution as one of these random variables, such that the distribution is determined solely by the colours of its endpoints. *A value* of a vertex is defined as a product of all labels on the path connecting the vertex to the root. We study how the total number of vertices with value of at least x grows as $x \downarrow 0$, and apply the results to some other relevant models.

1 Introduction

In [4] Volkov showed how the $5x+1$ problem can be approximated by a probabilistic model involving a binary tree with randomly labeled edges, with distributions of the random variables assigned to edges being determined by their directions, these random variables being independent.

Menshikov et al. [3] studied a similar model, where random variables assigned to edges of the tree were dependent both on the type of parent vertex and the type of the child, as described below. At the same time, the results in [3] did not give the answers to all the questions answered in [4], and this is the purpose of the current paper. We want to stress that answering these questions is not a straightforward application of the previous results, but requires some new additional arguments.

Let $d \geq 2$. We consider the d-ary regular rooted tree T_d with vertex set $\mathbb{V}$ (that is, the tree where every vertex has degree $d+1$ with the exception of the root, $u_0 \in \mathbb{V}$, which has degree d). For the vertices $u, w \in \mathbb{V}$, the following quantities are defined:

- $\ell(u)$ is the unique self-avoiding path connecting u to the root;
- $|u|$ is the number of edges in $\ell(u)$;
- $\mathbb{V}_n = \{u \in \mathbb{V} : \ |u| = n\}$ is the set of d^n vertices that lie at graph-theoretical distance n from the root;

*Department of Mathematics, University of Bristol, BS8 1TW, U.K.
†Corresponding author. E-mail: s.volkov@bristol.ac.uk

- $u \sim w$ means that u and w are connected by an edge.

Among d distinct colours we arbitrarily choose one to colour the root. All other vertices are coloured from left to right, so that all d children of each vertex have different colours. We denote by $c(u) \in \{1, 2, .., d\}$ the colour assigned to the vertex u.

Now we assign a random variable (label) to each edge as follows. First, consider d^2 strictly positive and non-degenerate random variables, $\tilde{\xi}_{ij}$, with $i, j \in \{1, 2, ...d\}$, of known joint distribution. Now for $u, w \in \mathbb{V}$ such that $u \sim w$ we assign the random variable, ξ_{uw} to the undirected edge $(u, w) \equiv (w, u)$, so that:

- for every edge (u, w) such that u is the parent of w, $\xi_{uw} \overset{\mathcal{D}}{=} \tilde{\xi}_{c(u)c(w)}$ where $X \overset{\mathcal{D}}{=} Y$ means that X and Y have the same distribution, and
- for any collection of edges of the tree $(u_1, w_1), (u_2, w_2), ..., (u_m, w_m)$, where u_i is the parent of w_i $\forall i \in 1, 2, ...m$ and $u_i \neq u_j$ whenever $i \neq j$, the random variables $\{\xi_{u_i w_i}\}_{i=1}^{m}$ are independent.

For $u \in \mathbb{V}$, we define value $\xi[u]$ to be the product of all the random variables assigned to the edges of $\ell(u)$. The main object of interest in the present paper is

$$Z(x) := \text{card}\{\text{u} \in \mathbb{V} : \xi[\text{u}] \geq \text{x}\}.$$

In [4] the ultimate object of interest was the complimentary quantity $Q(x) = \text{card}\{\text{u} \in \mathbb{V} : \xi[\text{u}] \leq \text{x}\}$, however, one can easily see that these two problems are equivalent once we replace $\tilde{\xi}_{ij}$ and x by its inverses $(\tilde{\xi}_{ij})^{-1}$ and x^{-1} respectively; we have chosen to study Z here in order to be consistent with notations in [3].

Similar to [3], we will randomize the colouring to avoid the disadvantage of the above colouring method, consisting in the fact that for different $u, w \in \mathbb{V}_n$ the distribution of $\xi[u]$ may differ from that of $\xi[w]$. In order to achieve equality of the distributions of $\xi[u]$ for all $u \in \mathbb{V}_n$, let the colouring be done recursively for $n = 1, 2, \ldots$ as follows. We first colour the root in any of the possible d colours; next, assuming that the vertices up to level $n-1$ (i.e., the vertices that belong in $\mathbb{V}_1, \mathbb{V}_2, \ldots, \mathbb{V}_{n-1}$) are already coloured, independently for each $v \in \mathbb{V}_{n-1}$ we colour each of its children in some colour so that no two children have the same colour, with all $d!$ colourings of the children of v being equally likely. As a result, each one of the $(d!)^{d^{n-1}}$ possible colourings of V_n has the same probability.

As before, to each edge (u, w) we assign a random variable ζ_{uw}, which distribution satisfies the conditions imposed on ξ_{uw}. Define $\zeta[u]$ in the same way as $\xi[u]$; then it is clear that at every level n the distribution of the *unordered* set $\{\zeta[u], u \in \mathbb{V}_n\}$ is the same as the distribution of $\{\xi[u], u \in \mathbb{V}_n\}$. This means that the two models will give the same results for a number of problems, while the randomized colouring ensures that for any $u, w \in \mathbb{V}_n$ $\zeta[u] \overset{\mathcal{D}}{=} \zeta[w]$, even though $\zeta[u]$ and $\zeta[w]$ could be dependent. In particular, $Z(x) = \text{card}\{\text{u} \in \mathbb{V} : \zeta[\text{u}] \geq \text{x}\}$.

2 Results from [3]

Let probability $\mathbb{P}$ and expectation $\mathbb{E}$ be with respect to the measure generated *both* by a random colouring $\mathbf{c} = \{c(u), u \in \mathbb{V}\}$ and a random environment $\zeta = \{\zeta_{uw},\ u, w \in$

$\mathbb{V}$ such that $u \sim w\}$. Define the $d \times d$ matrix $m(s)$, $s \in [0, \infty)$, as

$$m(s) := \begin{pmatrix} \mathbb{E}[\tilde{\xi}_{11}]^s & \mathbb{E}[\tilde{\xi}_{12}]^s & \dots & \mathbb{E}[\tilde{\xi}_{1d}]^s \\ \mathbb{E}[\tilde{\xi}_{21}]^s & \mathbb{E}[\tilde{\xi}_{22}]^s & \dots & \mathbb{E}[\tilde{\xi}_{2d}]^s \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[\tilde{\xi}_{d1}]^s & \mathbb{E}[\tilde{\xi}_{d2}]^s & \dots & \mathbb{E}[\tilde{\xi}_{dd}]^s \end{pmatrix}.$$

Let $\rho(s)$ be its largest eigenvalue, then $\rho(s)$ is positive by Perron-Frobenius theorem for matrices with strictly positive entries.

Let $\mathbb{D} = \left\{ s \in \mathbb{R} : \mathbb{E}[\tilde{\xi}_{ij}]^s < \infty \quad \forall \quad i, j \in \{1, 2, ..., d\} \right\}$ and Int($\mathbb{D}$) be its interior. Assume that the conditions below are satisfied:

$$\begin{array}{rcll} [0,1] & \subseteq & \mathbb{D}, & \\ 0 & \in & \mathrm{Int}(\mathbb{D}), & \\ \mathbb{E}|\log \tilde{\xi}_{ij}| & < & \infty & \forall \quad i, j \in \{1, 2, ..., d\}, \\ \mathbb{E}|\tilde{\xi}_{ij} \log \tilde{\xi}_{ij}| & < & \infty & \forall \quad i, j \in \{1, 2, ..., d\}. \end{array} \tag{1}$$

Theorem 1 (Theorem 2 in [3]) *Suppose $x > 0$,*

$$\lambda = \inf_{s \geq 0} \rho(s)$$

and conditions (1) are fulfilled. Then

(a) if $\lambda < 1$, then $Z(x) < \infty$ a.s.;

(b) if $\lambda > 1$, then $Z(x) = \infty$ a.s.

For a vertex $u \in \mathbb{V}_n$, let $u_0, u_1, \dots, u_{n-1}, u_n \equiv u$ be the consecutive vertices of the path $\ell(u)$. The proof of the above theorem is largely based on the following statement from [2].

Lemma 1 (Lemma 1 in [3]) *Let $S_n = \sum_{i=1}^n \log(\zeta_{u_{i-1}u_i})$ and $k_n(s) = \left(\mathbb{E}[e^{sS_n}]\right)^{1/n} = \left(\mathbb{E}\left[\prod_{i=1}^n \zeta^s_{u_{i-1}u_i}\right]\right)^{1/n}$. Suppose (1) is fulfilled. Then*

(a) $k(s) = \lim_{n\to\infty} k_n(s) \in [0, \infty]$ exists for all s;

(b) $\Lambda(s) = \log \rho(s) - \log d = \log k(s) \in (-\infty, +\infty]$ is convex;

(c) the rate function $\Lambda^*(z) = \sup_{s\geq 0}(sz - \Lambda(s))$, $z \in \mathbb{R}$, is convex, lower semi-continuous and differentiable in Int($\mathbb{D}$). Moreover,

$$\Lambda^*(z) = \begin{cases} s_0(z)z - \Lambda(s_0(z)), & \text{if } z \geq \Lambda'(0), \\ 0, & \text{if } z \leq \Lambda'(0), \end{cases}$$

where $s_0(z)$ is the solution of equation $z - \Lambda'(s) = 0$;

(d) for all $a > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{S_n}{n} \geq \log a\right) = -\Lambda^*(\log a).$$

3 Expectation of $Z(x)$

Here we will need one additional assumption:

$$\mathbb{E}\left[\tilde{\xi}_{ij}\right]^s \in \mathbf{C}^2(\mathbb{R}_+) \quad \forall \quad i,j \in \{1,2,...,d\} \tag{2}$$

as functions of s, which is required to ensure that $\Lambda \in \mathbf{C}^2(\mathbb{R}_+)$. Indeed, the characteristic polynomial $P(s,\lambda) = \det(m(s) - \lambda I)$ of $m(s)$ can be written as

$$P(s,\lambda) = \sum_{k=0}^{d} a_k(s)\lambda^k.$$

where $a_k(s) \in \mathbf{C}^2(\mathbb{R}_+)$, $k = 0,1,...,d$, are its coefficients and I is $d \times d$ identity matrix. By the Perron-Frobenius theorem, $\rho(s)$ is a *simple* root of this polynomial, hence it is *not* a root of the polynomial $\frac{\partial P(s,\lambda)}{\partial \lambda} = 0$. Hence

$$\left.\frac{\partial P(s,\lambda)}{\partial \lambda}\right|_{\lambda=\rho(s)} \neq 0$$

and by the implicit function theorem we obtain that $\rho(s)$ is continuously differentiable in s as $a_i(s)$ are, i.e. $\rho(s) \in \mathbf{C}^2(\mathbb{R}_+)$ and therefore $\Lambda \in \mathbf{C}^2(\mathbb{R}_+)$.

Suppose conditions (1) are fulfilled. By Theorem 1 if $\lambda < 1$ then $Z(x) < \infty$ a.s. Also, since

$$\Lambda^*(z) = \sup_{s\geq 0}(sz - \Lambda(s)) = \sup_{s\geq 0}(sz - \log\rho(s) + \log d)$$

we have

$$\Lambda^*(0) = \sup_{s\geq 0}(-\log\rho(s) + \log d) = -\log\left(\inf_{s\geq 0}\rho(s)\right) + \log d = -\log\lambda + \log d.$$

Therefore,

$$\lambda < 1 \iff \Lambda^*(0) > \log d. \tag{3}$$

From now on assume that indeed $\lambda < 1$ and hence $Z(x)$ is a.s. finite for all $x > 0$. Observe that $Z(x)$ increases to $+\infty$ as $x \downarrow 0$. We are now ready to give the main theorem describing the asymptotical behaviour of $\mathbb{E}[Z(x)]$, thus generalizing the result of Theorem 3 in [4] to a more general setup of [3] described above.

Theorem 2 *Suppose that conditions (1) and (2) are fulfilled, and moreover the following are true:*

(A1) $\lambda < 1$;

(A2) $\mu := -\Lambda'(0) > 0$ *(equivalently,* $\rho'(0) < 0$*).*

Then

$$\lim_{t\to\infty}\frac{\log\mathbb{E}\left[Z(e^{-t})\right]}{t} \text{ exists and is given by } M = \max_{u\in[0,\mu]}\frac{\log d - \Lambda^*(-u)}{u}.$$

Proof. Let
$$f(u) = \frac{\log d - \Lambda^*(-u)}{u}.$$
By the definition of the rate function $\Lambda^*(z) \geq 0$ for all $z \in \mathbb{R}$, and also $\Lambda^*(-\mu) = 0$. Since Λ^* is a differentiable and convex function we have $\Lambda^{*\prime}(-\mu) \equiv \frac{d\Lambda^*(z)}{dz}\Big|_{z=-\mu} = 0$. Also

$$\begin{aligned} \lim_{u \to +0} f(u) &= -\infty \quad \text{(because of A1 and (3));} \\ f(\mu) &= \frac{\log d - \Lambda^*(-\mu)}{\mu} = \frac{\log d}{\mu} > 0 \quad \text{(because of A2);} \\ f'(\mu) &= \frac{\mu \cdot \Lambda^{*\prime}(-\mu) - \log d + \Lambda^*(-\mu)}{\mu^2} = -\frac{\log d}{\mu^2} < 0. \end{aligned}$$

We conclude that $\max_{x \in [0, -\Lambda'(0)]} f(x)$ exists and is achieved strictly inside the interval $(0, \mu)$. Let $u^* \in (0, \mu)$ denote the point where the maximum of $f(u)$ is achieved.

Keeping in mind that $\zeta[\cdot]$ is the same for the vertices which appear at the same level of the tree, we derive an expression for $\mathbb{E}[Z(e^{-t})]$ similar to [4]:

$$\begin{aligned} \mathbb{E}[Z(e^{-t})] &= \sum_{u \in \mathbb{V}} \mathbb{P}\left(\zeta[u] \geq e^{-t}\right) = \sum_{n=0}^{\infty} \sum_{u \in \mathbb{V}_n} \mathbb{P}(\zeta[u] \geq e^{-t}) = \sum_{n=0}^{\infty} \sum_{u \in \mathbb{V}_n} \mathbb{P}\left(\log \zeta[u] \geq -t\right) \\ &= \sum_{n=0}^{\infty} d^n \cdot \mathbb{P}\left(\log\left(\prod_{i=1}^{n} \zeta_{u_{i-1}u_i}\right) \geq -t\right) = \sum_{n=0}^{\infty} d^n \cdot \mathbb{P}\left(S_n \geq -t\right), \end{aligned}$$

where
$$S_n = \sum_{i=1}^{n} \log\left(\zeta_{u_{i-1}u_i}\right).$$

Hence
$$\mathbb{E}[Z(e^{-t})] = \sum_{n=0}^{\infty} \exp\left\{n \log d + \log \mathbb{P}\left(S_n \geq -t\right)\right\} = \sum_{n=0}^{\infty} e^{tU_n}$$

where
$$U_n = \frac{\log d + \frac{1}{n} \log \mathbb{P}\left(S_n/n \geq -t/n\right)}{t/n}.$$

First we get the upper bound for $\mathbb{E}[Z(e^{-t})]$.

By Lemma 1 Λ^* is a continuous function and $\Lambda^*(0) > \log d$, therefore, there are $\epsilon \in (0, \mu)$ and $\bar{\delta} > 0$ such that for all $\delta \in (0, \bar{\delta})$ we have $\Lambda^*(-\epsilon) > \log d + 2\delta$. In turn, by part (d) of Lemma 1 there is an $n_0 = n_0(\epsilon, \delta) \in \mathbb{N}$ such that for all $n \geq n_0$

$$\frac{1}{n} \log \mathbb{P}\left(S_n/n \geq -\epsilon\right) \leq -\Lambda^*(-\epsilon) + \delta \leq -(\log d + \delta). \tag{4}$$

On the other hand, when $n \geq t/\epsilon$

$$\mathbb{P}\left(S_n/n \geq -\epsilon\right) \geq \mathbb{P}\left(S_n/n \geq -t/n\right). \tag{5}$$

Plugging the inequalities (4) and (5) into the expression for U_n for $n \geq \max\{n_0, t/\epsilon\}$ we obtain $U_n \leq -\frac{n\delta}{t}$. Assume that t is sufficiently large. Then $t/\epsilon > n_0$ yielding

$$\sum_{n=\lfloor \frac{t}{\epsilon} \rfloor + 1}^{\infty} e^{tU_n} \leq \sum_{n=0}^{\infty} e^{-n\delta} = \frac{1}{1 - e^{-\delta}}. \tag{6}$$

Secondly,

$$\sum_{n=0}^{\lfloor \frac{t}{\mu} \rfloor} d^n \cdot \mathbb{P}\left(S_n \geq -t\right) \leq \sum_{n=0}^{\lfloor \frac{t}{\mu} \rfloor} d^n \leq \left(\left\lfloor \frac{t}{\mu} \right\rfloor + 1 \right) e^{\frac{t \log d}{\mu}} \leq \left(\frac{t}{\mu} + 1 \right) e^{tM} \tag{7}$$

since $\frac{\log d}{\mu} = f(\mu) \leq M$.

To complete the first part of the proof for the upper bound, we need to study the case when

$$n \in \left[\frac{t}{\mu}, \frac{t}{\epsilon} \right] \iff \frac{t}{n} \in [\epsilon, \mu]. \tag{8}$$

The proof of the following statement is deferred until Section 4.3.

Proposition 1 *Let $a_1, a_2 \in \mathbb{R}$ be such that $a_1 < a_2$. Then for any $\delta > 0$ there is an $n_1 = n_1(a_1, a_2, \delta)$ such that*

$$\frac{1}{n} \log \mathbb{P}\left(\frac{S_n}{n} \geq a \right) \leq -\Lambda^*(a) + \delta \quad \textit{for all } a \in [a_1, a_2] \textit{ and } n \geq n_1.$$

Set

$$a = -\frac{t}{n}, \quad a_1 = -\mu, \quad a_2 = -\epsilon.$$

Note that (8) implies $a \in [a_1, a_2]$, hence the conditions of Proposition 1 are fulfilled, as long as t is large enough, namely $t > \mu n_1$. Consequently,

$$\frac{1}{n} \log \mathbb{P}\left(S_n/n \geq -t/n\right) \leq -\Lambda^*\left(-t/n\right) + \delta$$

yielding

$$U_n \leq \frac{\log d - \Lambda^*\left(-t/n\right) + \delta}{t/n} \leq f(t/n) + \frac{n\delta}{t} \leq M + \delta/\epsilon$$

since t/n satisfies (8). As a result

$$\sum_{n=\lfloor \frac{t}{\mu} \rfloor + 1}^{\lfloor \frac{t}{\epsilon} \rfloor} e^{tU_n} \leq \frac{t}{\epsilon} \cdot e^{t(M + \delta/\epsilon)}. \tag{9}$$

Consequently, combining (6), (7) and (9) together for t sufficiently large we can obtain the upper bound as follows:

$$\mathbb{E}[Z(e^{-t})] = \sum_{n=0}^{\lfloor \frac{t}{\mu} \rfloor} e^{tU_n} + \sum_{n=\lfloor \frac{t}{\mu} \rfloor+1}^{\lfloor \frac{t}{\epsilon} \rfloor} e^{tU_n} + \sum_{n=\lfloor \frac{t}{\epsilon} \rfloor+1}^{\infty} e^{tU_n}$$

$$\leq \left(\frac{t}{\mu}+1\right) e^{tM} + \left(\frac{t}{\epsilon}\right) e^{t(M+\delta/\epsilon)} + \frac{1}{1-e^{-\delta}}$$

$$= C(t,\epsilon,\mu,\delta,M)\,\epsilon^{-1}\,te^{t(M+\delta/\epsilon)} \tag{10}$$

where

$$\lim_{t\to\infty} C(t,\epsilon,\mu,\delta,M) = 1$$

for all $\delta > 0$. Taking the logarithm of (10) we obtain

$$\limsup_{t\to\infty} \frac{\log\left(\mathbb{E}[Z(e^{-t})]\right)}{t} \leq M + \delta/\epsilon$$

Thus by letting $\delta \to 0$ we have

$$\limsup_{t\to\infty} \frac{\log\left(\mathbb{E}[Z(e^{-t})]\right)}{t} \leq M. \tag{11}$$

Now, we obtain the lower bound for $\mathbb{E}[Z(e^{-t})]$. Recall that u^* is the value such that $f(u^*) = M$. Fix a small $\delta > 0$. By part (d) of Lemma 1 there is $n_2 = n_2(\delta)$ such that for all $n \geq n_2$

$$\frac{1}{n}\log\mathbb{P}\left(\frac{S_n}{n} \geq -u^*\right) \geq -\Lambda^*(-u^*) - \delta. \tag{12}$$

For any $t > n_2u^*$ define $n^* = n^*(t) = \lfloor t/u^* \rfloor \geq n_2$. Then $t/n^* \geq u^*$, moreover $t/n^* = u^*[1+O(1/t)]$. Therefore, using (12) we obtain

$$\begin{aligned} U_{n^*} &\geq \frac{\log d + \frac{1}{n^*}\log\mathbb{P}\left(S_{n^*}/n^* \geq -u^*\right)}{t/n^*} \geq \frac{\log d - \Lambda^*(-u^*) - \delta}{u^*[1+O(1/t)]} \\ &= M - \delta/u^* + O(1/t). \end{aligned}$$

Recalling

$$\mathbb{E}[Z(e^{-t})] = \sum_{n=0}^{\infty} e^{tU_n} \geq e^{tU_{n*}}$$

we obtain

$$\liminf_{t\to\infty} \frac{\log\mathbb{E}[Z(e^{-t})]}{t} \geq M - \delta/u^*.$$

Since $\delta > 0$ is arbitrary, this yields $\liminf_{t\to\infty} \frac{\log\mathbb{E}[Z(e^{-t})]}{t} \geq M$ which, together with (11), concludes the proof. ■

In fact, the result of Theorem 2 can be rewritten in a somewhat simpler form.

Corollary 1 *Suppose that all the assumptions made in Theorem 2 hold. Then*

$$\lim_{t\to\infty} \frac{\log \mathbb{E}\left[Z(e^{-t})\right]}{t} = \min\{s \in \mathbb{D} : \rho(s) = 1\}.$$

Before we present the proof, observe that $\rho(0) = d \geq 2$ and $\inf_{s\geq 0} \rho(s) \equiv \lambda < 1$, hence $\min\{s \in \mathbb{D} : \rho(s) = 1\}$ is well defined.

Proof. Form Lemma 1, part (b), it follows that we only need to show that

$$\min\{s \in \mathbb{D} : \Lambda(s) = -\log d\} = M$$

where M is defined in the statement of Theorem 2.

By Lemma 1, part (c),

$$\Lambda^*(z) = z s_0(z) - \Lambda(s_0(z)) \tag{13}$$

where $s_0(z)$ solves $\Lambda'(s_0(z)) = z$. Note that $s_0(z) = (\Lambda')^{-1}(z)$ is uniquely defined, since Λ is strictly convex due to non-degeneracy assumptions (see [3], Section 5.4, right after formula (5.10) there), yielding that $\Lambda'(s)$ is strictly increasing. Since $\Lambda'(s) \in \mathbf{C}(\mathbb{R}_+)$ from the arguments after equation (2), we conclude that $s_0(z)$ is continuously differentiable and increasing in z. This implies

$$\Lambda^{*\prime}(z) = s_0(z) \quad \text{for all } z. \tag{14}$$

Recall that

$$f(u) = \frac{\log d - \Lambda^*(-u)}{u}$$

and u^* is the point where the maximum of f on the segment $[0, \mu]$ is achieved; in the proof of Theorem 2 we have shown that $0 < u^* < \mu$. Using (13) and (14) have

$$\begin{aligned} f'(u) &= \frac{u\Lambda^{*\prime}(-u) - \log d + \Lambda^*(-u)}{u^2} = \frac{u s_0(-u) - \log d + [-u s_0(-u) - \Lambda(s_0(-u))]}{u^2} \\ &= -\frac{\log d + \Lambda(s_0(-u))}{u^2} = \frac{s_0(-u) - f(u)}{u}. \end{aligned} \tag{15}$$

We know $\Lambda(0) = 0$, and from (A1) it follows that $\inf_{s\geq 0} \Lambda(s) < -\log d$, hence from the strict convexity of Λ it follows the set $\{s \geq 0 : \ \Lambda(s) = -\log d\}$ contains either 1 or 2 points. Now, if $0 < s_1 < s_2$ are such that $\Lambda(s_1) = \Lambda(s_2) = -\log d$, from the convexity it follows $\Lambda(s) + \log d > 0$ for $s < s_1$ and $s > s_2$, while $\Lambda(s) + \log d < 0$ for $s \in (s_1, s_2)$. Suppose $s_1 = s_0(-u_1)$ and $s_2 = s_0(-u_2)$, then $u_1 > u_2$ (recall that $s_0(z)$ is increasing), and $f'(u) < 0$ for $u < u_2$ and $u > u_1$ while $f'(u) > 0$ for $u \in (u_2, u_1)$. This implies that $u^* = u_1$ is the point where the maximum is really achieved. On the other hand, from (15) we see that $f'(u) = 0$ implies $f(u) = s_0(-u)$ thus yielding $M = f(u_1) = s_0(-u_1) = s_1$ which concludes the proof. ■

4 Applications and remaining proof

The construction studied in this paper relates to many other probabilistic models; see [3]. These applications include random walks in random environment, first-passage percolation, multi-type branching walks among others. Here, we will only focus on the two of them for which Theorem 2 provides additional information.

4.1 First-passage percolation

Consider the coloured tree T_d as constructed in Section 1. To each edge (u, w), where u is the parent of w we assign a random variable τ_{uw} which denotes the *passage time* from vertex u to vertex w and can be one of the d^2 possible types $\tilde{\tau}_{ij}$, $i, j = 1, \ldots, d$; the type is determined by the colours of the edge's endpoints. We assume for simplicity that all the passage times are independent. We want to study

$$R(t) = \mathrm{card}\{\mathrm{u} \in \mathbb{V} : \sum_{(\mathrm{v},\mathrm{w})\in\ell(\mathrm{u})} \tau_{\mathrm{vw}} \leq \mathrm{t}\}$$

that is, the number of vertices of the tree which can be reached by a particle traveling at unit speed by time t; as in Section 5.3 of [3], we allow the passage times to be negative, indicating a sort of 'speeding up' of the motion. Proposition 3 in [3] provides a criterion for finiteness of $R(t)$. Using our Theorem 2 and Corollary 1 we obtain a much finer result:

Proposition 2 *Let $\tilde{\xi}_{ij} = e^{-\tilde{\tau}_{ij}}$, $i, j = 1, \ldots, d$. Suppose that $m(s)$, $\rho(s)$, $\mathbb{D}$, and λ are the same as in Section 2. If $\lambda < 1$ and $\rho'(0) < 0$ then*

$$\lim_{t\to\infty} \frac{\mathbb{E}[R(t)]}{t} = \min\{s \in \mathbb{D} : \rho(s) = 1\}.$$

4.2 Multi-type branching random walks on $\mathbb{R}$

Suppose there are d different types of particles and d^2 positive random variables, τ_{ij}, $i, j = 1, 2, \ldots, d$, whose joint distribution is non-degenerate, and define the following process on $\mathbb{R}$. The process starts at time $n = 0$ with one particle of type $j \in \{1, 2, \ldots, d\}$ located at point 0, write this as $X_1^{(0)} = 0$. At time $n = 1$ this particle splits into d other particles which have different types and take their position $X_1^{(1)}, X_2^{(1)}, \ldots, X_d^{(1)}$ on the real line. The distributions of the jumps $X_k^{(1)} - X_m^{(0)}$ are assumed to be independent for different k's and m's. Now, at time $n = 2$ the first generation particles split into other particles, following the same rules as the original particle, giving a total of d^2 new particles located somewhere on $\mathbb{R}$. If we let this procedure to continue, at time n we will get exactly d^n particles with positions $X_1^{(n)}, X_2^{(n)}, \ldots, X_{d^n}^{(n)} \in \mathbb{R}$. Suppose that the jump from an ancestor to a descendant, say $X_k^{(n)} - X_m^{(n-1)}$, has the distribution of τ_{ij} provided the particle at $X_m^{(n-1)}$ is of type i and the particle at $X_k^{(n-1)}$ is of type j, thus the jump distribution depends on the types of both the parent and the offspring. Such a model was considered in [1] and [3].

Again, set $\tilde{\xi}_{ij} = e^{-\eta_{ij}}$ and let $\rho(s)$ and λ be the same as in Section 2.

Proposition 3 (Proposition 5 in [3]) *Let $x_0 \in \mathbb{R}$ be the unique solution of the equation* $\inf_{s\geq 0} e^{sx_0}\rho(s) = 1$. *Then*

$$\lim_{n\to\infty} \frac{\min\{X_k^{(n)},\ k = 1, 2, \ldots, d^n\}}{n} = x_0 \quad a.s.$$

Observe that the definition $\tilde{\xi}_{ij}$ above implies that $Z(e^{-t})$ corresponds to the number of particles of all generations that lie to the left of t. Hence, our Theorem 2 and Corollary 1 give the following result about the expected number of visits to $(-\infty, t]$ by particles of all generations of our branching random walk:

Proposition 4 *Suppose that $\lambda < 1$ and $\rho'(0) < 0$. Then*

$$\lim_{t\to\infty} \frac{\log\left(\mathbb{E}\left[\sum_{n=1}^{\infty} \mathrm{card}\left\{\mathrm{i} \in \{1,2,\ldots,\mathrm{d^n}\} :\ \mathrm{X_i^{(n)}} \le \mathrm{t}\right\}\right]\right)}{t} = \min\{s \in \mathbb{D} : \rho(s) = 1\}.$$

4.3 Proof of Proposition 1

Firstly, we know that Λ^* is continuous on a compact set $[a_1, a_2] \iff \Lambda^*$ is uniformly continuous on $[a_1, a_2]$ by uniform continuity theorem.

Fix $\delta > 0$. Then we can choose $\tau > 0$ small so that, for $x^{'}, x^{''} \in [a_1, a_2]$

$$|\Lambda^*(x') - \Lambda^*(x'')| \le \frac{\delta}{2} \quad \text{whenever} \quad |x' - x''| \le \tau. \tag{16}$$

Then we choose an $m \in \mathbb{Z}$ and a sequence of real numbers $x_1, x_2, \cdots, x_m$ such that,

$$a_1 = x_1 < x_2 < \ldots < x_{m-1} < x_m = a_2 \text{ and}$$
$$x_{i+1} - x_i < \tau \ \forall\ i \in \{1, 2, \cdots, m-1\}.$$

By Lemma 1, for each $i \in \{1, 2, \cdots, m\}$ there is an n_i such that

$$\frac{1}{n} \cdot \log \mathbb{P}\left(\frac{S_n}{n} \ge x_i\right) \le -\Lambda^*(x_i) + \frac{\delta}{2} \quad \forall n \ge n_i. \tag{17}$$

Define $\tilde{n} := \max\{n_1, n_2, \cdots, n_m\} < \infty$.

Now, $\forall\ a \in (a_1, a_2)$ there is a $j \in \{1, 2, \cdots, m-1\}$ such that $x_j \le a \le x_{j+1}$. Consequently, for all $n \ge \tilde{n}$

$$\begin{aligned}\frac{1}{n} \cdot \log \mathbb{P}\left(\frac{S_n}{n} \ge a\right) &\le \frac{1}{n} \cdot \log \mathbb{P}\left(\frac{S_n}{n} \ge x_j\right) \overset{(\text{by } 17)}{\le} -\Lambda^*(x_j) + \frac{\delta}{2} \\ &\le \left[-\Lambda^*(a) + \frac{\delta}{2}\right] + \frac{\delta}{2} = -\Lambda^*(a) + \delta.\end{aligned}$$

where the final inequality follows from (16) and the fact that $|a - x_{j+1}| < \tau$. ■

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