AVERAGING SEQUENCES

FERNANDO ALCALDE CUESTA AND ANA RECHTMAN

Abstract. In the spirit of Goodman-Plante average condition for the existence of a transverse invariant measure for foliations, we give an averaging condition to find tangentially smooth measures with prescribed Radon-Nikodym cocycle. Harmonic measures are examples of tangentially smooth measures for foliations; we also present sufficient hypothesis on the averaging condition in order to obtain a harmonic measure.

1. Introduction

Averaging sequences for foliations were introduced in the pioneering work of J. F. Plante [20] on the influence that the existence of transverse invariant measures exerts on the structure of a foliation. Although only the case of sub-exponential growth was dealt with in [20], Plante's approach is clearly reminiscent of the classic work of E. Følner on groups. Using the same kind of ideas, S. E. Goodman and J. F. Plante exhibited an averaging condition which guarantees the existence of transverse invariant measures for compact foliated spaces [10].

In this paper we formulate a more general averaging condition which gives rise to a tangentially smooth measure for a compact foliated space $(M, \mathcal{F})$ (Theorem 4.10). This condition may be related to the η-Følner condition of [2], in the same spirit as Følner, but using a modified Riemannian metric along the leaves. The modification is done by replacing any complete Riemannian metric along the leaves with the product with a certain tangentially smooth function. When this function is harmonic, we obtain a harmonic measure.

We can use the discrete approach to study foliations, which is the equivalence relation defined on any total transversal. Such an equivalence relation is also the orbit equivalence relation under the holonomy pseudogroup. In fact, this is the point of view adopted at first by S. E. Goodman and J. F. Plante in [20] and [10]. Similarly, we start by showing an averaging condition for orbit equivalence relations $\mathcal{R}$ defined by finitely generated pseudogroups acting on compact spaces and continuous cocycles $\delta : \mathcal{R} \to \mathbb{R}_+^*$ (Theorem 3.3). Under some additional conditions, if δ is harmonic, the measure obtained is harmonic. As in the classic case, our result is reminiscent of Kaimanovich's characterization of amenable equivalence relations [13].

Given a continuous cocycle $\delta : \mathcal{R} \to \mathbb{R}_+^*$, the *Radon-Nikodym problem* is to determine the existence of probability measures which are quasi-invariant and admit δ as

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their Radon-Nikodym derivative. In the foliated setting, discrete equivalence relations and cocycles can be replaced by compact foliated spaces and modular forms. According to Theorem 3.3, the existence of an δ-averaging sequence gives a positive answer to the Radon-Nikodym problem. Similarly, according to Theorem 4.10 that involves a boundedness condition, any η-averaging sequence defines a tangentially smooth measure whose modular form is equal to η.

2. Preliminaries

2.1. **Foliations and equivalence relations.** A compact p-dimensional space M admits a d-dimensional foliation $\mathcal{F}$ of class C^r, with $2 \leq r \leq \infty$ or $r = \omega$, if there exists a cover of M by open sets U_i homeomorphic to the product of an open disc P_i in $\mathbb{R}^d$ centered at the origin 0 and a locally compact separable metrizable space T_i. Thus, if we denote the corresponding foliated chart by $\varphi_i : U_i \to P_i \times T_i$, each U_i splits into *plaques* $\varphi_i^{-1}(P_i \times \{y\})$. Each point $y \in T_i$ can also be identified with the point $\varphi_i^{-1}(0, y)$ in the *local transversal* $\varphi_i^{-1}(\{0\} \times T_i)$. In addition, the change of charts $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is given by

$$\varphi_j \circ \varphi_i^{-1}(x, y) = (\varphi_{ij}^y(x), \gamma_{ij}(y)) \tag{2.1}$$

where γ_{ij} is an homeomorphism between open subsets of T_i and T_j and φ_{ij}^y is a C^r-diffeomorphism depending continuously on y in the C^r-topology. We say that $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ is a *good foliated atlas* if it satisfies the following conditions:

(i) the cover $\mathcal{U} = \{U_i\}_{i \in I}$ is locally finite, hence finite;
(ii) each open set U_i is a relatively compact subset of a foliated chart;
(iii) if $U_i \cap U_j \neq \emptyset$, there is a foliated chart containing $\overline{U_i \cap U_j}$, implying that each plaque of U_i intersects at most one plaque of U_j.

Each foliated chart U_i admits a tangentially C^∞-smooth Riemannian metric $g_i = \varphi_i^* g_0$ induced from a C^∞-smooth Riemannian metric g_0 on $\mathbb{R}^p$. If $\mathcal{F}$ is C^∞, we can glue together these local Riemannian metrics g_i to a global one g using a tangentially C^∞-smooth partition of unity. In fact, any C^1 compact foliation equipped with a C^1 foliated atlas $\mathcal{A}$ admits a complete Riemannian metric along the leaves because there is a C^∞ foliated atlas C^1-equivalent to $\mathcal{A}$. We refer to Lemma 2.6 of [1].

A discrete equivalence relation $\mathcal{R}$ is defined by $\mathcal{F}$ on the total transversal $T = \sqcup T_i$: the equivalence classes are the traces of the leaves on T. We can see $\mathcal{R}$ as the orbit equivalence relation defined by the action of the pseudogroup Γ, called the *holonomy pseudogroup of* $\mathcal{F}$, generated by the local diffeomorphisms γ_{ij}. These diffeomorphisms form a finite generating set, which we will denote $\Gamma^{(1)}$, which defines a *graphing* of $\mathcal{R}$. This means that each equivalence class $\mathcal{R}[y]$ is the set of vertices of a graph, and there is an edge joining two vertices z and w if there is $\gamma \in \Gamma^{(1)}$ such that $\gamma(z) = w$. We can define a graph metric $d_\Gamma(z, w) = \min \{ n \,/\, \exists \gamma \in \Gamma^{(n)} : g(z) = w \}$, where $\Gamma^{(n)}$ are the elements that can be expressed as words of a length of at most n in terms of $\Gamma^{(1)}$. This approach makes it possible to turn $\mathcal{R}$ into an *étale equivalence relation* since the graphs of the elements of Γ form a topology whereby the maps $((y, \gamma(y)), (\gamma(y), \gamma'(\gamma(y))) \in \mathcal{R} * \mathcal{R} \mapsto (y, \gamma' \circ \gamma(y)) \in \mathcal{R}$ and $(y, \gamma(y)) \in \mathcal{R} \mapsto (\gamma(y), y) \in \mathcal{R}$ are continuous, and the left and right projections $\beta : (y, z) \in \mathcal{R} \mapsto y \in T$ and $\alpha : (y, z) \in \mathcal{R} \mapsto z \in T$ are local homeomorphisms. A *transverse invariant measure* for $\mathcal{F}$ is a measure on T that is invariant under the action of Γ. It is quite rare for a measure of this kind to exist.

2.2. **Compactly generated pseudogroups.** In the last section, we obtained we obtained a pseudogroup from a foliated atlas. Here we will recall the *Haefliger equivalence* for pseudogroups obtained from different atlases and its metric counterpart in the compact case [11]. For any compact foliated space $(M, \mathcal{F})$ the holonomy pseudogroup Γ is *compactly generated* in the sense of [11], meaning that:

(i) T contains a relatively compact open set T_1 meeting all the orbits;
(ii) the reduced pseudogroup $\Gamma|_{T_1}$ (whose elements have domain and range in T_1) admits a finite generating set (called a *compact generation system*): each element $\gamma : A \to B$ extends to an element $\overline{\gamma}$ of Γ whose domain contains the closure of A.

Observe that T is covered by the domains of a family of elements of Γ with range in T_1. The union of these elements and their inverses defines the *fundamental equivalence* between the holonomy pseudogroup Γ and the reduced pseudogroup $\Gamma|_{T_1}$. The notion of *Haefliger equivalence* is modelled by this definition:

Definition 2.1. Two pseudogroups Γ_1 and Γ_2 acting on the spaces T_1 and T_2, respectively, are *Haefliger equivalent* if they are reductions (to open sets meeting all the orbits) of a same pseudogroup Γ acting on the disjoint union $T = T_1 \sqcup T_1$.

In general, the quasi-isometric type of the orbits is not preserved by Haefliger equivalence. However, if T_1 and T_2 are two relatively compact open subsets of T meeting all the orbits of Γ, the reduced pseudogroups $\Gamma|_{T_1}$ and $\Gamma|_{T_2}$ become *Kakutani equivalent* (*i.e.* the orbits of $\Gamma|_{T_1}$ and $\Gamma|_{T_2}$ are quasi-isometric in the sense of Gromov) with respect to some good compact generation system [16].

In this context, any probability measure ν_K on the compact set $K = \overline{T_1}$ that is preserved by the action of $\Gamma|_K$ extends to a unique Borel measure ν on T which is Γ-invariant and finite on compact sets. We refer to Lemma 3.2 of [20].

2.3. **Existence of transverse invariant measures.** In this section we will discuss a sufficient condition for the existence of a transverse invariant measure, which serves as motivation for Theorem 3.3. In [10], Goodman and Plante formulate:

Proposition 2.2 (Goodman-Plante [10])**.** *Let $\{A_n\}$ be an averaging sequence for Γ, i.e. a sequence of finite subsets A_n of T such that for all $\gamma \in \Gamma^{(1)}$ (and then for all $\gamma \in \Gamma$),*

$$\lim_{n\to\infty} \frac{|\Delta_\gamma A_n|}{|A_n|} = 0$$

where $\Delta_\gamma A = A \,\Delta\, \gamma(A) = (A \setminus \gamma(A)) \cup (\gamma(A) \setminus A)$ and $|A|$ denotes the cardinality of A. Then $\{A_n\}$ gives rise to a transverse invariant measure ν whose support is contained in the limit set $\lim_{n\to\infty} A_n = \{y \in T \mid \exists y_n \in A_n : y = \lim_{n\to\infty} y_n\}$.

The idea of the proof is the following. Assuming that T is compact, we may construct a Γ-invariant probability measure on T from the sequence of probability measures ν_n defined by $\nu_n(B) = |B \cap A_n|/|A_n|$ for every Borel set $B \subset T$. According to Riesz's representation theorem, each measure ν_n can be identified to the functional I_n on the space $C(T)$ of continuous real-valued functions on T given by

$$I_n(f) = \frac{1}{|A_n|} \sum_{y \in A_n} f(y).$$

By passing to a subsequence, if necessary, I_n converges in the weak topology to a positive functional I which determines a unique Borel regular measure ν such that $I(f) = \int_T f d\nu$ for every $f \in C(T)$. The averaging condition implies that I and ν are Γ-invariant since for every $\gamma \in \Gamma$ and every $f \in C(T)$ with support on the range of γ, we have

$$|I(f \circ \gamma) - I(f)| \leq \|f\|_\infty \lim_{n\to\infty} \frac{|\Delta_\gamma A_n|}{|A_n|} = 0.$$

Finally, it is clear that $\nu(T) = 1$ and $supp(\nu) = \lim_{n\to\infty} A_n$.

In the non-compact case, for any good compact generation system, the holonomy pseudogroup Γ is Kakutani equivalent to its reduction $\Gamma|_K$ to a compact total transversal K. Then any averaging sequence A_n for Γ defines an averaging sequence $A_n \cap K$ for $\Gamma|_K$, and we obtain a probability measure ν_K on K that is invariant under $\Gamma|_K$. Now, we can extend ν_K to a unique Borel measure ν on T which is Γ-invariant and finite on compact sets.

Example 2.3. Consider a graph with bounded geometry, as for example any orbit $\Gamma(x)$ of the holonomy pseudogroup of a compact foliated space. This graph is said to be *Følner* if it contains a sequence of finite subsets of vertices A_n such that $|\partial A_n|/|A_n| \to 0$, where ∂A_n denotes the boundary set with respect to the graph structure. Since $\Delta_\gamma A \subset \partial A \cup \gamma^{-1}(\partial A)$ for any $\gamma \in \Gamma^{(1)}$, we get that $|\Delta_\gamma A_n| \leq 2|\partial A_n|$, and we have an averaging sequence. In particular, any orbit $\Gamma(x)$ having sub-exponential growth is an example of Følner graph, in this case we have

$$\liminf_{n\to\infty} \frac{|A_{n+1} - A_{n-1}|}{|A_n|} = 0,$$

where $A_n = \Gamma^{(n)}(x)$.

Using the one-to-one correspondence between foliated cycles and transverse invariant measures stablished by D. Sullivan [21], it is not difficult to show the following continuous version of Goodman-Plante's result:

Proposition 2.4 (Goodman-Plante [10])**.** *Let $\{V_n\}$ be an averaging sequence for $\mathcal{F}$, i.e. a sequence of compact domains V_n (of dimension d) in the leaves such that*

$$\lim_{n\to\infty} \frac{area(\partial V_n)}{vol(V_n)} = 0$$

where area denotes the $(d-1)$-volume and vol the d-volume with respect to the complete Riemannian metric along the leaves. Then $\{V_n\}$ gives rise to a transverse invariant measure ν whose support is contained in the saturated limit set $\lim_{n\to\infty} V_n = \{ p \in M \,/\, \exists p_n \in V_n : p = \lim_{n\to\infty} p_n \}$.

Recall that a *foliated d-form* $\alpha \in \Omega^d(\mathcal{F})$ is a family of differentiable d-forms over the plaques of $\mathcal{A}$ depending continuously on the transverse parameter and which agree on the intersection of each pair of foliated charts. A *foliated r-cycle* is a continuous linear functional $\xi : \Omega^d(\mathcal{F}) \to \mathbb{R}$ strictly positive on strictly positive forms and null on exact forms with respect to the leafwise exterior derivative $d_\mathcal{F}$. Thus, Sullivan's result identifies the space of transverse invariant measures with the positive cone $H_d^+(\mathcal{F})$ in the d-th homology group of foliated currents. Any averaging sequence V_n gives us the sequence of foliated currents

$$\xi_n(\alpha) = \frac{1}{\mathrm{vol}(V_n)} \int_{V_n} \alpha$$

where α is a foliated d-form. By passing to a subsequence, if necessary, we have a limit current $\xi = \lim_{n\to\infty} \xi_n$. Since the boundaries of the domains V_n vanish asymptotically, Stokes' theorem implies that ξ is a foliated d-cycle [21].

3. Averaging sequences in the discrete setting

The main objective of this section is to prove the existence of a harmonic measure for an étale equivalence relation $\mathcal{R}$ using some kind of averaging sequence. The equivalence relation $\mathcal{R}$ is given by the action of a pseudogroup Γ on a compact space T, but all the results are still valid if we take a compactly generated pseudogroup acting on a locally compact Polish space. The main result is proved in section 3.2.

3.1. **Quasi-invariants measures.** Let ν be a quasi-invariant measure on T. Integrating the counting measures on the fibers of the left projection $\beta(y,z) = y$ with respect to ν gives the *left counting measure* $d\widetilde{\nu}(y,z) = d\nu(y)$. The same is valid for the right projection $\alpha(y,z) = z$ and we get the *right counting measure* $d\widetilde{\nu}^{-1}(y,z) = d\widetilde{\nu}(z,y) = d\nu(z)$. Then $\widetilde{\nu}$ and $\widetilde{\nu}^{-1}$ are equivalent measures if and only if ν is quasi-invariant, in which case the Radon-Nikodym derivative is given by $\delta(y,z) = d\widetilde{\nu}/d\widetilde{\nu}^{-1}(y,z)$. The map δ is known as the *Radon-Nikodym cocycle* of $(\mathcal{R}, T, \nu)$. We will denote $|\cdot|_z$ the measure on $\mathcal{R}[y]$ given by $|y|_z = \delta(y,z)$.

More generally, a *cocycle with values in* $\mathbb{R}^*_+$ is a map $\delta : \mathcal{R} \to \mathbb{R}^*_+$ satisfying $\delta(x,y)\delta(y,z) = \delta(x,z)$ for all $(x,y),(y,z) \in \mathcal{R}$. As we mentioned in the introduction, given a continuous cocycle $\delta : \mathcal{R} \to \mathbb{R}^*_+$, the *Radon-Nikodym problem* is to determine the set of probability measures ν on T which are quasi-invariant and admit δ as their Radon-Nikodym derivative.

3.2. **Discrete averaging sequences.** We are interested in giving a sufficient condition to solve the Radon-Nikodym problem in the discrete setting. We will state this condition using the notion of modified averaging sequence of V. A. Kaimanovich (see [13] and [15]):

Definition 3.1. Let $\delta : \mathcal{R} \to \mathbb{R}^*_+$ be a cocycle of $\mathcal{R}$. Let $\{A_n\}$ be a sequence of finite subsets of T such that $A_n \subset \mathcal{R}[y_n]$ for each $n \in \mathbb{N}$. We will say that $\{A_n\}$ is a δ-*averaging sequence for* Γ if

$$\lim_{n\to\infty} \frac{|\Delta_\gamma A_n|_{y_n}}{|A_n|_{y_n}} = 0$$

for all $\gamma \in \Gamma^{(1)}$. An equivalence class $\mathcal{R}[y]$ is δ-*Følner* if $\mathcal{R}[y]$ contains an δ-averaging sequence $\{A_n\}$ such that $|\partial A_n|_y/|A_n|_y \to 0$ as $n \to +\infty$.

Definition 3.2. A measure ν on T is *harmonic* or *stationary* (for the simple random walk on $\mathcal{R}$) if it satisfies any one of the following equivalent properties (see [19]):
(i) for every bounded measurable function $f : T \to \mathbb{R}$, we have $\int \Delta f \, d\nu = 0$;
(ii) $D^*\nu = \nu$;
(iii) the Radon-Nikodym cocycle $\delta : \mathcal{R} \to \mathbb{R}^*_+$ is harmonic, *i.e.* for ν-almost every $y \in T$ and every $z \in \mathcal{R}[y]$, we have

$$\delta(z,y) = \frac{1}{deg(z)} \sum_{w\sim z} \delta(w,y) \tag{3.1}$$

where $w \sim z$ means that w is a neighbor of z in the graph $\mathcal{R}[y]$ and $deg(z)$ is the number of neighbors of z.

Recall that $D : L^\infty(T,\nu) \to L^\infty(T,\nu)$ is the Markov operator defined by

$$Df(y) = \frac{1}{deg(y)} \sum_{z\sim y} f(z),$$

D^* is the dual operator acting on the space of positive Borel measures on T, and $\Delta : L^\infty(T,\nu) \to L^\infty(T,\nu)$ is the Laplace operator defined by $\Delta f(y) = Df(y) - f(y)$. Finally, we will use $\mathcal{D}$ to denote the set of discontinuities of the degree function *deg*.

Theorem 3.3. *Let $\mathcal{R}$ be the orbit equivalence relation defined by a finitely generated pseudogroup Γ acting on a compact space T. Let $\delta : \mathcal{R} \to \mathbb{R}^*_+$ be a continuous cocycle. Any δ-averaging sequence $\{A_n\}$ gives rise to a positive Borel measure ν on T whose support is contained in the limit set of $\{A_n\}$, which is quasi-invariant and has δ as Radon-Nikodym cocycle. Moreover, if we assume that δ is harmonic and $\nu(\mathcal{D}) = 0$, then ν is a harmonic measure.*

Proof. We start by constructing a sequence of probability measures ν_n given by $\nu_n(B) = |B \cap A_n|_{y_n}/|A_n|_{y_n}$ for every Borel subset B of T. By passing to a subsequence, the sequence ν_n converges in the weak topology to a positive Borel measure ν on T. First, we will prove that ν is a quasi-invariant measure having a Radon-Nikodym cocycle equal to δ. For every local transformation $\gamma \in \Gamma$ an every function $f \in C(T)$ with support on the range of γ, we have

$$\int f(z)\, d(\gamma_*\nu)(z) = \int f(\gamma(y))\, d\nu(y) = \lim_{n\to\infty} \frac{1}{|A_n|_{y_n}} \sum_{y\in A_n} f(\gamma(y))\delta(y, y_n)$$

and

$$\begin{aligned} \int f(y)\delta(z, y)\, d\nu(y) &= \lim_{n\to\infty} \frac{1}{|A_n|_{y_n}} \sum_{y\in A_n} f(y)\delta(\gamma(y), y)\delta(y, y_n) \\ &= \lim_{n\to\infty} \frac{1}{|A_n|_{y_n}} \sum_{y\in A_n} f(y)\delta(\gamma(y), y_n) \end{aligned}$$

where $z = \gamma(y)$. Therefore

$$\begin{aligned} 0 &\leq \left| \int f(z)\, d(\gamma_*\nu)(z) - \int f(y)\delta(z, y)\, d\nu(y) \right| \\ &\leq \lim_{n\to\infty} \frac{1}{|A_n|_{y_n}} \left| \sum_{y\in A_n} f(\gamma(y))\delta(y, y_n) - f(y)\delta(\gamma(y), y_n) \right| \\ &\leq \lim_{n\to\infty} \|f\|_\infty \frac{|\Delta_\gamma A_n|_{y_n}}{|A_n|_{y_n}} \;=\; 0 \end{aligned}$$

and thus

$$\int f(z)\, d(\gamma_*\nu)(z) = \int f(y)\delta(z, y)\, d\nu(y),$$

proving the claim.

On the other hand, if $\nu(\mathcal{D}) = 0$, then

$$\int \Delta f\, d\nu = \lim_{n\to\infty} \int \Delta f\, d\nu_n$$

for all $f \in C(T)$. If δ is harmonic, we have

$$\begin{aligned}
\int \Delta f(y)\, d\nu_n(y) &= \frac{1}{|A_n|_{y_n}} \sum_{y\in A_n} \left(\frac{1}{deg(y)} \sum_{z\sim y} f(z) - f(y) \right) \delta(y,y_n) \\
&= \frac{1}{|A_n|_{y_n}} \sum_{y\in A_n} \frac{1}{deg(y)} \sum_{z\sim y} f(z)\delta(y,y_n) - f(y) \left(\frac{1}{deg(y)} \sum_{z\sim y} \delta(z,y_n) \right) \\
&= \frac{1}{|A_n|_{y_n}} \sum_{y\in A_n} \frac{1}{deg(y)} \sum_{z\sim y} f(z)\delta(y,y_n) - f(y)\delta(z,y_n)
\end{aligned}$$

and then

$$\begin{aligned}
0 &\leq \left| \int \Delta f(y)\, d\nu(y) \right| \\
&\leq \lim_{n\to\infty} \frac{1}{|A_n|_{y_n}} \left| \sum_{y\in A_n} \sum_{z\sim y} f(z)\delta(y,y_n) - f(y)\delta(z,y_n) \right| \\
&\leq \lim_{n\to\infty} \|f\|_\infty \sum_{\gamma\in\Gamma^{(1)}} \frac{|\Delta_\gamma A_n|_{y_n}}{|A_n|_{y_n}} \leq \lim_{n\to\infty} 2\,\|f\|_\infty\, |\Gamma^{(1)}|\, \frac{|\partial A_n|_{y_n}}{|A_n|_{y_n}} = 0. \qquad \square
\end{aligned}$$

In general, the above theorem remains valid when the Laplace operator Δ preserves continuous functions. This is always true when $\mathcal{D} = \emptyset$, as in the following case:

Corollary 3.4. *Let $\mathcal{R}$ be the orbit equivalence relation defined by a group of finite type Γ acting freely on a compact space T. Let $\delta : \mathcal{R} \to \mathbb{R}^*_+$ be a continuous harmonic cocycle. Any δ-averaging sequence $\{A_n\}$ gives rise to a harmonic measure ν on T supported by the limit set of $\{A_n\}$.*

Arguing as in the invariant case, we can extend Theorem 3.3 to any compactly generated pseudogroup Γ acting on a locally compact Polish space T. Moreover, in the 0-dimensional case, the degree function is again continuous. This applies in particular to solenoids [3] and foliated spaces defined by repetitive graphs (introduced in [8] and studied in [1], [4] and [17]):

Corollary 3.5. *Let $\mathcal{R}$ be the orbit equivalence relation defined by compactly generated pseudogroup Γ acting on a locally compact separable 0-dimensional space T. Let $\delta : \mathcal{R} \to \mathbb{R}^*_+$ be a continuous harmonic cocycle. Any δ-averaging sequence $\{A_n\}$ gives rise to a harmonic measure ν on T supported by the limit set of $\{A_n\}$.*

4. Averaging sequences in the continuous setting

We are interested in stating the previous results in the continuous setting, namely for a foliated space $(M, \mathcal{F})$. Instead of working with quasi-invariant measures, we are going to use tangentially smooth measures. These were introduced in [2] and form a larger class than harmonic measures. As previously mentioned, transverse invariant measures for foliations are rather rare, but harmonic measures always exist. Harmonic measures were introduced by L. Garnett in [7].

4.1. **Tangentially smooth measures.** Consider now a regular Borel measure μ on M. Using a C^r foliated atlas $\mathcal{A}$, we can give a local decomposition $\mu = \int \lambda_i^y d\nu_i(y)$ on each foliated chart U_i, where λ_i^y is a measure on the plaque $\varphi_i^{-1}(P_i \times \{y\})$ and ν_i a measure on T_i. Assume that $\mathcal{F}$ is a C^r-foliation.

Definition 4.1 ([2])**.** A measure μ on M is *tangentially smooth* if for every $i \in I$ and ν_i-almost every $y \in T_i$, the measures λ_i^y are absolutely continuous with respect to the Riemannian volume *dvol* restricted to the plaque passing through y, and the density functions $h_i(x,y) = d\lambda_i^y/dvol(x,y)$ are smooth functions of class C^{r-1} on the plaques.

Observe that the local decomposition of μ is not necessarily unique. Let $\mu|_{U_i} = \int \lambda_i^y d\nu_i(y) = \int \bar{\lambda}_i^y d\bar{\nu}_i(y)$ be two decompositions. Then we obtain

$$\int_{T_i}\int_{P_i\times\{y\}} h_i(x,y)\,dvol(x,y)\,d\nu_i(y) = \int_{T_i}\int_{P_i\times\{y\}} \bar{h}_i(x,y)\,dvol(x,y)\,d\bar{\nu}_i(y),$$

and we can consider the Radon-Nikodym derivative $\delta_i(y) = d\nu_i/d\bar{\nu}_i(y)$ such that $\bar{h}_i(x,y) = \delta_i(y)h_i(x,y)$. This situation arises naturally in the intersection of two foliated charts U_i and U_j. Indeed, if $U_i \cap U_j \neq \emptyset$, we have that $\mu|_{U_i\cap U_j} = \int \lambda_i^y d\nu_i(y) = \int \lambda_j^y d\nu_j(y)$. Thus, as before, we deduce that

$$\delta_{ij}(y) = d\nu_i/d((\gamma_{ji})_*\nu_j)(y) = \frac{h_j(\varphi_{ij}^y(x), \gamma_{ij}(y))}{h_i(x,y)}. \tag{4.1}$$

Then the functions h_i verify that $\log h_j - \log h_i = \log \delta_{ij}$ on $U_i \cap U_j$. Since δ_{ij} is a function on T_i, we have that $d_{\mathcal{F}} \log h_i = d_{\mathcal{F}} \log h_j$. Then $\eta = d_{\mathcal{F}} \log h_i$ is a well-defined foliated 1-form of class C^{r-2} along the leaves, which makes it possible to estimate the transverse measure distortion under the holonomy.

Definition 4.2. The foliated 1-form η is the *modular form* of μ.

Since the functions h_i coincide on the intersections of the plaques modulo multiplication by a constant, they define a primitive of the induced 1-form on the holonomy covering of each leaf L. If $\mathcal{F}$ has no essential holonomy, the functions $\log h_i$ can be glued together to obtain a measurable primitive $\log h$ of η.

According to [7], any harmonic measure is an example of tangentially smooth measure since the densities h_i are positive harmonic functions of class C^{r-1} on the plaques. In particular, any transverse invariant measure combined with the Riemannian volume on the leaves gives a harmonic measure which is called *completely invariant*. A harmonic measure μ is completely invariant if and only if $\eta = 0$ (we refer to corollary 5.5 of A. Candel's paper [5]).

4.2. **Modular form associated with a cocycle.** We have just associated a 1-foliated form to any tangentially smooth measure. Since our objective is to construct a measure for $(M, \mathcal{F})$ starting with a continuous cocycle and an averaging sequence, we will now describe the construction of a continuous modular form $\eta \in \Omega^1(\mathcal{F})$ associated with any continuous cocycle $\delta : \mathcal{R} \to \mathbb{R}_+^*$. We will start by considering the tangentially C^r-smooth continuous functions $c_{ki} : U_i \cap U_k \to \mathbb{R}$ given by $c_{ki}(\varphi_k^{-1}(x,y)) = log\,\delta_{ki}(y)$ where $\delta_{ki}(y) = \delta(y, \gamma_{ki}(y))$ for all $(x,y) \in P_k \times T_k$. By choosing a tangentially C^r-smooth partition of unity $\{\rho_i\}_{i=1}^m$ subordinated to the foliated atlas $\mathcal{A}$, we can glue the functions c_{ki} obtaining tangentially C^r-smooth continuous functions $c_i : U_i \to \mathbb{R}$ given by $c_i = \sum_{k=1}^m \rho_k c_{ki}$. The cocycle condition implies that $c_{ij} = c_{kj} - c_{ki}$, so that

$$c_j - c_i = \sum_{k=1}^m \rho_k c_{kj} - \sum_{k=1}^m \rho_k c_{ki} = \left(\sum_{k=1}^m \rho_k \right) c_{ij} = c_{ij}.$$

Hence, for each $i = 1, \dots, m$ we can define a tangentially C^{r-1}-smooth continuous foliated 1-form $\eta_i = \sum_{k=1}^m (d_{\mathcal{F}} \rho_k)\, c_{ki}$ on U_i. Each local 1-form η_i is exact

$$\eta_i = \sum_{k=1}^m (d_{\mathcal{F}} \rho_k)\, c_{ki} = d_{\mathcal{F}} c_i = d_{\mathcal{F}} \log h_i$$

where $h_i = e^{c_i} : U_i \to \mathbb{R}_+^*$ is a continuous function of class C^r along the leaves.

Proposition 4.3. *There is a well-defined continuous foliated* 1*-form* $\eta \in \Omega^1(\mathcal{F})$*, that will be called the* modular form of δ*, such that* $\eta|_{U_i} = \eta_i$*.*

Proof. For each pair $i, j \in \{1, \dots, m\}$, we have that:

$$\eta_j - \eta_i = \sum_{k=1}^m (d_{\mathcal{F}} \rho_k)\, c_{kj} - \sum_{k=1}^m (d_{\mathcal{F}} \rho_k)\, c_{ki} = \left(\sum_{k=1}^m d_{\mathcal{F}} \rho_k \right) c_{ij} = 0$$

on $U_i \cap U_j$. Then the 1-form η is well defined and continuous. □

Remark 4.4. The modular form η of a continuous cocycle δ admits a continuous primitive $\log h$ on the residual set of leaves without holonomy. In general, by passing to the holonomy covers of the leaves, we may find a global continuous primitive on the holonomy groupoid $Hol(\mathcal{F})$, see [2].

4.3. **Modular form of a harmonic measure.** In the case when μ is a harmonic measure (and assuming that M is compact), we know from the Ph.D. thesis of B. Deroin [6] (see Lemma 4.19 on page 116) that the modular form η is bounded. Let us look at the proof in order to specify properties of the primitive $\log h$ and the cocycle δ. Let $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ be a good C^r foliated atlas of $(M, \mathcal{F})$, and h_i the local density functions of μ. Assume that $\mathcal{A}$ is a refinement of a good atlas $\mathcal{A}' = \{(U_i', \phi_i')\}_{i \in I}$, and h_i' are the corresponding local densities. Thus, every plaque of U_i is relatively compact in a plaque of U_i'. In fact, using a vertical reparameterization, we can suppose that $\phi_i^{-1}(P_i \times \{y\}) \subset (\phi_i')^{-1}(P_i' \times \{y\})$ for every $y \in T_i$. There exists a relatively compact open set $V \subset P_i'$ such that $\phi_i^{-1}(P_i \times \{y\}) \subset (\phi_i')^{-1}(V \times \{y\})$ for every $y \in T_i$. Since h_i is harmonic, the Harnack inequality implies the existence of a constant $C_i > 0$ such that

$$\frac{1}{C_i} h_i(x_0, y) \leq h_i(x, y) \leq C_i h_i(x_0, y), \tag{4.2}$$

for all $x, x_0 \in P_i$ and for all $y \in T_i$. Since the atlases $\mathcal{A}$ and $\mathcal{A}'$ are finite, we have the following result:

Proposition 4.5. *If* μ *is a harmonic measure, then* η *is a bounded foliated* 1*-form having a measurable uniformly tangentially Lipschitz primitive* $\log h$*.* □

In fact, if we replace the plaques $\phi_i^{-1}(P_i \times \{y\})$ by disjoint disks E_y around the points $\phi_i^{-1}(0, y)$ and if we denote the plaques $(\phi_i')^{-1}(V \times \{y\})$ as V_y, then the axis T of $\mathcal{A}$ is $*$*-recurrent* in the sense of [18], see also [12]. For each point $p \in M$, let L_p be the leaf of $\mathcal{F}$ passing through p. According to Theorem 5 of [18], there is an assignment of a probability measure π_p on $L_p \cap T$ satisfying:

(i) $\pi_p(q) > 0$ for each point $q \in L_p \cap T$;
(ii) $f(p) = \sum_{q \in L_p \cap T} \pi_p(q) f(q)$ for each bounded harmonic function f on L_p.

If both p and q belong to the geometric realization of T, and and as a result they are identified with two points y and z of T, we obtain a random walk on $\mathcal{R}$ with transition probabilities $\pi(y,z) = \pi_p(q)$ for all $(y,z) \in \mathcal{R}$. Thus, the Radon-Nikodym cocycle $\delta(y,z) = \delta(y, \gamma_{ij}(y)) = \delta_{ij}(y)$ given by (4.1) has the following property:

Proposition 4.6. *If μ is a harmonic measure, then the Radon-Nikodym cocycle is π-harmonic, i.e. $\delta(z,y) = \sum_{w \in \mathcal{R}[y]} \pi(z,w)\delta(w,y)$ for ν-almost every point $y \in T$ and every $z \in \mathcal{R}[y]$.* □

Reciprocally, if we choose V in such way that the sets V_y are pairwise disjoint, then Theorem 1 of [12] tells us that the modular form of a π-harmonic cocycle has a primitive $\log h$ with h harmonic. Thus, by replacing the simple random walk with the random walk whose transition kernel is π, the discrete and continuous approaches become equivalent from the point of view of harmonicity:

Proposition 4.7. *A tangentially smooth measure μ is harmonic if and only if the measure ν (well defined up to equivalence) is π-harmonic.* □

4.4. **Continuous averaging sequences.** In the present setting, we can reformulate the *Radon-Nikodym problem* as the problem of determining tangentially smooth measures μ on M which admit η as their modular form. The aim of this section is to establish Theorem 3.3 for foliations. First, we need a continuous analogue of Definition 3.1. Consider a d-dimensional foliation $\mathcal{F}$ of class C^r on a compact space M, endowed with a tangentially C^r-smooth Riemannian metric g, and a continuous cocycle $\delta : \mathcal{R} \to \mathbb{R}^*_+$. The modular form η admits a continuous tangentially C^r-smooth primitive $\log h$ on the residual set of leaves without holonomy. In restriction to each leaf without holonomy L_y passing through $y \in T$, we can multiply g by the normalized density function $h/h(y)$ in order to obtain a *modified metric* $(h/h(y))g$.

Definition 4.8. Let $\{V_n\}$ be a sequence of compact domains with boundary contained in a sequence of leaves without holonomy L_{y_n}. We will say that $\{V_n\}$ is a *η-averaging sequence for $\mathcal{F}$* if

$$\lim_{n\to\infty} \frac{\mathrm{area}_\eta(\partial V_n)}{\mathrm{vol}_\eta(V_n)} = 0$$

where area_η denotes the $(d-1)$-volume and vol_η the d-volume with respect to the modified metric along L_{y_n}. A leaf L_y is *η-Følner* if it contains an η-averaging sequence $\{V_n\}$ such that $\mathrm{area}_\eta(\partial V_n)/\mathrm{vol}_\eta(V_n) \to 0$ as $n \to +\infty$.

Remarks 4.9. (i) The isoperimetric ratio $\mathrm{area}_\eta(\partial V_n)/\mathrm{vol}_\eta(V_n)$ does not depend on the choice of y and h in the second definition. This justifies the notation, which is slightly different from that used in [2].
(ii) When μ is a completely invariant harmonic measure, the normalized density function is equal to 1 and thus the modified volume and the Riemannian volume coincide. Hence, we recover the common definition of averaging sequence.
(iii) In general, for harmonic measures, Harnack's inequalities (4.2) imply that the modified volume of the plaques and the modified area of their boundaries remain uniformly bounded. Repeating the same argument as in the classical case, we have that *the leaf L_y is η-Følner if and only if the graph $\mathcal{R}[y]$ is δ-Følner.*

Theorem 4.10. *Let $(M, \mathcal{F})$ be a compact foliated space of class C^r, $2 \leq r \leq \infty$ or $r = \omega$, and let $\mathcal{R}$ be the equivalence relation induced by $\mathcal{F}$ on a total transversal T. Consider a continuous cocycle $\delta : \mathcal{R} \to \mathbb{R}_+^*$, and let η be the modular form of δ. Assume that $\mathcal{F}$ admits a foliated atlas such that the modified volume of the plaques is bounded. Any η-averaging sequence $\{V_n\}$ for $\mathcal{F}$ gives rise to a tangentially smooth measure μ whose support is contained in the limit set of $\{V_n\}$ and whose modular form is equal to η. In particular, if we assume that η has a primitive $\log h$ such that h is harmonic, then μ is a harmonic measure.*

Proof. As in the discrete case, we will start by constructing a sequence of foliated d-currents

$$\xi_n(\alpha) = \frac{1}{\mathrm{vol}_\eta(V_n)} \int_{V_n} \frac{h}{h(y_n)} \alpha,$$

where α is a foliated d-form. By passing to a subsequence, the sequence ξ_n converges to a foliated d-current ξ. Let μ be the measure on M associated with the current ξ. For every function $f \in C(T)$, we have $\int f \, d\mu = \xi(f\omega)$ where $\omega = dvol$ is the volume form along the leaves.

Now, we will prove that μ is a tangentially smooth measure with modular form η. Consider a good C^r foliated atlas $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ obtained by refinement from a given good atlas, and whose plaques have bounded modified volume. Since the modified volume of the plaques of $\mathcal{A}$ and the modified area of their boundaries remain bounded, the traces $A_n = V_n \cap T$ of the domains V_n on the total transversal T form a δ-averaging sequence. In fact, since V_n is covered by the plaques P_y of $\mathcal{A}$ centered at the points y of A_n, we have that:

$$\mathrm{vol}_\eta(V_n) = \int_{V_n} \omega_\eta \leq \sum_{y \in A_n} \int_{P_y} \omega_\eta = \sum_{y \in A_n} \left(\int_{P_y} \frac{h(x,y)}{h(0,y)} dvol(x,y) \right) \delta(y, y_n)$$

where ω_η is the modified volume form along the leaves and $h(x, y)$ denotes the density function restricted to a foliated chart U_y containing the plaque P_y. Then there is a constant $C > 0$ such that $\mathrm{vol}_\eta(V_n) \leq C|A_n|_{y_n}$ Actually, we can choose $C > 0$ such that $\frac{1}{C} \leq \mathrm{vol}_\eta(V_n)/|A_n|_{y_n} \leq C$. Thus, by passing to a subsequence, we may assume that the ratio $\mathrm{vol}_\eta(V_n)/|A_n|_{y_n}$ converges to a constant $c > 0$. Now, as stated in the proof of Theorem 3.3, we may also assume that the sequence of measures $\nu_n(B) = |B \cap A_n|_{y_n}/|A_n|_{y_n}$ converge to a quasi-invariant measure ν on T whose Radon-Nikodym derivative is equal to δ. Combined with the modified Riemannian volume along the leaves, this transverse measure gives us a tangentially smooth measure μ' on M. Thus, for every function $f \in C(M)$ with support in U_i, we have

$$\int f \, d\mu' = \int_{T_i} \int_{P_i \times \{y\}} f(x,y) \frac{h_i(x,y)}{h_i(0,y)} dvol(x,y) \, d\nu(y).$$

Then

$$\begin{aligned} \int f \, d\mu' &= \lim_{n \to +\infty} \frac{1}{|A_n|_{y_n}} \sum_{y \in V_n \cap T_i} \left(\int_{P_i \times \{y\}} f(x,y) \frac{h_i(x,y)}{h_i(0,y)} \, dvol(x,y) \right) \delta(y, y_n) \\ &= \lim_{n \to +\infty} \frac{1}{|A_n|_{y_n}} \sum_{y \in V_n \cap T_i} \int_{P_i \times \{y\}} f \, \omega_\eta. \end{aligned} \tag{4.3}$$

On the other hand, by definition, we have

$$\begin{aligned} \int f\,d\mu &= \xi(f\omega) = \lim_{n\to+\infty} \frac{1}{\mathrm{vol}_\eta(V_n)} \int_{V_n} f\omega_\eta \\ &= \lim_{n\to+\infty} \frac{1}{\mathrm{vol}_\eta(V_n)} \sum_{y\in V_n\cap T_i} \int_{P_i\times\{y\}} f\,\omega_\eta \end{aligned} \tag{4.4}$$

Comparing identities (4.3) and (4.4), we deduce that $\mu = \frac{1}{c}\,\mu'$ is a tangentially smooth measure with Radon-Nikodym cocycle δ.

To conclude, we will prove that μ is harmonic when h is harmonic. Recall that according to Theorem 1 of [12], see also Proposition 4.7, this is always the case when δ is π-harmonic. We will start by denoting $h_n = h/h(y_n)$ as the normalized density function on the leaf L_{y_n}. Since the Laplace operator $\Delta_{\mathcal{F}}$ preserves continuous functions, we have that

$$\int \Delta_{\mathcal{F}} f\,d\mu = \lim_{n\to\infty} \frac{1}{\mathrm{vol}_h(V_n)} \int_{V_n} (\Delta_{\mathcal{F}} f)\,h_n\,\omega,$$

for all $f \in C(T)$. Green's formula implies that

$$\int_{V_n} (\Delta_{\mathcal{F}} f)\,h_n\,\omega = \int_{V_n} ((\Delta_{\mathcal{F}} f)\,h_n - f\,(\Delta_{\mathcal{F}} h_n)\,\omega = \int_{\partial V_n} h_n\,i_{grad(f)}\omega - f\,i_{grad(h_n)}\omega.$$

Since h_n is harmonic, we have

$$\int_{\partial V_n} \iota_{grad(h_n)}\omega = \int_{V_n} \mathrm{div}(grad(h_n))\,\omega = \int_{V_n} (\Delta_{\mathcal{F}} h_n)\,\omega = 0$$

and then

$$0 \leq \left| \int_{\partial V_n} f\iota_{grad(h_n)}\omega \right| \leq \|f\|_\infty \int_{\partial V_n} \iota_{grad(h_n)}\omega = 0$$

for all $n \in \mathbb{N}$. On the other hand, since f is bounded, there exists a constant $k > 0$ depending only on f such that we have

$$0 \leq \left| \frac{1}{\mathrm{vol}_h(V_n)} \int_{\partial V_n} h_n\,\iota_{grad(f)}\omega \right| \leq \lim_{n\to\infty} k\frac{\mathrm{area}_\eta(\partial V_n)}{\mathrm{vol}_\eta(V_n)} = 0$$

and therefore

$$\int \Delta_{\mathcal{F}} f\,d\mu = \lim_{n\to\infty} \frac{1}{\mathrm{vol}_h(V_n)} \int_{V_n} (\Delta_{\mathcal{F}} f)\,h_n\,\omega = 0. \qquad \square$$

According to Remark 4.4, the notion of η-Følner remains valid for the holonomy covers of the leaves of $\mathcal{F}$. Thus, it suffices to replace $\mathcal{F}$ with the lifted foliation in the holonomy groupoid $Hol(\mathcal{F})$ in order to globalize the previous result.

5. Examples

5.1. **Discrete averaging sequences for amenable non Følner actions.** There are amenable actions of non amenable discrete groups whose orbits contain averaging sequences [15]. For example, let $\partial\Gamma$ be the space of ends of the free group Γ with two generators γ_1 and γ_2 whose elements are infinite words $x = \gamma_1\gamma_2\dots$ with letters γ_n in $\Phi = \{\gamma_1^{\pm1}, \gamma_2^{\pm1}\}$. If ν denotes the equidistributed probability measure on $\partial\Gamma$ (such that all cylinders consisting of infinite words with fixed first n letters have the same measure), then Γ acts essentially freely on $\partial\Gamma$ by sending each generator γ and each infinite word $x = \gamma_1\gamma_2\dots$ to $\gamma.x = \gamma\gamma_1\gamma_2\dots$. Since this action is amenable, according to Theorem 2 of [13] (see also Proposition 4.1 of [2]), we know that ν-almost every orbit is δ-Følner (where δ is the Radon-Nikodym derivative of ν). We will recall here an explicit construction by V. A. Kaimanovich in [15].

For each $x \in \partial\Gamma$, let $b_x : \Gamma \to \mathbb{R}$ be the *Busemann function* defined by

$$b_x(\gamma) = lim_{n\to+\infty}\big(d_\Gamma(\gamma, x_{[n]}) - d_\Gamma(1, x_{[n]})\big)$$

where d_Γ is the Cayley graph metric, $x_{[n]}$ is the word consisting of first n letters of x and 1 is the identity element. The level sets $H_k(x) = \{\, \gamma \in \Gamma \,/\, b_x(\gamma) = k \,\}$ are the *horospheres* centered at x. The Radon-Nikodym derivative of ν is given by

$$\delta(\gamma^{-1}.x, x) = \frac{d\gamma.\nu}{d\nu}(x) = 3^{-b_x(\gamma)}$$

where $\gamma.\nu$ is the translation of ν by γ. Since $|\cdot|_x = \delta(\cdot, x)$ is a harmonic measure on $\Gamma.x$, ν is also a harmonic measure. In fact, as stated in Theorem 17.4 of [14], ν is the unique harmonic probability measure on $\partial\Gamma$.

Let A_n^x be the set of all points $\gamma^{-1}.x$ in $\Gamma.x$ such that $0 \le b_x(\gamma) = d_\Gamma(1,\gamma) \le n$. Since $|A_n^x \cap H_k(x)|_x = \sum_{b_x(\gamma)=d_\Gamma(1,\gamma)=k} \delta(\gamma^{-1}.x, x) = 3^k \frac{1}{3^k} = 1$ for all $0 \le k \le n$, we have that $|A_n|_x = n+1$. But $\partial A_n^x = \{1\} \cup (A_n^x \cap H_n(x))$ and so $|\partial A_n^x|_x = 2$. The δ-averaging sequence $\{A_n^x\}$ defines a harmonic measure (which is equal to ν up to multiplication by a constant).

5.2. **Averaging sequences for hyperbolic surfaces.** The geodesic and horocycle flows are classical examples of flows on the unitary tangent bundle of a compact hyperbolic surface. They are given by the right actions of the diagonal subgroup

$$D = \left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

and the unipotent subgroup

$$H^+ = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \,\middle|\, s \in \mathbb{R} \right\}$$

of $G = PSL(2,\mathbb{R})$ on the quotient $\Gamma\backslash G$ by the left action of a uniform lattice Γ. If $\mathbb{H}$ denotes the hyperbolic plane, we can identify $\Gamma\backslash G$ with the unitary tangent bundle of the compact hyperbolic surface $\Gamma\backslash\mathbb{H}$. The right action of the normalizer A of H^+ in $PSL(2,\mathcal{R})$ defines a foliation $\mathcal{F}$ by Riemann surfaces on $\Gamma\backslash G$. Since A is an amenable group, $\mathcal{F}$ is an amenable non Følner foliation. Moreover, there is an invariant measure μ on $\Gamma\backslash G$. In [7], L. Garnett proved that μ is a harmonic measure by describing its density function on a foliated chart.

We can identify G/A with the boundary $\partial\mathbb{H}$ by sending each coset of A in G to the center of the horocycle defined by the corresponding coset of H^+ in G. For each point $z \in \partial\mathbb{H}$, there is a unique probability measure ν_z on $\partial\mathbb{H}$ which is invariant by the action of all isometries of $\mathbb{H}$ fixing z. This measure is the image of the normalized Lebesgue measure on the circle of the tangent plane at z under the exponential map, and is called the *visual measure* at z. According to Proposition 2 of [7], the normalized density function is given by $d\nu_z/d\nu_{z_0}(x)$ where $z, z_0 \in \mathbb{H}$ and $x \in \partial\mathbb{H}$. In particular, for $x = \infty$, we have that

$$\frac{d\nu_z}{d\nu_{z_0}}(\infty) = \frac{y}{y_0}$$

where $z = x+iy$ and $z_0 = x_0+iy_0$. In the leaf passing through $x = \infty$, the sequence $V_n^\infty = \{\, z \in \mathbb{H} \,|\, -1 \leq x \leq 1\,,\, e^{-n} \leq y \leq 1 \,\}$ becomes a η-averaging sequence (where η is the modular form of μ). Indeed, on the one hand, we have that

$$\text{area}_\eta(V_n^\infty) = \int_{V_n^\infty} \frac{d\nu_z}{d\nu_i}(\infty)\, dvol(z) = \int_{V_n^\infty} y\,\frac{dx \wedge dy}{y^2} = \int_1^1 dx \int_{e^{-n}}^1 \frac{dy}{y} = 2n.$$

On the other hand, the modified length of a smooth curve $\sigma(t) = x(t) + iy(t)$ (with $0 \leq t \leq l$) is given by $\text{length}_\eta(\sigma) = \int_0^l \sqrt{x'(t)^2 + y'(t)^2}dt$, and so we have that

$$\text{length}_\eta(\partial V_n^\infty) = 2(2 + (1 - e^n)) \leq 6.$$

As before, this η-averaging sequence defines a harmonic measure (which is equal to μ up to multiplication by a constant). In fact, all leaves are η-Følner since for each point $x \in \partial\mathbb{H}$ obtained as the image of ∞ under $g \in G$, the sets $V_n^x = g(V_n^\infty)$ form a η-averaging sequence in the leaf passing through x.

5.3. **Averaging sequences for torus bundles over the circle.** To conclude, we will present other examples of foliations on homogeneous spaces studied by É. Ghys and V. Sergiescu in [9]. Each matrix $A \in SL(2,\mathbb{Z})$ with $|tr(A)| > 2$ defines a natural representation $\varphi : \mathbb{Z} \to Aut(\mathbb{Z}^2)$ which extends to a representation $\Phi : \mathbb{R} \to Aut(\mathbb{R}^2)$ given by $\Phi(t) = A^t$. If $\lambda > 1$ and $\lambda^{-1} < 1$ are the eigenvalues of A, then Φ is conjugated to the representation Φ_0 given by

$$\Phi_0(t) = \begin{pmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{pmatrix}.$$

Let T_A^3 be the homogeneous space obtained as the quotient of the Lie group $G = \mathbb{R}^2 \rtimes_\Phi \mathbb{R}$ with group law $(x,y,t).(x',y',t') = ((x,y)+A^t(x',y'), t+t')$ by the uniform lattice $\Gamma = \mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}$ with a similar law. Observe that G is isomorphic to the solvable group $Sol^3 = \mathbb{R}^2 \rtimes_{\Phi_0} \mathbb{R}$ with group law $(x,y,t).(x',y',t') = (x+\lambda^t x', y+\lambda^{-t}y', t+t')$ (where x and y are the first and second coordinate with respect to the eigenbasis) and T_A^3 is diffeomorphic to the quotient of Sol^3 by a uniform lattice Γ_0. The right action of the image A of the monomorphism

$$(a,b) \in \mathbb{R} \ltimes \mathbb{R}^* \mapsto \left(a, 0, \frac{\log b}{\log \lambda}\right) \in Sol^3$$

defines a foliation $\mathcal{F}$ on T_A^3. The Lebesgue measure on T_A^3 defined by the volume form $\Omega = dx \wedge dy \wedge dt$ is a tangentially smooth measure. Since the Riemannian volume along the right orbits is given by

$$\frac{da \wedge db}{b^2} = (\log \lambda)\, \lambda^{-t} dx \wedge dt$$

the density function is equal to $\frac{\lambda^t}{\log \lambda}$. In the orbit of the identity element, the sequence $V_n = \{\, (a,b) \in A \,/\, -1 \leq a \leq 1\,,\, e^{-n \log \lambda} \leq b \leq 1 \,\}$ becomes a η-averaging sequence (where η is the modular form of μ). Indeed, on one hand, we have that

$$\text{area}_\eta(V_n) = \int_{V_n} \frac{1}{\log \lambda} \lambda^t (\log \lambda) \lambda^{-t} dx \wedge dt = \int_1^1 dx \int_{-n}^0 dt = 2n.$$

On the other hand, the modified length of a smooth curve $\sigma(t) = (a(t), b(t)$ (with $0 \leq t \leq L$) is given by $\text{length}_\eta(\sigma) = \int_0^L \sqrt{a'(t)^2 + b'(t)^2}dt$, and so we have that

$$\text{length}_\eta(\partial V_n) = 2(2 + (1 - e^{n \log \lambda})) \leq 6.$$

By replacing the orbit corresponding to $y = 0$ with another orbit, it is easy to see that all leaves are η-Følner. As in the previous example, all η-averaging sequences define (up to multiplication by a constant) the same harmonic measure: the Lebesgue measure.

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Departamento de Xeometría e Topoloxía, Universidade de Santiago de Compostela, E-15782 Santiago de Compostela (Spain)

E-mail address: `fernando.alcalde@usc.es`

Mathematics Department, Northwestern University, 2033 Sheridan Road Evanston, IL 60208-2730, USA

E-mail address: `rechtman@math.northwestern.edu`