

# A Dilution Test for the Convergence of Subseries of a Monotone Series

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**1. INTRODUCTION.** Cauchy's condensation test allows to determine the convergence of a monotone series by looking at a weighted subseries that only involves terms of the original series indexed by the powers of two. It is natural to ask whether the converse is also true: *Is it possible to determine the convergence of an arbitrary subseries*

$$\sum_{k \geq 1} a_{s(k)} = a_{s(1)} + a_{s(2)} + a_{s(3)} + \dots$$

*of a monotone series  $\sum_{n \geq 1} a_n$  by looking at a suitably weighted version of the original series?* In this note we show that the answer is affirmative and introduce a new convergence test particularly designed for this purpose.

**2. CAUCHY'S AND SCHLÖMILCH'S CONDENSATION TESTS.** Consider a series which is monotone in the sense that its terms satisfy  $a_1 \geq a_2 \geq \dots \geq 0$ . Cauchy's condensation test (e.g. [3, Theorem 2.3]) states that a monotone series  $\sum_{n \geq 1} a_n$  converges if and only if

$$\sum_{k \geq 0} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 16a_{16} + \dots$$

converges, thereby allowing to determine the convergence of a monotone series by only looking at its terms indexed by the powers of two. Schlömilch's extension (e.g. [3, Theorem 2.4]) allows to replace the powers of two by a more general subsequence  $s(1) < s(2) < s(3) < \dots$  of the positive integers, assuming that the forward differences

$$\Delta s(k) = s(k+1) - s(k) \tag{1}$$

do not grow too fast.

**Theorem 1 (Schlömilch).** *For any monotone series  $\sum_{n \geq 1} a_n$  and subsequence of the integers such that for some  $c > 0$ ,*

$$\frac{\Delta s(k+1)}{\Delta s(k)} \leq c \quad \text{for all } k \geq 1, \tag{2}$$

*the series  $\sum_{n \geq 1} a_n$  converges if and only if  $\sum_{k \geq 1} a_{s(k)} \Delta s(k)$  converges.*

Cauchy's condensation test can be recovered as a special case of Theorem 1 by substituting  $s(k) = 2^{k-1}$ . We will next present a short proof of Schlömilch's result to highlight its structural similarity to our new test given in Section 3.

*Proof of Theorem 1.* Because the terms of  $\sum_{n \geq 1} a_n$  are nonincreasing, we see that

$$a_{s(k+1)}\Delta s(k) \leq a_{s(k)} + \cdots + a_{s(k+1)-1} \leq a_{s(k)}\Delta s(k) \quad (3)$$

for all  $k \geq 1$ . Inequalities (3) combined with assumption (2) imply that

$$c^{-1}a_{s(k+1)}\Delta s(k+1) \leq a_{s(k)} + \cdots + a_{s(k+1)-1} \leq a_{s(k)}\Delta s(k).$$

By summing the above display over  $k$  we now find that

$$c^{-1} \sum_{k \geq 2} a_{s(k)}\Delta s(k) \leq \sum_{n \geq s(1)} a_n \leq \sum_{k \geq 1} a_{s(k)}\Delta s(k),$$

so that all three series above either converge or diverge together.  $\square$

**3. DILUTION TEST.** Cauchy's and Schlömilch's condensation tests are designed for determining the convergence of a monotone series by looking at a weighted subseries of the original series. We will now reverse this line of thought and prove a converse to these results, which allows to determine the convergence of a subseries of a monotone series using a weighted version of the original series. Suitable weights can be defined in terms of the forward differences (1) and the counting function

$$S(n) = \#\{k : s(k) \leq n\}$$

of a sequence  $s(1) < s(2) < \cdots$ . It is interesting to note that the growth condition of the forward differences in Theorem 1 is not needed below. The weights  $\Delta s(S(n))$  in Theorem 2 measure the distance between points of the subsequence nearest to  $n$ ; see Figure 1.

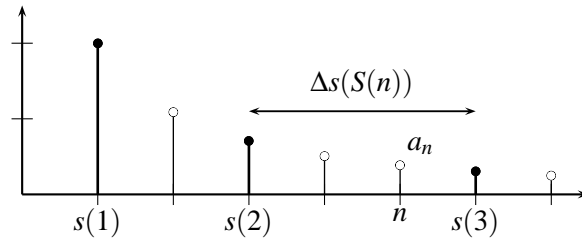


Figure 1: The weight  $\Delta s(S(n))$ .

**Theorem 2.** For any monotone series  $\sum_{n \geq 1} a_n$  and any infinite subsequence of the integers, the subseries  $\sum_{k \geq 1} a_{s(k)}$  converges if and only if

$$\sum_{n \geq s(1)} \frac{a_n}{\Delta s(S(n))} < \infty. \quad (4)$$

*Proof.* The monotonicity of the series implies the validity of (3). After dividing the terms in (3) by  $\Delta s(k)$ , we find that

$$a_{s(k+1)} \leq \frac{a_{s(k)}}{\Delta s(k)} + \cdots + \frac{a_{s(k+1)-1}}{\Delta s(k)} \leq a_{s(k)}. \quad (5)$$

Because  $S(n)$  indexes the last member of the sequence  $s(1) < s(2) < \cdots$  not exceeding  $n$ , it follows that  $S(n) = k$  for all  $n$  such that  $s(k) \leq n \leq s(k+1) - 1$ . This is why (5) may be rephrased as

$$a_{s(k+1)} \leq \sum_{n=s(k)}^{s(k+1)-1} \frac{a_n}{\Delta s(S(n))} \leq a_{s(k)}.$$

After summing the above display over  $k$ , we find that

$$\sum_{k \geq 2} a_{s(k)} \leq \sum_{n \geq s(1)} \frac{a_n}{\Delta s(S(n))} \leq \sum_{k \geq 1} a_{s(k)},$$

which shows that all series above either converge or diverge together.  $\square$

**4. THINNING OUT A DIVERGENT SERIES INTO A CONVERGENT ONE.** Given a divergent monotone series  $\sum_{n \geq 1} a_n$ , one may ask whether it can be made convergent by deleting some of its terms. If  $\lim_{n \rightarrow \infty} a_n > 0$ , this is obviously not possible, while if  $\lim_{n \rightarrow \infty} a_n = 0$  this can always be done by selecting terms of the series along a sparse enough subsequence. Indeed, in this case the series may even be thinned out to sum to an arbitrary positive real number (Banerjee and Lahiri [1]).

For a divergent monotone series  $\sum_{n \geq 1} a_n$  such that  $a_n \rightarrow 0$ , a more specific question is to quantify a sufficient degree or sparsity required for the thinning subsequence. The following corollary of the dilution test presents a sufficient condition.

**Theorem 3.** *Consider a monotone divergent series  $\sum_{n \geq 1} a_n$  such that  $\sum_{n \geq 1} a_n^p$  converges for some  $p > 1$ . A sufficient condition for the convergence of the subseries  $\sum_{k \geq 1} a_{s(k)}$  is that*

$$\sum_{k \geq 1} s(k)^{-1/p} < \infty. \quad (6)$$

*Proof.* Observe that the series (4) in Theorem 2 can be written as  $\sum_{k \geq 1} A_k$ , where

$$A_k = \frac{1}{\Delta s(k)} \sum_{n=s(k)}^{s(k+1)-1} a_n$$

is the average of the terms  $a_{s(k)}, \dots, a_{s(k+1)-1}$ . Because all these terms are less than or equal to the terms  $a_1, \dots, a_{s(k)}$ , we see that  $A_k$  is bounded from above by the average

$$B_k = \frac{1}{s(k)} \sum_{n=1}^{s(k)} a_n.$$

Jensen's inequality implies that

$$B_k^p \leq \frac{1}{s(k)} \sum_{n=1}^{s(k)} a_n^p \leq \frac{1}{s(k)} \sum_{n=1}^{\infty} a_n^p,$$

so that

$$\sum_{n \geq s(1)} \frac{a_n}{\Delta s(S(n))} = \sum_{k \geq 1} A_k \leq \sum_{k \geq 1} B_k \leq \left( \sum_{n=1}^{\infty} a_n^p \right)^{1/p} \sum_{k \geq 1} s(k)^{-1/p}.$$

The now claim follows as a consequence of Theorem 2.  $\square$

**5. SPARSE SUBSERIES OF THE HARMONIC SERIES.** The harmonic series  $\sum_{n \geq 1} \frac{1}{n}$  is probably the best-known example of a divergent series; see [3, Section 3] for a lively discussion. Kempner has shown [4] that, rather surprisingly, we obtain a convergent series by deleting from the harmonic series all terms whose decimal representation contains the digit '9'. Kempner's curious series has afterwards attracted lots of interest, with several articles generalizing and sharpening the basic result; see for example Schmelzer and Baillie [6] and references therein.

The following corollary of Theorem 3 shows that the harmonic series converges over any polynomially sparse subsequence. A subsequence of the integers is called *polynomially sparse* if its density among the first  $n$  positive integers decreases fast enough as  $n$  grows, according to

$$S(n)/n \leq cn^{-\alpha} \tag{7}$$

for some  $c > 0$  and  $\alpha \in (0, 1)$ . A simple counting argument (e.g. Behforooz [2]) may be used to verify that Kempner's no-'9' fulfills (7) with  $c = 10$  and  $\alpha = 1 - \frac{\log 9}{\log 10}$ .

**Theorem 4.** *The harmonic series converges over any polynomially sparse subsequence.*

*Proof.* Fix an integer  $k \geq 1$ , and let  $c$  and  $\alpha$  be such that (7) holds for all  $n$ . The definition of the counting function implies that  $s(k) \geq n + 1$  for all integers  $n$  such that  $S(n) < k$ , and in particular for all integers  $n$  such that  $n < (k/c)^\beta$ , where  $\beta = 1/(1 - \alpha)$ . By letting  $n$  be the largest integer strictly less than  $(k/c)^\beta$ , we see that  $s(k) \geq (k/c)^\beta$ . Therefore, condition (6) of Theorem 3 is valid for any  $p \in (1, \beta)$ . The claim now follows by Theorem 3, because the series  $\sum_{n \geq 1} n^{-p}$  converges for all  $p > 1$  (this well-known fact is usually proved by using Cauchy's condensation test).  $\square$

Theorem 4 may also be proved as a consequence of a stronger result specialized to the harmonic series (Powell and Šalát [5]): The harmonic series over a subsequence converges if and only if the counting function of the subsequence satisfies  $\sum_{n \geq 1} S(n)/n^2 < \infty$ .

**6. CONCLUDING REMARK.** Many nonmonotone series  $\sum_{n \geq 1} a_n$  encountered in applications admit a monotone majorant series  $\sum_{n \geq 1} b_n$ . In this case, the dilution test can be applied to subseries of the majorant series; if  $\sum_{k \geq 1} b_{s(k)}$  converges, then so does the corresponding subseries  $\sum_{k \geq 1} a_{s(k)}$  of the original nonmonotone series.

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