Conformally flat Lorentzian manifolds with special holonomy

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Abstract

Connected holonomy groups of conformally flat Lorentzian manifolds are classified. It is shown that among conformally flat Lorentzian manifolds there are two classes of spaces with special holonomy: pp-waves with a certain potential and some spaces with the holonomy group $\mathrm{Sim}(n)$, the local structure of these spaces is found.

1 Introduction and the main result

It is known [10] that a conformally flat Riemannian manifold is either a product of two spaces of constant sectional curvature, or it is a product of a space of constant sectional curvature with an interval, or its restricted holonomy group is the identity component of the orthogonal group. The last condition represents the general case and among various manifolds satisfying the last condition one can emphasize only the spaces of constant sectional curvature.

In the case of pseudo-Riemannian manifolds can appear an additional possibility for the holonomy group. Namely, the holonomy group can be weakly irreducible (this means that it does not preserve any non-degenerate proper vector subspace of the tangent space) and not irreducible in the same time, i.e. it may preserve a degenerate vector subspace of the tangent space.

The main result of the present paper is the complete local description of conformally flat Lorentzian manifolds (M, g) with weakly irreducible not irreducible holonomy groups. Let $\dim M = n + 2$. The holonomy algebra (i.e. the Lie algebra of the holonomy group) $\mathfrak{g} \subset \mathfrak{so}(1, n + 1)$ of such manifold preserves an isotropic line of the tangent space (identified with the Minkowski space $\mathbb{R}^{1,n+1}$). Hence $\mathfrak{g}$ is contained in the maximal subalgebra of $\mathfrak{so}(1, n + 1)$ preserving an isotropic line. This algebra is denoted by $\mathfrak{sim}(n)$ and it admits the decomposition

$$\mathfrak{sim}(n) = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n.$$

Manifold (M,g) with such holonomy algebra admits (locally) a distribution of isotropic lines and they are called the Walker manifolds. On such manifold there exist the so called Walker coordinates $v,x^1,...,x^n,u$ and the metric g has the form

$$g = 2dvdu + h + 2Adu + H(du)^2, \tag{1}$$

where $h = h_{ij}(x^1,...,x^n,u)dx^i dx^j$ is an u-dependent family of Riemannian metrics, $A = A_i(x^1,\dots,x^n,u)dx^i$ is an u-dependent family of one-forms, and H is a local function on M [11]. The vector field ∂_v defines the parallel distribution of isotropic lines.

Theorem 1 *Let (M,g) be a conformally flat Lorentzian manifold of dimension $n+2 \geq 4$. Then the holonomy algebra $\mathfrak{g}$ of (M,g) is weakly irreducible and not irreducible if and only if one of the following holds:*

1) $\mathfrak{g} = \mathbb{R}^n \subset \mathfrak{sim}(n)$, i.e. (M,g) is a pp-wave, and locally there exist coordinates $v,x^1,...,x^n,u$ and a function $a(u)$ such that

$$g = 2dvdu + \sum_{i=1}^n (dx^i)^2 + a(u)\sum_{i=1}^n (x^i)^2(du)^2,$$

and $a(u) \neq 0$ for some system of coordinates;

2) $\mathfrak{g} = \mathfrak{sim}(n)$ and locally there exist coordinates $v,x^1,...,x^n,u$ and functions $a(u)$, $B_i(u)$, $C_i(u)$, $D(u)$ such that

$$g = 2dvdu + \sum_{i=1}^n (dx^i)^2 + 2Adu + (vH_1 + H_0)(du)^2,$$

where

$$\begin{aligned}
A &= A_i dx^i,\\
A_i &= \frac{1}{4}\left(2B_j(u)x^j x^i - B_i(u)\sum_{j=1}^n (x^j)^2\right),\\
H_1 &= B_j(u)x^j,\\
H_0 &= \frac{1}{16}\sum_{k=1}^n B_k^2(u)\sum_{i,j=1}^n (x^i x^j)^2 + a(u)\sum_{i=1}^n (x^i)^2 + C_i(u)x^i + D(u),
\end{aligned}$$

and $\sum_i B_i^2(u) \neq 0$ for some system of coordinates.

The Ricci operator of the first metric has the form

$$\mathrm{Ric} = \frac{1}{2}a(u)\partial_v \otimes du,$$

in particular, $\mathrm{Ric}^2 = 0$.

In [3] complete conformally flat Lorentzian manifolds (M, g) satisfying the condition

$$[R(X,Y), \mathrm{Ric}] = 0. \tag{2}$$

are studied. It is shown that these manifolds are exhausted by the spaces of constant sectional curvature, by the products of two spaces of constant sectional curvature, and by products of spaces of constant sectional curvature with intervals.

The Ricci operator of the first metric obtained in this paper satisfies (2), but the metric is not complete [6], i.e. the assumption of completeness in [3] is essential.

For the second metric it holds $\mathrm{Ric}^2 \neq 0$ and $\mathrm{Ric}^3 = 0$. Condition (2) is not satisfied.

In [4] pseudo-Riemannian conformally flat manifolds (M, g) satisfying (2) are studied. It is shown that in addition to the obvious cases, (M, g) may be a complex sphere or a space satisfying $\mathrm{Ric}^2 = 0$. Various examples of conformally flat manifolds with $\mathrm{Ric}^2 = 0$ are constructed in [5].

Remark that an important fact is that a simply connected conformally flat spin Lorentzian manifold admits the spaces of conformal Killing spinors of maximal dimension [1].

2 Decomposability of conformally flat pseudo-Riemannian manifolds

In [10] Kurita proved the following theorem for the case of Riemannian manifolds.

Theorem 2 *Let (M, g) be an n-dimensional conformally flat Riemannian manifold. Then its local restricted holonomy group H_x ($x \in M$) is in general $\mathrm{SO}(n)$. If $H_x \neq \mathrm{SO}(n)$, then for some coordinate neighborhood U of x one of the following holds:*

1) H_x is identity and the metric is flat in U;

2) $H_x = \mathrm{SO}(k) \times \mathrm{SO}(n-k)$ and U is a direct product of a k-dimensional manifold of constant sectional curvature K and an $(n-k)$-dimensional manifold of constant sectional curvature $-K$ ($K \neq 0$);

3) $H_x = \mathrm{SO}(n-1)$ and U is a direct product of a straight line (or a segment) and an $(n-1)$-dimensional manifold of constant sectional curvature.

We generalize this theorem for the case of pseudo-Riemannian manifolds. We also make it more precise.

Theorem 3 *Let (M,g) be a conformally flat pseudo-Riemannian manifold of signature (r,s) with the restricted holonomy group $\mathrm{Hol}^0(M,g)$. If (M,g) is not flat, then one of the following holds:*

1) $\mathrm{Hol}^0(M,g) = \mathrm{SO}(r,s)$*;*

2) $\mathrm{Hol}^0(M,g)$ *is weakly irreducible and not irreducible (in particular, it preserves a degenerate subspace of the tangent space);*

3) $\mathrm{Hol}^0(M,g) = \mathrm{SO}(r_1,s_1) \times \mathrm{SO}(r-r_1,s-s_1)$ *and each point $x \in M$ has a neighborhood that is either flat or it is a product of a pseudo-Riemannian manifold of constant sectional curvature K and signature (r_1,s_1) and a pseudo-Riemannian manifold of constant sectional curvature $-K$ ($K \neq 0$) and signature $(r-r_1,s-s_1)$;*

4) $\mathrm{Hol}^0(M,g) = \mathrm{SO}(r-1,s)$ *(resp., $H_x = \mathrm{SO}(r,s-1)$) and each point $x \in M$ has a neighborhood that is either flat or it is a product of a pseudo-Riemannian manifold of constant sectional curvature and signature $(r-1,s)$ (resp., $(r,s-1)$) and the space $(L,-dt^2)$ (resp., (L,dt^2)), L is the straight line or a segment.*

Proof. Let (M,g) be a pseudo-Riemannian manifold of signature (r,s) and dimension $d=r+s$. The vector bundle $\mathfrak{so}(TM)$ of skew-symmetric endomorphisms of the tangent bundle TM can be identified with the space of bivectors $\wedge^2 TM$ in such a way that

$$(X\wedge Y)Z = g(X,Z)Y - g(Y,Z)X$$

for all vector fields X,Y,Z on M. The Weyl tensor W of the pseudo-Riemannian manifold (M,g) is defined by the equality

$$W = R + R_L, \tag{3}$$

where the tensor R_L is defined by

$$R_L(X,Y) = LX\wedge Y + X\wedge LY, \tag{4}$$

$$L = \frac{1}{d-2}\left(\mathrm{Ric} - \frac{s}{2(d-1)}\,\mathrm{id}\right)$$

is the Schouten tensor and s is the scalar curvature.

Suppose that the restricted holonomy group $\mathrm{Hol}^0(M,g)$ is not weakly irreducible. The Wu decomposition Theorem [12] states that each point of M has a neighborhood U such that $(U,g|_U)$ is a product

$$(U,g|_U) = (M_1\times M_2, g_1+g_2)$$

of two pseudo-Riemannian manifolds (M_1, g_1) and (M_2, g_2). Let d_1 and d_2 be the dimensions of these manifolds. For the curvature tensors, Ricci operators and the scalar curvatures it holds

$$R = R_1 + R_2, \quad \mathrm{Ric} = \mathrm{Ric}_1 + \mathrm{Ric}_2, \quad s = s_1 + s_2.$$

First suppose that $d \geq 4$. In this case $W = 0$ and we get

$$R_1 + R_2 = -R_L. \tag{5}$$

Assume that $d_1 \geq d_2$ and $d_1 \geq 2$. The curvature tensor R_1 can be written in the form $R_1 = W_1 - R_{L_1}$. Considering (5) restricted to TM_1, we get that $W_1 = 0$ and

$$\frac{1}{d_1 - 2}\left(\mathrm{Ric}_1 - \frac{s_1}{2(d_1-1)}\,\mathrm{id}\right) = \frac{1}{d-2}\left(\mathrm{Ric}_1 - \frac{s_1 + s_2}{2(d-1)}\,\mathrm{id}\right). \tag{6}$$

If $d_2 \geq 2$, then taking the trace in (6), we get

$$\frac{s_1}{d_1(d_1-1)} = -\frac{s_2}{d_2(d_2-1)}.$$

Substituting this back to (6), we obtain

$$\mathrm{Ric}_1 = \frac{s_1}{d_1}\,\mathrm{id}\,. \tag{7}$$

Since s_1 is a function on M_1 and s_2 is a function on M_2, the both functions must be constant. Next,

$$R_1(X, Y) = \frac{s_1}{d_1(d_1-1)} X \wedge Y. \tag{8}$$

The same holds for the second manifold. For the sectional curvatures we get

$$k_1 = \frac{s_1}{d_1(d_1-1)} = -\frac{s_2}{d_2(d_2-1)} = -k_2.$$

If $d_2 = 1$, than (6) is equivalent to (7) and this implies (8). From this and the Schur Theorem it follows that k_1 is constant. If $d_1 = 2$, then the curvature tensor R_1 satisfies $R_1(X, Y) = f X \wedge Y$ for some function f on M_1. The proof in this case is the same.

If $d = 3$, then $d_1 = 2$ and $d_2 = 1$. It holds $R = R_1$ and $R_1(X, Y) = f X \wedge Y$ for some function f on M_1. In this case (M, g) is conformally flat if and only if the Cotton tensor C defined by

$$C(X, Y, Z) = g((\nabla_Z L)X, Y) - g((\nabla_Y L)X, Z)$$

is zero. This implies that f is constant, i.e. (M_1, g_1) has constant sectional curvature.

Now we have to prove that if $\mathrm{Hol}^0(M, g)$ is irreducible, then it coincides with $\mathrm{SO}(r, s)$. Suppose that $\mathrm{Hol}^0(M, g)$ is irreducible and it is different from $\mathrm{SO}(r, s)$ and $\mathrm{U}(\frac{r}{2}, \frac{s}{2})$. Then the manifold is Einstein [2]. Since (M, g) is in addition conformally flat, (M, g) has constant sectional curvature and its connected holonomy group must be either trivial or $\mathrm{SO}(r, s)$, i.e. we get a contradiction. Thus we need only to prove that $\mathrm{Hol}^0(M, g) \neq \mathrm{U}(\frac{r}{2}, \frac{s}{2})$. This will follow from the following (probably known) statement.

Proposition 1 *If a pseudo-Kählerian manifold is conformally flat, then it is flat.*

Proof. Let (M,g) be a pseudo-Kählerian conformally flat manifold of dimension $2n$. Then its curvature satisfies

$$-R(X,Y)=R_L(X,Y)=LX\wedge Y+X\wedge LY.$$

Since $R(X,Y)$ commutes with J, we get

$$\begin{aligned}g(LX,JZ)Y-g(Y,JZ)LX+g(X,JZ)LY-g(LY,JZ)X\\ =J(g(LX,Z)Y-g(Y,Z)LX+g(X,Z)LY-g(LY,Z)X).\end{aligned}$$

Fix a local basis $X_1,...,X_{2n}$ of vector fields such that $g(X_i,X_j)=\epsilon_i\delta_{ij}$, where $\epsilon_i=\pm1$. Note that for any vector field it holds $X=\sum_i\epsilon_i g(X,X_i)X_i$. Putting in the above equation $X=\epsilon_iX_i$, $Z=X_i$ and taking the sum over i, we get

$$(2d-4)LJY=-J(\operatorname{tr}L)Y.$$

Hence,

$$(2d-4)L=-(\operatorname{tr}L)\operatorname{id}.$$

This implies $L=0$, i.e. $R=0$. □

This proves Theorem 3. □

3 The Weyl curvature tensor of Walker metrics

In order to prove Theorem 1 we give some information about the curvature tensor of the Walker metric (1). For the fixed coordinates $v,x^1,...,x^n,u$ consider the fields of frames

$$p=\partial_v,\quad X_i=\partial_i-A_i\partial_v,\quad q=\partial_u-\frac{1}{2}H\partial_v.$$

Consider the distribution $E=\operatorname{span}\{X_1,...,X_n\}$. From the results of [8] it follows that the curvature tensor R of the metric g can be written in the form

$$\begin{aligned}R(p,q)&=-\lambda p\wedge q-p\wedge\vec{v},\qquad R(X,Y)=R_0(X,Y)-p\wedge(P(Y)X-P(X)Y),\\ R(X,q)&=-g(\vec{v},X)p\wedge q+P(X)-p\wedge T(X),\qquad R(p,X)=0\end{aligned}$$

for all $X,Y\in\Gamma(E)$. Here λ is a function, $\vec{v}\in\Gamma(E)$, $T\in\operatorname{End}(E)$ is symmetric, $T^*=T$, R_0 is the family of the curvature tensors of the family of Riemannian metrics $h(u)$, and the tensor $P\in E^*\otimes\mathfrak{so}(E)$ satisfies

$$g(P(X)Y,Z)+g(P(Y)Z,X)+g(P(Z)X,Y)=0\text{ for all }X,Y,Z\in\Gamma(E).$$

These element may be found in terms of the coefficients of the metric (1). For example,

$$\lambda = -\frac{1}{2}\partial_v^2 H, \quad \vec{v} = -\frac{1}{2}\left(\partial_i\partial_v H - A_i\partial_v^2 H\right) X_j h^{ij}. \tag{9}$$

The expressions for the other elements are more difficult and we will give them only in some partial cases.

The Ricci operator has the following form:

$$\begin{aligned} \mathrm{Ric}(p) &= -\lambda p, \quad \mathrm{Ric}(X) = g(X, \widetilde{\mathrm{Ric}}\, P - \vec{v})p + \mathrm{Ric}(h)(X), && (10) \\ \mathrm{Ric}(q) &= (\mathrm{tr}\, T)p + \widetilde{\mathrm{Ric}}(P) - \vec{v} - \lambda q, && (11) \end{aligned}$$

where $\widetilde{\mathrm{Ric}}\, P = h^{ij} P(X_i)X_j$. For the scalar curvature we get $s = 2\lambda + s_0$, where s_0 is the scalar curvature of h. Using this, we may compute the tensor R_L,

$$\begin{aligned} R_L(p, X) &= \frac{1}{n} p \wedge \left(\mathrm{Ric}(h) - \frac{(n-1)\lambda - s_0}{n+1}\,\mathrm{id}\right) X, && (12) \\ R_L(p, q) &= \frac{1}{n}\left(\frac{2n\lambda - s_0}{n+1} p \wedge q + p \wedge (\vec{v} - \widetilde{\mathrm{Ric}}\, P)\right), && (13) \\ R_L(X, Y) &= \frac{1}{n}\Big(p \wedge (g(X, v - \widetilde{\mathrm{Ric}}\, P)Y - g(Y, v - \widetilde{\mathrm{Ric}}\, P)X) && (14) \\ &\quad + \left(\mathrm{Ric}(h) - \frac{s}{2(n+1)}\right) X \wedge Y + X \wedge \left(\mathrm{Ric}(h) - \frac{s}{2(n+1)}\right) Y\Big), \\ R_L(X, q) &= \frac{1}{n}\Big((\mathrm{tr}\, T)p \wedge X + g(X, \vec{v} - \widetilde{\mathrm{Ric}}\, P)p \wedge q + X \wedge (\vec{v} - \widetilde{\mathrm{Ric}}\, P) && (15) \\ &\quad + \left(\mathrm{Ric}(h) - \frac{(n-1)\lambda - s_0}{n+1}\,\mathrm{id}\right) X \wedge q\Big). \end{aligned}$$

The Weyl tensor W can be computed using this and (3).

4 Proof of Theorem 1

Suppose that the metric (1) is conformally flat, i.e. $W = 0$. Using the computations of the previous section, it is easy to show that this is equivalent to the following equations:

$$\lambda = 0, \quad R_0 = 0, \quad P(X) = \vec{v} \wedge X, \quad T = f\,\mathrm{id}_E, \tag{16}$$

where f is a function. In particular, it holds $\widetilde{\mathrm{Ric}}\, P = -(n-1)\vec{v}$. Since $R_0 = 0$, each metric in the family h is flat, hence, changing the coordinates, we may assume that

$$h = \delta_{ij} dx^i dx^j.$$

From (9) and the equality $\lambda = 0$ it follows that $\partial_v^2 H = 0$, hence

$$H = vH_1 + H_0, \quad \partial_v H_1 = \partial_v H_0 = 0.$$

Using (9), we get

$$\vec{v} = -\frac{1}{2}\partial_i H_1 \delta^{ij} X_j.$$

For the case of h independent of u the curvature tensor of the metric (1) is computed in [9]. Let $P(X_k)X_j = P^i_{jk}X_i$ and $T(X_j) = \sum_i T_{ij}X_j$. Then

$$P^i_{jk} = R^i_{jkq}, \quad T_{ij} = -R^i_{qjq}.$$

Using the computations from [9], for our metric we obtain

$$P^i_{jk} = \frac{1}{2}\partial_k F_{ij}, \tag{17}$$

$$T_{ij} = -\frac{1}{2}\partial_i\partial_j(vH_1 + H_0) + \frac{1}{4}\sum_k F_{ik}F_{jk} + \frac{1}{4}H_1(\partial_i A_j + \partial_j A_i) + \frac{1}{2}(A_i\partial_j H_1 + A_j \partial_i H_1), \tag{18}$$

where

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

is the differential of the 1-form A.

Consider the following two cases:

1) $\vec{v} = 0$ for any coordinate system;

2) $\vec{v} \neq 0$ for some coordinate system.

Case 1). We have $\vec{v} = 0$ for any coordinate system. Then the curvature tensor satisfies $R(E, E) = 0$. Consequently, (M, g) is a pp-wave (see e.g. [7]), i.e. locally g can be written in the form

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + H(du)^2, \quad \partial_v H = 0$$

i.e. $A = 0$, $H_1 = 0$, and $H_0 = H$. We obtain the equation

$$f\delta_{ij} = -\frac{1}{2}\partial_i\partial_j H_0.$$

Taking the trace, we may find f, and we obtain

$$\frac{1}{n}\Delta H_0 \delta_{ij} = \partial_i \partial_j H_0. \tag{19}$$

The general solution of this equation has the form

$$H_0 = a(u)\sum_{i=1}^{n}(x^i)^2 + C_i(u)x^i + D(u).$$

Using some transformations of coordinates, we get

$$H_0 = a(u)\sum_{i=1}^{n}(x^i)^2.$$

Case 2). Suppose that $\vec{v} \neq 0$. Since $P(X) = \vec{v} \wedge X$, we get

$$P^i_{jk} = \delta_{ki}\vec{v}_j - \delta_{kj}\vec{v}_i,$$

where $\vec{v} = \sum_j \vec{v}_j X_j$. We obtain the system of equations

$$\partial_k F_{ij} = -\delta_{ki}\partial_j H_1 + \delta_{kj}\partial_i H_1. \tag{20}$$

These equations can be rewritten in the form

$$\partial_i(\partial_k A_j - \delta_{kj}H_1) - \partial_j(\partial_k A_i - \delta_{ki}H_1).$$

This system of equation is equivalent to

$$dG^k = 0,$$

where we define the 1-forms

$$G^k = G^k_i dx^i, \quad G^k_i = \partial_k A_i - \delta_{ki}H_1.$$

We conclude that there exist functions f^k such that

$$G^k_i = \partial_i f^k.$$

Our system of equations takes the form

$$\partial_k A_i - \delta_{ki}H_1 = \partial_i f^k.$$

This implies

$$F = dA = -df, \quad \text{where} \quad f = \sum_k f^k dx^k$$

and

$$A = -f + d\varphi$$

for some function φ. Since $\partial_v^2 H = 0$, the gauge transformation

$$v \mapsto v - \phi$$

changes the metric in the following way [9]:

$$A \mapsto A + d\phi, \quad H_1 \mapsto H_1, \quad H_0 \mapsto H_0 + H_1\phi + 2\partial_u\phi. \tag{21}$$

Hence, changing the coordinates, we get $dA = -df$. Our equations take the form

$$\partial_i A_j + \partial_j A_i = \delta_{ij} H_1. \tag{22}$$

Conversely, this system of equations implies (20).

Consider now (18). Since $T_{ij} = f\delta_{ij}$ for some function f, we get that $f = vf_1 + f_0$, where $\partial_v f_1 = \partial_v f_0 = 0$. Applying ∂_v to (18), we get

$$f_1 \delta_{ij} = -\frac{1}{2}\partial_i \partial_j H_1.$$

Taking the trace, we get $f_1 = -\frac{1}{2n}\Delta H_1$, where Δ is the Euclidean Laplacian. We get the equation

$$n\Delta H_1 \delta_{ij} = \partial_i \partial_j H_1.$$

Clearly, this implies

$$H_1 = a(u)\sum_i (x^i)^2 + B_i(u)x^i + c(u).$$

From (22) it follows that for each i it holds

$$\partial_i A_i = \frac{1}{2} H_1.$$

Integrating this equation, we get

$$A_i = \frac{1}{2}\left(a(u)x^i \sum_{j\neq i}(x^j)^2 + \frac{a(u)}{3}(x^i)^3 + \sum_{j\neq i} B_j(u)x^j x^i + \frac{B_i(u)}{2}(x^i)^2 + c(u)x^i + c_i(x^k, u)\right), \quad \partial_i c_i = 0.$$

Let $i \neq j$. Substituting the obtained A_i to (22), we get

$$4a(u)x^i x^j + B_j(u)x^i + B_i(u)x^j + \partial_j c_i + \partial_i c_j = 0.$$

Applying ∂_i, we get

$$4a(u)x^j + B_j(u) + \partial_i^2 c_j = 0.$$

Applying ∂_j, we get $a(u) = 0$. We conclude that

$$\partial_i^2 c_j = -B_j(u), \quad \partial_j c_j = 0.$$

This implies

$$c_j = -\frac{B_j(u)}{2}\sum_{k\neq j}(x^k)^2 + d_{jk}(u)x^k + f_j(u), \quad d_{jj}(u) = 0.$$

Using (22) for $i \neq j$, we get

$$d_{ij}(u) = -d_{ji}(u).$$

Thus,

$$H_1 = B_i(u)x^i + c(u), \tag{23}$$

$$A_i = \frac{1}{2}\left(B_j(u)x^j x^i - \frac{B_i(u)}{2}\sum_j (x^j)^2 + c(u)x^i + d_{ik}(u)x^k + f_i(u)\right). \tag{24}$$

Since $\vec{v} \neq 0$, it holds

$$\sum_j B_j^2(u) \neq 0.$$

Consider the coordinate transformation with the inverse one

$$v = \tilde{v}, \quad x^i = \tilde{x}^i + b^i(\tilde{u}), \quad u = \tilde{u}$$

such that $B_i(u)b^i(u) + c(u) = 0$. After that $H_1 = B_i(u)x^i$, i.e. we may assume that $c(u) = 0$. Next, consider the coordinate transformation with the inverse one

$$v = \tilde{v}, \quad x^i = A^i_j(\tilde{u})\tilde{x}^j, \quad u = \tilde{u},$$

where $A^i_j(u)$ is a family of orthogonal matrices. It is easy to check that

$$\tilde{H}_1 = B_i(u)A^i_j(u)\tilde{x}^j, \quad \tilde{A}_i = \sum_k A^k_i(u)(\partial_u A^k_l(u))\tilde{x}^l + A^k_i(u)A_i.$$

The obtained metric has the same form and it holds

$$\tilde{B}_i(u) = B_j(u)A^j_i(u), \quad \tilde{d}_{ij}(u) = \sum_k A^k_i(u)\partial_u A^k_j(u) + \frac{1}{2}A^r_i(u)d_{rk}(u)A^k_j(u), \quad \tilde{f}_i(u) = A^k_i(u)f_k(u).$$

Consider the equation $\tilde{d}_{ij}(u) = 0$. Since $\sum_k A^k_i(u)A^k_j(u) = \delta_{ij}$, it can be written in the form

$$\partial_u A^k_i(u) = A^j_i(u)\frac{1}{2}d_{jk}(u).$$

Since $d_{jk}(u)$ is skew-symmetric, $\frac{1}{2}d_{jk}(u)$ is a curve in the Lie algebra $\mathfrak{so}(n)$. Then $A^k_i(u)$ satisfying the above equation is nothing else es the development of the curve $\frac{1}{2}d_{jk}(u)$ in the Lie group $\mathrm{SO}(n)$. Thus, applying such transformation, we may assume that $d_{ij}(u) = 0$. Applying (21), we may assume that $f_i(u) = 0$. Thus,

$$H_1 = B_i(u)x^i, \quad A_i = \frac{1}{2}\left(B_j(u)x^j x^i - \frac{B_i(u)}{2}\sum_j (x^j)^2\right).$$

Note that

$$F_{ij} = B_i(u)x^j - B_j(u)x^i$$

and (22) holds. The equation $T_{ij} = f\delta_{ij}$ takes the following form:

$$f_0\delta_{ij} = -\frac{1}{2}\partial_i\partial_j H_0 + \frac{1}{4}\sum_k B_k^2(u)x^i x^j + \frac{1}{4}H_1^2\delta_{ij}.$$

It can be rewritten in the form

$$\frac{1}{n}\left(-\frac{1}{2}\Delta H_0+\frac{1}{4}\sum_k B_k^2(u)\sum_l (x^l)^2\right)=-\frac{1}{2}\partial_i\partial_j H_0+\frac{1}{4}\sum_k B_k^2(u)x^ix^j.$$

Clearly, the function

$$H_0=\frac{1}{16}\sum_{k=1}^n B_k^2(u)\sum_{i,j=1}^n (x^ix^j)^2$$

is a partial solution of this equation. On the other hand,

$$a(u)\sum_{i=1}^n (x^i)^2+C_i(u)x^i+D(u)$$

is the general solution of the corresponding homogeneous system.

Let us compute the holonomy algebra of the obtained metric. Let $x\in M$ be a point such that $\vec{v}_x\neq 0$. The condition on the curvature tensor shows that

$$R_x(p_x,q_x)=-p_x\wedge\vec{v}_x,\quad R_x(X,Y)=p_x\wedge((X\wedge Y)\vec{v}_x).$$

This shows that $p_x\wedge E_x\subset\mathfrak{g}$. Next,

$$R_x(\vec{v}_x,q_x)=-g(\vec{v}_x,\vec{v}_x)p_x\wedge q_x-p_x\wedge T_x(\vec{v}_x),$$

which implies $\mathbb{R}p_x\wedge q_x\subset\mathfrak{g}$. Finally,

$$R_x(X,q_x)=-g(\vec{v}_x,X)p_x\wedge q_x+\vec{v}_x\wedge X-p_x\wedge T_x(X).$$

Since the bivectors of the form $\vec{v}_x\wedge X$ generate the Lie algebra $\mathfrak{so}(E_x)$, we conclude that

$$\mathfrak{g}=\mathbb{R}p_x\wedge q_x+\mathfrak{so}(E_x)+p_x\wedge E_x\simeq\mathfrak{sim}(n).$$

This proves the theorem. □

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