H-product and the related dimension of graphs

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Abstract

We consider the graph together with the arbitrary partition of its vertex set into n subsets (n-partitioned graph). On the set of n-partitioned graphs distinguished up to isomorphism we define the binary algebraic operation $\circ_H$ (H-product of graphs), determined by the digraph H. We prove, that every operation $\circ_H$ defines the unique factorization as a product of prime factors. We define the dimension of graphs based on the considered operations. It is proved, that a graph G has the dimension at most k if and only if its vertex set could be partitioned into k cliques and stable sets, and one can associate with this partition the graph and the digraph such, that the graph is bipartite and the digraph is acyclic. The finite list of forbidden induced subgraphs for the class of graphs with the dimension 2 is obtained.

1 Introduction

The decomposition methods are widely and fruitfully used in different areas of combinatorics and graph theory. One of the possible approaches to graph decomposition is to define the binary operation on the set of graphs or related objects and to represent the considered object as a product of prime elements with respect to this operation.

There are a number of such operations known in graph theory. The most well-known are *cartesian product*, *direct* (or *categorical*, or *tensor) product*, *lexicographic product*. In all those examples the vertex set of the product of two graphs G and H is defined as the cartesian product $V(G) \times V(H)$.

There are several natural directions of research connected with such decompositions. In this paper we consider two of them.

1) The uniqueness of decomposition. Is the operation under study defines the *unique* factorization as a product of prime factors?

2) The dimension. One can define the *dimension* of the graph G as the minimal k such that G can be naturally embedded into the product of k "elementary" graphs (or, in a different way, into the product of "elementary" graphs of size k). What are the properties of this dimension? What is the structure of graphs with the fixed dimension?

For the cartesian product of graphs it is known, that the decomposition into prime factors is unique for connected graphs [17][22], but could be not unique for disconnected graphs [11]. The direct product does not define the unique decomposition, but at least the number of prime factors for all factorizations of a fixed graph is the same [11].

The complete graphs could be considered as "elementary" graphs in 2). For the direct product the dimension of a graph G is defined as the smallest integer k such that G is isomorphic to an induced subgraph of a product of k complete graphs. This dimension is called the *Prague dimension* or *product dimension* of graphs, and it was first introduced and studied by L. Lovasz, J. Nesetril, A. Pultr and V. R*ö*dl [14][15][16]. For the properties of this dimension see [10].

The cartesian product of complete graphs is a *Hamming graph*. The problem of graph embedding into Hamming graphs is well-studied, and different characterizations and properties of induced subgraphs [12] and isometric subgraphs [1][4][5][12][23][24] of Hamming graphs were obtained. In contrast to the direct product, not every graph is embeddable (as unduced or isometric subgraph) into the cartesian product of complete graphs. Note, that R. Graham and P. Winkler showed [9], that every graph has a canonical isometric embedding into the cartesian product of so-called irreducible factors.

We develop a different approach to the graph products. We consider the graph together with some arbitrary partition of its vertex set into n subsets, and we call this object *n-partitioned graph*. On the set of all n-partitioned graphs (distinguished up to an isomorphism) we define the binary algebraic operation $\circ_H$ (*H-product* of graphs) determined by the digraph H with $V(H) = \{1, ..., n\}$. For the two

n-partitioned graphs $T = (G, A_1, ..., A_n)$ and $S = (F, B_1, ..., B_n)$ their product $S = T \circ_H S$ is the n-partitioned graph $(R, A_1 \cup B_1, ..., A_n \cup B_n)$, where A_i and B_j are completely adjacent in F, if (i, j) is an arc of H, and completely nonadjacent, otherwise.

The idea of this approach is related to the well-studied idea of M-*partitions* introduced by T. Feder, P. Hell, S. Klein and R. Motwani in [6]. Suppose that M is the $k \times k$ symmetric matrix with the elements from the set $\{0, 1, *\}$. An M-partition of the graph G is a partition $V(G) = A_1 \cup ... \cup A_k$ such that each A_i is either a clique (if $M_{i,i} = 1$), or stable set (if $M_{i,i} = 0$), or an arbitrary set (if $M_{i,i} = *$); and A_i and A_j are either completely adjacent (if $M_{i,j} = 1$), or completely nonadjacent (if $M_{i,j} = 0$), or can have arbitrary set of edges between them (if $M_{i,j} = *$). The matrix M could be considered as an adjacency matrix of a *trigraph* [7], which consists of the set of k vertices $\{v_1, ..., v_k\}$, any two vertices v_i, v_j are connected either by a *non-edge* (if $M_{i,j} = 0$), or *weak edge* (if $M_{i,j} = *$), or *strong edge* (if $M_{i,j} = 1$). In this terms our H-decomposable graphs are M-partitionable graphs, where $M_{i,i} = *$ for all i and the graph formed by strong edges and non-edges of trigraph defined by M is complete bipartite with the parts of equal size (or, in other terms, H-decomposable graphs are the graphs admitting homomorphism to trigraphs with the above-mentioned properties).

But we consider this idea from the different point of view - as the study of binary algebraic operation, and the main questions considered in this paper are the questions 1) and 2). This paper is the result of the continuation of the previous research of the author and his colleagues. In the paper [18] they define the *operator decomposition of graphs* and studied its properties. This decomposition is, in fact, H_0-decomposition for the digraph H_0 shown in the figure 1. Some special cases of operator decomposition based on the notions of split graph and polar graph were introduced earlier by R. Tyshkevich and A. Chernyak [21]. Operator decomposition and its special cases appear to be very useful for different graph theory problems. In particular, the applications to the *reconstruction conjecture* [18] and the characterization of the structure of *unigraphs* (graphs defined up to isomorphism by their degree sequences)[20] could be mentioned. Another examples could be found in [3] and [13].

This paper consists of 3 parts. In the first part we define the H-product. We show, that for every digraph H the operation $\circ_H$ defines the unique factorization as a product of prime factors. Namely, for every digraph H every n-partitioned graph has the unique H-

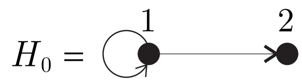

Figure 1: The digraph H_0

decomposition up to the permutation of staying together commutative factors.

In the second part we define the dimension of graphs based on the binary algebraic operations defined in the first part. The idea of this dimension came from the well-known notion of threshold graph. One of the equivalent definitions of threshold graph states, that threshold graphs are exactly the graphs, which could be represented via the product $\circ_{H_0}$ of "elementary" factors, where by "elementary" factors we mean the 2-partitioned graphs consisting of one vertex. We define H-threshold graphs as graphs, which could be represented via the product $\circ_H$ of one-vertex factors. We show, that every graph is H-threshold for some digraph H. So, it is natural to look for such representation defined by the operation $\circ_H$, which is as simple as possible. We define H-threshold dimension of the graph G as a minimum size of a digraph H such, that G is H-threshold. We give the structural characterization of graphs with the fixed H-threshold dimension. We show, that a graph G has H-threshold dimension at most k if and only if its vertex set $V(G)$ could be partitioned into k cliques and stable sets V_1,...,V_k and we can associate with this partition the graph $R(V_1, ..., V_k)$ and the digraph $F(V_1, ..., V_k)$ such, that $R(V_1, ..., V_k)$ is bipartite and $F(V_1, ..., V_k)$ is acyclic.

In the third part of the paper we give the structural characterization and the characterization by the finite list of forbidden induced subgraphs of graphs with H-threshold dimension 1 and 2.

All graphs considered are finite, undirected, without loops and multiple edges. At the same time further in this paper the loops (but not multiple arcs) are allowed in digraphs. The vertex and the edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The arc set of a digraph H is denoted by $A(H)$. Further, denote by $G[A]$ the subgraph induced by the set $A \subseteq V(G)$. For the convenience of reading the edges of graphs will be denoted as uv, and the arcs of digraphs - as (u, v). Write $u \sim v$ (resp. $u \not\sim v$) if $uv \in E(G)$ (resp. $uv \notin E(G)$).

If $X, Y \subseteq V(G)$, we will write $X \sim Y$ ($X \not\sim Y$) if for every $x \in X$ and $y \in Y$ $x \sim y$ ($x \not\sim y$). Let $N_Y(x) = \{y \in Y : y \sim x\}$.

For a digraph H and $v \in V(H)$ let $N_{in}(v) = \{u \in V(H) \setminus \{v\} : (u, v) \in A(H)\}$ and $N_{out}(v) = \{w \in V(H) \setminus \{v\} : (v, w) \in A(H)\}$ be the *in-neighborhood* and the *out-neighborhood* of v, respectively.

A graph G is called *split* [8], if its vertex set could be partitioned into a clique A and a stable set B. The graph G is *bipartite*, if if its vertex set could be partitioned into two stable sets A and B. The vertex set of the complement of bipartite graph could be partitioned into two cliques A and B. The partition (A, B) in all those cases is called *a bipartition*.

The important and well-known subclass of split graphs is the class of *threshold graphs*. There is a number of known equivalent definitions of threshold graphs. Let us quote some of them.

Theorem 1. *[13]*

The following definitions of threshold graphs are equivalent:

1) A graph G is threshold, if there exist nonnegative weights $(\alpha_v : v \in V(G))$ and a threshold β such, that the set $A \subseteq V(G)$ is stable if and only if the sum of weights of its members does not exceed β.

2) A graph G is threshold, if it is split with a bipartition (A, B), and the sets $\{N_A(b) : b \in B\}$ and $\{N_B(a) : a \in A\}$ are ordered by inclusion.

3) A graph G is threshold, if it is $(2K_2, C_4, P_4)$-free.

4) A graph G is threshold, if it could be iteratively constructed from one-vertex graph K_1 by adding at every iteration dominating or isolated vertex.

A graph is called *bipartite chain* [25], if it is bipartite with a bipartition (A, B), and the sets $\{N_A(b) : b \in B\}$ and $\{N_B(a) : a \in A\}$ are ordered by inclusion.

Let H be a digraph and let $(v_1, ..., v_n)$ be the ordering of its vertices. This ordering is called *acyclic ordering* or *topological sort*, if all arcs of H have the form (v_i, v_j), where $i < j$. A digraph is *acyclic*, if it does not contain directed cycles. The following property of acyclic graphs is well-known.

Proposition 1. *[2]*

A digraph is acyclic if and only if there exists an acyclic ordering of its vertices.

For a sequence $\pi = (a_1, ..., a_n)$ denote by $inv(\pi)$ the sequence $(a_n, ..., a_1)$.

2 H-product of graphs

Let H be a digraph with the vertex set $V(H) = \{1, ..., n\}$ and arc set $A(H)$. The *n-partitioned graph* is a $(n+1)$-tuple $T = (G, A_1, ..., A_n)$, where G is a graph and $(A_1, ..., A_n)$ is a partition of its vertex set into disjoint subsets: $V(G) = A_1 \cup ... \cup A_n$, $A_i \cap A_j = \emptyset$ for all $i \neq j$. Some of sets A_i could be empty. G is called *the basic graph* of T. Denote the set of vertices and the set of edges of T by $V(T)$ and $E(T)$, respectively.

The isomorphism f of n-partitioned graphs T and $S = (F, B_1, ..., B_n)$ is an isomorphism of G and F such that $f(A_i) = B_i$, $i = 1, ..., n$. Let Σ_n be the set of all n-partitioned graphs distinguished up to isomorphism.

On the set Σ_n define a binary algebraic operation $\circ_H : \Sigma_n \times \Sigma_n \to \Sigma_n$ (*H-product* of n-partitioned graphs) as follows:

$$(G, A_1, ..., A_n) \circ_H (F, B_1, ..., B_n) = (R, A_1 \cup B_1, ..., A_n \cup B_n), \quad (1)$$

where $V(R) = V(G) \cup V(F)$ (we assume without lost of generality that $V(G) \cap V(F) = \emptyset$), $E(R) = E(G) \cup E(H) \cup \{xy : x \in A_i, y \in B_j, (i, j) \in A(H)\}$.

For the convenience we will further sometimes denote the operation $\circ_H$ simply by $\circ$, if it is clear, what digraph H we mean. The operation, which was introduced and studied in [18], is the particular case of $\circ_H$ for a digraph $H = H_0$ shown in the figure 1.

It is easy to check, that for every digraph H the operation $\circ_H$ is associative. So, the set Σ_n with the operation $\circ_H$ is a semigroup.

The digraph H is *symmetric*, if $(i, j) \in A(H)$ whenever $(j, i) \in A(H)$. It is clear that the operation $\circ_H$ is commutative if and only if H is symmetric.

The n-partitioned graph $T \in \Sigma_n$ is called *H-decomposable*, if $T = T_1 \circ_H T_2$, $T_1, T_2 \in \Sigma_n$, and *H-prime*, otherwise. It is clear, that every n-partitioned graph $T \in \Sigma_n$ could be represented as a product $T = T_1 \circ_H ... \circ_H T_k$, $k \geq 1$, of prime elements. Such a representation is called an *H-decomposition* of T.

Theorem 2. *For every digraph H every n-partitioned graph $T \in \Sigma_n$ has the unique H-decomposition up to the permutation of staying together commutative multipliers.*

Proof. Let further the digraph H is fixed. It is evident, that if two n-partitioned graphs have the H-decompositions, which differ only by some permutations of staying together commutative multipliers, then they are isomorphic. So let us prove the inverse proposition. It is evident for prime n-partitioned graphs. Further apply the induction by the number of vertices.

Let

$$U = T_1 \circ ... \circ T_k,\ W = R_1 \circ ... \circ R_l, \tag{2}$$

$U \cong V$, $k, l \geq 2$. Let

$$U = (G, X_1, ..., X_n),\ W = (F, Y_1, ..., Y_n).$$

We may assume that $X_i \cup Y_i \neq \emptyset$ for all $i = 1, ..., n$.

Let $f : V(U) \to V(W)$ is the isomorphism of U and W. We will use the following notation. For the set $X \subseteq V(U)$ let $f(X) = \{f(x) : x \in X\}$, for the subgraph G' of G let $f(G') = W[f(V(G'))]$ and for the n-partitioned graph $T = (G', A_1, ..., A_n)$, where G' is a subgraph of G, let $f(T) = (f(G'), f(A_1), ..., f(A_n))$.

Setting $T_1 = (G', A_1, ..., A_n)$, $S = T_2 \circ ... \circ T_k = (G'', S_1, ..., S_n)$, $R_1 = (F', B_1, ..., B_n)$, $Q = R_2 \circ ... \circ R_l = (F'', Q_1, ..., Q_n)$, we have

$$U = T_1 \circ S,\ W = R_1 \circ Q. \tag{3}$$

By the definition of the isomorphism $f(A_i \cup S_i) = B_i \cup Q_i$.

Suppose that there exists $i \in \{1, ..., n\}$ such that $f(A_i) \cap B_i \neq \emptyset$, $f(A_i) \cap Q_i \neq \emptyset$. Then

$$f(T_1) = T' \circ T'',$$

where

$$T' = (F[f(V(T_1)) \cap V(R_1)], f(A_1) \cap B_1, ..., f(A_n) \cap B_n),$$

$$T'' = (F[f(V(T_1)) \cap V(Q)], f(A_1) \cap Q_1, ..., f(A_n) \cap Q_n).$$

Here $V(T'), V(T'') \neq \emptyset$ by the assumption. It contradicts the fact that T_1 is prime.

Analogously, the existence of $i \in \{1, ..., n\}$ such that $f^{-1}(B_i) \cap A_i \neq \emptyset$, $f^{-1}(B_i) \cap S_i \neq \emptyset$ contradicts the fact, that R_1 is prime.

So, further we can assume that for every $i = 1, ..., n$ $f(A_i) \subseteq B_i$ or $f(A_i) \subseteq Q_i$.

Suppose that there exist $i, j \in \{1, ..., n\}$, $i \neq j$ such that $f(A_i) \subseteq B_i$ and $f(A_j) \subseteq Q_j$. Then $f(T_1) = T' \circ T''$, where T', T'' are defined as above and by the assumption $V(T'), V(T'') \neq \emptyset$. Again the contradiction with the indecomposibility of T_1 is obtained.

So, there are two possibilities:

1) For every $i = 1, ..., n$ $f(A_i) \subseteq B_i$. Then the facts proved above imply, that $f(A_i) = B_i$, $f(S_i) = Q_i$ for every $i = 1, ..., n$. Thus $T_1 \cong R_1$, $S \cong Q$. After applying induction assumption for S and Q we get the theorem proved.

2) For every $i = 1, ..., n$ $f(A_i) \subseteq Q_i$. Then $B_i \subseteq f(S_i)$.

Let $f(S_i) \cap Q_i = \emptyset$ for all $i = 1, ..., n$. Then $f(S_i) = B_i$, $f(A_i) = Q_i$ for every $i = 1, ..., n$. So

$$U \cong T_1 \circ R_1 \cong W \cong R_1 \circ T_1,$$

and the statement of the theorem is true.

Consider the case, when there exist $i \in \{1, ..., n\}$ such that $f(S_i) \cap Q_i \neq \emptyset$. Let

$$Z = (F[f(V(S)) \cap V(Q)], f(S_1) \cap Q_1, ..., f(S_n) \cap Q_n).$$

By the assumption $V(Z) \neq \emptyset$. Then $f(S) = R_1 \circ Z$, $Q = f(T_1) \circ Z$ and thus

$$S \cong R_1 \circ Z, \ Q \cong T_1 \circ Z.$$

So, T_1 is the first factor in some H-decomposition of Q. Applying the induction assumption to Q, we may assume without lost of generality, that $T_1 = R_2$ and $Z = R_3 \circ ... \circ R_l$. So,

$$T_2 \circ ... \circ T_k \cong S \cong R_1 \circ R_3 \circ ... \circ R_l.$$

By the induction assumption applied to S, we have $k = l$ and under the respective ordering $R_1 \cong T_2$, $T_3 \cong R_3$,...,$T_k \cong R_k$.

To complete the proof, it remains to show, that T_1 and R_1 commutate. To do it, it is sufficient to prove, that for every pair $i, j \in \{1, ..., n\}$, $i \neq j$, such that $(i, j) \in A(H)$ and $(j, i) \notin A(H)$ one of the following four conditions hold: either $A_i \cup A_j = \emptyset$, or $A_j \cup B_j = \emptyset$, or $A_i \cup B_i = \emptyset$, or $B_i \cup B_j = \emptyset$.

We have $f(A_i) \sim f(S_j)$, $f(A_j) \not\sim f(S_i)$ (because $A_i \sim S_j$, $A_j \not\sim S_i$ and f is an isomorphism).

But then, since $f(A_i) \subseteq Q_i$, $f(A_j) \subseteq Q_j$, $B_i \subseteq f(S_i)$, $B_j \subseteq f(S_j)$, we have $f(A_i) \not\sim f(S_j)$, $f(A_j) \sim f(S_i)$.

This two facts imply, that one of the following is true:
1) $A_i \cup A_j = \emptyset$;
2) $A_i \cup S_i = \emptyset$, which implies, that $B_i = \emptyset$;
3) $A_j \cup S_j = \emptyset$, which implies, that $B_j = \emptyset$;
4) $S_i \cup S_j = \emptyset$, which implies, that $B_i = \emptyset$, $B_j = \emptyset$.
The theorem is proved. $\square$

3 H-threshold graphs and H-threshold dimension of graphs

Denote by K_i^k the k-partitioned graph $(K_1, \emptyset, ..., \emptyset, \{v\}, \emptyset, ..., \emptyset)$ (the only nonempty set of the partition is the ith set).

Let H be a digraph on k vertices. Let us call a graph G *H-threshold graph*, if it is basic for the n-partitioned graph of the form $K_{i_1}^k \circ_H ... \circ_H K_{i_n}^k$. In this case for the simplicity of the notation we will write $G = K_{i_1}^k \circ_H ... \circ_H K_{i_n}^k$ (though strictly speaking the left part of this equality is the graph and the right part is k-partitioned graph, and so we should understand this notation as follows: there exists a k-partitioned graph $T = (G, A_1, ..., A_k)$ such that $T = K_{i_1}^k \circ_H ... \circ_H K_{i_n}^k$).

To illustrate the notion of H-threshold graph, note that the graphs P_4 and C_4 are H-threshold for different H (see the figure 3)

It is easy to see, that by Theorem 1, 4) threshold graphs are exactly H_0-threshold graphs for the digraph H_0 shown in the figure 1

Proposition 2. *Every graph G is H-threshold for some digraph H.*

Proof. Let $V(G) = \{1, ..., n\}$. Define H as follows: $V(H) = \{1, ..., n\}$, $(i, j) \in A(H)$ if and only if $ij \in E(G)$, $i < j$ (i.e. H is obtained from G by assigning an arbitrary orientation on every edge of G). It is easy to see, that $G = K_1^n \circ_H K_2^n \circ_H ... \circ_H K_n^n$. $\square$

The digraph H constructed in the proof of Proposition 2 has $|V(G)|$ vertices. But, for example, threshold graphs are H-threshold for the digraph H with only 2 vertices. So, it is natural to consider the minimum order of a digraph, for which a graph G is H-threshold. Here we introduce the corresponding graph parameter.

The *H-threshold dimension* of a graph G is the parameter $HThrDim(G) = min\{k : G\ is\ H-threshold, |V(H)| = k\}$. By the Proposition 2 every graph has the H-threshold dimension. It is clear, that for every graph G on n vertices $HThrDim(G) \leq n$.

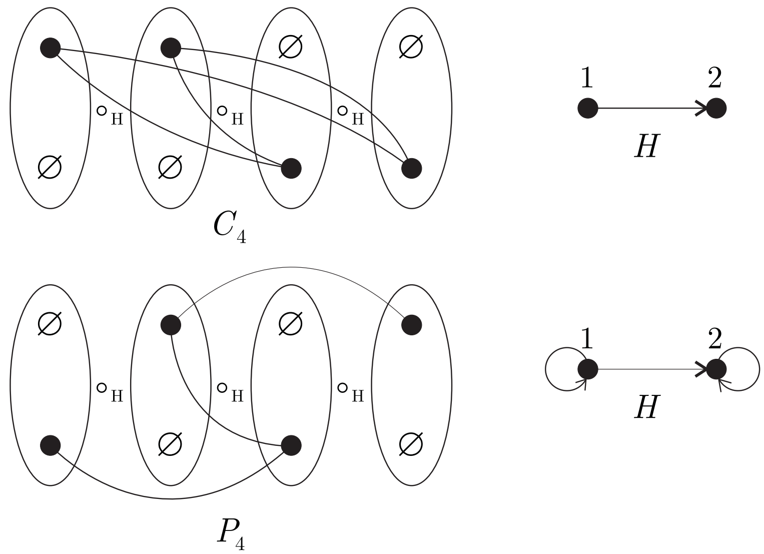

Figure 2: C_4 and P_4 as H-threshold graphs

Here are some simplest properties of H-threshold dimension.

Proposition 3. *For every graph G $HThrDim(G) = HThrDim(\overline{G})$.*

Proof. Suppose, that G is H-threshold for a digraph H with the vertex set $V(H) = \{1, ..., k\}$, i.e. $G = K^k_{i_1} \circ_H ... \circ_H K^k_{i_n}$. Let $\{v_j\} = V(K^k_{i_j})$, $j = 1, ..., n$. Consider the vertices v_p and v_q. Suppose, that $p < q$. Then $v_p \sim v_q$ if and only if one of the following conditions hold:

1) $i_p = i_q$ and $(i_p, i_p) \in A(H)$;

2) $i_p \neq i_q$ and $(i_p, i_q) \in A(H)$.

Define $\overline{H}$ be the complement of H, i.e. the digraph with the same vertex set and with the arc set $A(\overline{H}) = \{(i, j) : (i, j) \notin A(H)\}$. Then $\overline{G} = K^k_{i_1} \circ_{\overline{H}} ... \circ_{\overline{H}} K^k_{i_n}$, where $\{v_j\} = V(K^k_{i_j})$, $j = 1, ..., n$. □

Now we are going to give the characterization of graphs with $HThrDim(G) \leq k$. But firstly we need some auxiliary definitions and lemmas.

Let S be the family of sets $S = (\{X^1_1, X^1_2\}, ..., \{X^n_1, X^k_2\})$, where $X^i_j \subseteq \{1, ..., k\} \setminus \{i\}$, $i = 1, ..., k$, $j = 1, 2$ (some of sets X^i_j could be empty). Let us call S a *digraphical family*, if there exists a digraph D on the vertex set $V(D) = \{1, ..., k\}$ such, that $S = (\{N_{in}(1), N_{out}(1)\}, ..., \{N_{in}(k), N_{out}(k)\})$. D is called *a realization* of S.

The evident necessary condition for the digraphicity of S is $i \in X_1^j \cup X_2^j$ whenever $j \in X_1^i \cup X_2^i$. Let us call the family S with this property *proper*.

Suppose that S is the proper family. Define the graph $R(S)$ as follows: $V(R(S)) = S$, $X_q^i \sim X_p^j$ if and only if either $i = j$, $q \neq p$ or $i \in X_p^j$, $j \in X_q^i$, $i, j = 1, ..., k$, $q, p = 1, 2$.

Lemma 1. *The proper family S is digraphical if and only if the graph $R(S)$ is bipartite.*

Proof. Suppose that D is a realization of S. Let

$$l(X_q^i) = \begin{cases} 1, \ if \ X_q^i = N_{out}(i) \\ 2, \ if \ X_q^i = N_{in}(i). \end{cases}$$

By the definition $l(X_1^i) \neq l(X_2^i)$, $i = 1, ..., k$. If $j \in X_q^i = N_{in}(i)$, then $i \in X_p^j = N_{out}(j)$, and so $l(X_q^i) \neq l(X_p^j)$. This l is a proper 2-coloring of $R(S)$.

Inversely, let l be a proper 2-coloring of $R(S)$. Define the digraph D on the vertex set $\{1, ..., k\}$ as follows: $(i, j) \in A(G)$ if and only if $i \in X_p^j$, $j \in X_q^i$, $l(X_q^i) = 1$, $l(X_p^j) = 2$.

Since l is a proper 2-coloring, this definition correctly defines a digraph, and for every $i = 1, ..., k$ if, for example, $l(X_1^i) = 1$, $l(X_2^i) = 2$, then $X_1^i = N_{out}(i)$, $X_2^i = N_{in}(i)$. □

Corollary 1. *If D_1 and D_2 are two different realizations of S, then D_1 could be obtained from D_2 by the reversal of all arcs of some of its connected components.*

Let

$$V(G) = V_1 \cup ... \cup V_k \tag{4}$$

is a partition of the vertex set of the graph G, where each V_i is a clique or a stable set.

We will say, that the partition (4) satisfies the *neighborhood ordering* property, if for every $i, j = 1, ...k$ the set $\{N_{V_j}(u) : u \in V_i\}$ is ordered by inclusion. In other words, for every $i, j = \{1, ..., k\}$ there exists a permutation $\pi_j^i = (u_1, ..., u_{r_i})$ of the set V_i such, that $N_{V_j}(u_1) \supseteq N_{V_j}(u_2) \supseteq ... \supseteq N_{V_j}(u_{r_i})$.

Let the partition (4) satisfies the neighborhood ordering property. We will say that 4 satisfies *codirectional property*, if for every $i \in$

$\{1, ..., k\}$ and for every $j, l \in \{1, ..., k\} \setminus \{i\}$ either $\pi^i_j = \pi^i_l$ or $\pi^i_j = inv(\pi^i_l)$.

Suppose, that (4) satisfies the codirectional property. Then by the definition for every $i = 1, ..., k$ there exist two permutations $\psi_1(i)$, $\psi_2(i) = inv(\psi_1(i))$ such that for every $j \in \{1, ..., k\} \setminus \{i\}$ $\pi^i_j = \psi_1(i)$ or $\pi^i_j = \psi_2(i)$. Let

$$Y^i_r = \{j : \pi^i_j = \psi_r(i)\},\ r = 1, 2.$$

Put

$$X^i_r = Y^i_r \setminus \{j : V_i \sim V_j \text{ or } V_i \not\sim V_j\},\ r = 1, 2.$$

Let $S = S(V_1, ..., V_k) = (\{X^1_1, X^1_2\}, ..., \{X^n_1, X^k_2\})$.

Suppose that S is a digraphical family (i.e. by the Lemma 1 $R(S) = R(V_1, ..., V_k)$ is a bipartite graph) and D is its realization. Define the digraph $F = F(V_1, ..., V_k) = F_D(V_1, ..., V_k)$ as follows: $V(F) = V(G)$, $A(F) = A_1 \cup A_2 \cup A_3$, where

$$A_1 = \{(u, v) : u \in V_i, v \in V_j, uv \in E(G), (i, j) \in A(D)\};$$

$$A_2 = \{(v, u) : u \in V_i, v \in V_j, uv \notin E(G), (i, j) \in A(D)\};$$

$$A_3 = \{(u^i_1, u^i_2), ..., (u^i_{r_i-1}, u^i_{r_i}) : i = 1, ..., k\}.$$

Here $(u^i_1, ..., u^i_{r_i}) = \psi_l(i)$, such that $X^i_l = N_{out}(i)$.

In other words, the graph F is constructed in the following way. Firstly consider every pair V_i, V_j such, that neither $V_i \sim V_j$ nor $V_i \not\sim V_j$. Suppose that $(i, j) \in A(D)$. Consider the set $E_{i,j}$ of edges of the complete bipartite graph with the parts V_i and V_j. If the edge $uv \in E_{i,j}$ belongs to $E(G)$, then orientate it in the direction from V_i to V_j; otherwise orientate it in the direction from V_j to V_i. Next turn every set V_i into the oriented path, whose orientation is agreed with the permutations π^i_j by the above formula.

Now we are ready to formulate the characterization of graphs with the H-threshold dimension $HThrDim(G) \leq k$.

Theorem 3. *Let G be a graph. $HThrDim(G) \leq k$ if and only if there exists a partition (4) such that*

1) *it satisfies the codirectional property;*

2) *the family $S = S(V_1, ..., V_k)$ is digraphical (i.e. the graph $R(S) = R(V_1, ..., V_k)$ is bipartite);*

3) *the digraph $F = F(V_1, ..., V_k)$ is acyclic.*

Proof. Let us prove sufficiency first. Suppose, that D is a realization of S, which defines F. Let us expand D by adding the set of arcs $\{(i, i) : V_i \ \textit{is a clique}\} \cup \{(i, j), (j, i) : V_i \sim V_j\}$. Denote the obtained graph by H.

Let $(v_1, ..., v_n)$ be a acyclic ordering of the digraph F. We will show, that $G = K^k_{i_1} \circ_H ... \circ_H K^k_{i_n}$, where $V(K^k_{i_j}) = \{v_j\}$, $v_j \in V_{i_j}$.

Let $K^k_{i_1} \circ_H ... \circ_H K^k_{i_n} = Z$. Consider the edge $ab \in E(G)$. Let us show, that $ab \in E(Z)$. If V_i is a clique in G, then $(i, i) \in A(H)$, which implies, that V_i is a clique in Z. Analogously, if $V_i \sim_G V_j$, then $(i, j), (j, i) \in A(H)$, and so by the definition of the operation $\circ_H$ $V_i \sim_Z V_j$.

So, it remains to consider the case, when $a \in V_i$, $b \in V_j$, $i \neq j$ and neither $V_i \sim V_j$ nor $V_i \not\sim V_j$. In this case without lost of generality $(i, j) \in A(D)$. Then $(i, j) \in A(F)$ by the definition of F. Then in the acyclic ordering a goes before b, i.e. $a = V(K^k_{i_r})$, $b = V(K^k_{i_s})$, $r < s$. It together with the fact, that $(i, j) \in A(H)$, implies that $ab \in E(Z)$.

Conversely, let $ab \in E(Z)$. Let $a = v_r$, $b = v_s$, $r < s$ (i.e. a precedes b in the acyclic ordering), $a \in V_i$, $b \in V_j$. So we know, that $V_i \not\sim V_j$ does not hold. By the definition of the operation $\circ_H$ $(i, j) \in A(H)$. If $i = j$, then V_i is a clique, and so $ab \in V(G)$. So let further $i \neq j$ and it is not true, that $V_i \sim V_j$. Then $(i, j) \in A(Q)$ and by the definition a and b are adjacent in F. Since a precedes b in the acyclic ordering, $(a, b) \in A(F)$. So the arc (a, b) is directed from V_i to V_j, which implies, that $ab \in E(G)$.

Now we will prove necessity. Assume, that $G = K^k_{i_1} \circ_H ... \circ_H K^k_{i_n}$, where $\{v_j\} = V(K^k_{i_j})$. Then

$$V(G) = V_1 \cup ... \cup V_k. \tag{5}$$

where $V_i = \{v : \{v\} = V(K^k_i)\}$, $i = 1, ..., k$. If $(i, i) \in A(H)$, then V_i is a clique, otherwise it is a stable set.

Suppose, that $V_i = \{v_{l_1}, ..., v_{l_i}\}$, $l_1 < l_2 < ... < l_i$. If $(i, j) \in A(H)$, then $N_{V_j}(v_{l_1}) \supseteq N_{V_j}(v_{l_2}) \supseteq ... \supseteq N_{V_j}(v_{l_i})$, otherwise $N_{V_j}(v_{l_i}) \supseteq ... \supseteq N_{V_j}(v_{l_1})$. In the first case let $\pi^i_j = (v_{l_1}, ..., v_{l_i})$, in the second case

$\pi_j^i = (v_{l_i}, ..., v_{l_1})$. So, the partition (5) satisfies the neighborhood ordering property and the codirectional property.

Let D be a digraph obtained from H by deleting loops and arcs of the set $\{(i,j) : V_i \sim V_j\}$. Then in the digraph D

$$N_{out}(i) = \{j : \pi_j^i = (v_{l_1}, ..., v_{l_i}) \text{ and neither } V_i \sim V_j \text{ nor } V_i \not\sim V_j\};$$

$$N_{in}(i) = \{j : \pi_j^i = (v_{l_i}, ..., v_{l_1}) \text{ and neither } V_i \sim V_j \text{ nor } V_i \not\sim V_j\}.$$

So, D is a realization of $S(V_1, ..., V_k)$.

It remains to show, that $(v_1, ..., v_n)$ is a acyclic ordering of $F = F(V_1, ..., V_k)$. All arcs with both ends in V_l, $l = 1, ..., k$, have the form (v_i, v_{i+1}). So, let us consider $v_i \in V_l$, $v_j \in V_s$, $l \neq s$ such, that v_i and v_j are adjacent in F. By the definition of F neither $V_l \sim V_s$ nor $V_l \not\sim V_s$. Then l and s are adjacent in H. Let $(l, s) \in A(H)$. If $(v_i, v_j) \in A(F)$, then $v_i v_j \in E(G)$, which could be only if $i < j$. If $(v_j, v_i) \in A(F)$, then $v_i v_j \notin E(G)$, which could be only if $j < i$. □

Remark 1. *If the partition (4) is given, it could be tested in a polynomial time, if it satisfies the conditions of the Theorem 3. In case of the positive answer, the proofs of the Lemma 1 and Theorem 3 contain the algorithm for reconstruction of the graph H such that G is H-threshold graph.*

The definition of the digraph $F(V_1, ..., V_k)$ depends on the realization D of the family $S(V_1, ..., V_k)$. But the family $S(V_1, ..., V_k)$ can have different realizations. The next proposition shows, that from the point of view of the Theorem 3 it does not matter, which realization to choose.

Proposition 4. *Let D_1, D_2 be two realizations of $S(V_1, ..., V_k)$ for a partition (4). If $F_{D_1}(V_1, ..., V_k)$ is acyclic, then $F_{D_2}(V_1, ..., V_k)$ is also acyclic.*

Proof. Suppose, that $F_{D_1}(V_1, ..., V_k)$ is acyclic. By the corollary from the Lemma 1 D_1 and D_2 have the same sets of connected components. It is easy to see that $\{i_1, ..., i_j\}$ is a connected component of D_l if and only if $V_{i_1} \cup ... \cup V_{i_j}$ is a connected component of $F_{D_l}(V_1, ..., V_k)$, $l = 1, 2$. So, it follows from the definition of F and the Corollary 1 (???), that $F_{D_2}(V_1, ..., V_k)$ could be obtained from $F_{D_l}(V_1, ..., V_k)$ by the reversal of all arcs of some of its connected components. So, $F_{D_2}(V_1, ..., V_k)$ is acyclic. □

4 Graphs with $HThrDim(G) \le 2$

It is clear, that graphs with $HThrDim(G) = 1$ are exactly complete and empty graphs. For every threshold graph G $HThrDim(G) \le 2$. But the set of graphs with $HThrDim(G) \le 2$ is not reduced to the threshold graphs. For example, on the figure 3 we can see, that C_4 and P_4 have the threshold dimension 2. Formally this fact follows from the next proposition.

Proposition 5. *$HThrDim(G) \le 2$ if and only if G or $\overline{G}$ is either threshold, or bipartite chain.*

Proof. By the Theorem 3 the necessity is straightforward, so let us prove the sufficiency. By the definition there exists the partition $V(G) = V_1 \cup V_2$ such that $V_1(V_2)$ is either clique or stable set.

It is evident, that this partition satisfies the codirectional property. It is also clear, that the realization of the family $S(V_1, V_2)$ is either empty digraph (if $V_1 \sim V_2$ or $V_1 \not\sim V_2$) or the digraph D with $A(D) = \{(1, 2)\}$.

Let us prove that $F = F(V_1, V_2)$ is acyclic. If $V_1 \sim V_2$ or $V_1 \not\sim V_2$, then F is empty. Otherwise let $A(D) = \{(1, 2)\}$.

Let $V_1 = \{u_1, ..., u_r\}$, $V_2 = \{v_1, ..., v_s\}$, where $N_{V_2}(u_1) \supseteq N_{V_2}(u_2) \supseteq ... \supseteq N_{V_2}(u_r)$, $N_{V_1}(v_s) \supseteq N_{V_2}(v_{s-1}) \supseteq ... \supseteq N_{V_2}(v_1)$. Then all arcs of F with both ends in V_1 (V_2) have the form (u_i, u_{i+1}), $i = 1, ..., r-1$ ((v_i, v_{i+1}), $i = 1, ..., s-1$). Therefore if there exists a directed cycle in F, it should contain arcs (u_j, v_l), (v_p, u_i), $i \le j$, $l \le p$ (since F contains no loops we may assume without lost of generality, that $i \ne j$). By the definition of F, it means that $u_jv_l \in E(G)$, $u_iv_p \notin E(G)$. Since $N_{V_2}(u_i) \supseteq N_{V_2}(u_j)$ we have $u_iv_l \in E(G)$. If $l = p$, then we have the contradiction. If $l \ne p$ then, as $N_{V_1}(v_p) \supseteq N_{V_1}(v_l)$, we again have $u_iv_p \in E(G)$. This contradiction finishes the proof. □

Theorem 4. *Let G be a graph. $HThrDim(G) \le 2$ if and only if neither G nor $\overline{G}$ contains one of the graphs from the set $L = \{C_5, P_5, House, P_3 \cup P_2, W_4, Bull, X, Y, Z\}$ as an induced subgraph.*

Proof. It is straightforward to check, that every graph from the set L do not satisfy the Proposition 5. So we will prove the sufficiency.

Let us prove firstly, that G is either split, or bipartite, or a complement of bipartite. After that we will prove, that for each its part the neighborhoods of its vertices in the another part are ordered by inclusion.

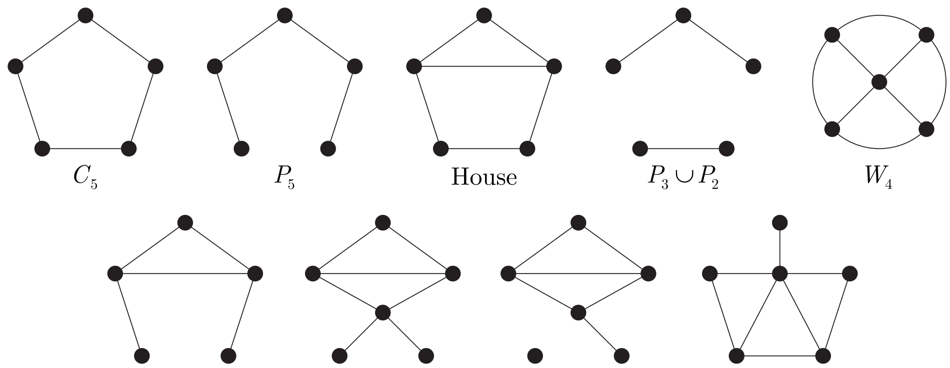

Figure 3: The set L

Suppose, that neither G nor $\overline{G}$ is bipartite. We will show, that G is split.

Let A be a maximum clique of G and such, that a subgraph induced by the set $B = V(G) \setminus A$ have the smallest possible number of edges. We will prove, that B is a stable set.

Suppose the contrary, i.e. there exist $x, y \in B$ such, that $x \sim y$. Since A is maximum, there exist vertices of A, which are not adjacent to x and y. If all vertices of A, except, possibly, one vertex u, adjacent to both x and y, then $A \setminus \{u\} \cup \{x, y\}$ is a clique, which contradicts the maximality of A. So, there exist $u, v \in A$ such, that $u \not\sim x$, $v \not\sim y$.

It is easy to see, that $|A| \geq 3$. Indeed, if $|A| = 2$, then G is triangle-free. It, together with the fact, that G is $\{C_5, P_5\}$-free, imply that G doesn't contain odd cycles. Let $w \in A \setminus \{u, v\}$.

Because $\overline{G}$ is not bipartite, there exists $z \in B \setminus \{x, y\}$ such, that $z \not\sim y$ or $z \not\sim x$. We may assume, that $w \not\sim z$, since A is a maximum clique.

Let us call the induced cycle $C = C_4$ *bad*, if there exists a vertex $a \in V(G) \setminus C$ such, that $|N(a) \cap C| \geq 2$. By the assumption of the theorem G does not contain bad P_4's.

If $u \sim y$ and $v \sim x$, then G contains bad C_4. Therefore the following cases are possible: 1) $u \not\sim y$, $v \not\sim x$ and 2) $u \sim y$, $v \not\sim x$. Consider this cases.

1) $u \not\sim y$, $v \not\sim x$.

Let without lost of generality $z \not\sim y$. If $z \sim x$, then without lost of generality $z \sim v$ (since $G[u, v, y, x, z] \neq P_3 \cup P_2$). As $G[y, x, z, v, w] \neq$

P_5, C_5, $w \sim x$. But then $\{w, v, z, x\}$ form bad C_4.

So it is proved, that $z \nsim x$. Moreover, it is shown, that for every $t \in B \setminus \{x, y\}$ $t \sim \{x, y\}$ or $t \nsim \{x, y\}$.

Let $T_1 = \{t \in B \setminus \{x, y\} : t \sim \{x, y\}\}$, $T_2 = \{t \in B \setminus \{x, y\} : t \nsim \{x, y\}\}$. We know from the considerations above, that $T_2 \neq \emptyset$.

Let $t \in T_2$. As $G[u, v, y, x, t] \neq P_3 \cup P_2$, without lost of generality $t \sim v$. Then, since $G[t, v, u, y, x] \neq P_3 \cup P_2$, $t \sim u$. So, we have $T_2 \sim \{u, v\}$.

Lemma 2. *For every $q \in A \setminus \{u, v\}$ $q \sim T_2$ or $q \sim \{x, y\}$. Moreover, T_2 is a clique.*

Proof. Suppose, that there exists $t \in T_2$ such, that $q \nsim t$. The statement, that $q \sim \{x, y\}$ follows from the fact, that $G[t, v, q, y, x] \neq P_3 \cup P_2, P_5$. If there exist $t_1, t_2 \in T_2$ such, that $t_1 \nsim t_2$, then $G[t_1, v, t_2, y, x] = P_3 \cup P_2$. □

Let $Q_1 = \{q \in A \setminus \{u, v\} : q \sim T_2\}$, $Q_2 = (A \setminus \{u, v\}) \setminus Q_1$. By the Lemma 2 $Q_2 \sim \{x, y\}$. Moreover, as A is maximal clique, $Q_2 \neq \emptyset$.

Lemma 3. *$Q_2 \sim T_1$. Moreover, T_1 is a clique.*

Proof. Suppose, that there exist $t_1 t_2 \in T_1$ such, that $t_1 \nsim t_2$. Since $G[u, v, y, t_1, t_2] \neq P_3 \cup P_2$, without lost of generality $t_2 \sim v$. Then either $t_2 \sim u$ or $t_1 \sim v$, because $G[u, v, t_2, y, t_1] \neq P_5, C_5$. But $t_1 \nsim v$, because otherwise v, t_2, x, t_1 form bad C_4. So $t_1 \nsim v$, $t_2 \sim u$. Analogously, it is easy to see, that $t_1 \nsim \{u\}$.

By the maximality of the clique A, there exists $q \in A \setminus \{u, v\}$ such, that $q \nsim t_2$. As $G[q, v, t_2, y, t_1] \neq P_5, C_5$, $q \sim x$. But then $G[q, v, t_2, x]$ is a bad C_4. So it is proved, that T_1 is a clique.

Let us show now, that $T_1 \sim Q_2$. Suppose the contrary, i.e. let there exist $t \in T_1$, $q \in Q_2$ such, that $t \nsim q$. By the definition of Q_2 there exist $z \in T_2$ such, that $q \nsim z$. Since $G[z, u, q, x, t] \neq P_5, C_5$, $t \sim u$. But then $G[u, q, x, t]$ is a bad C_4. □

By Lemma 2 and Lemma 3 $V_1 = Q_2 \cup T_1 \cup \{x, y\}$ and $V_2 = Q_1 \cup T_2 \cup \{u, v\}$ are cliques, $V_1 \cup V_2 = V(G)$. The contradiction with the fact, that $\overline{G}$ is not bipartite, is obtained. So, the case 1) is considered.

2) $u \sim y$, $v \nsim x$.

Lemma 4. *For every $z \in B \setminus \{x, y\}$ $z \sim \{x, y\}$ or $z \nsim \{x, y\}$.*

Proof. Assume, in contrary, that there are exist $z \in B \setminus \{x, y\}$ such, that the lemma is not satisfied for it.

Let $z \sim x$, $z \not\sim y$. Since $G[v, u, y, x, z] \neq P_5, C_5$, then $z \sim u$. Consider $w \in A \setminus \{u, v\}$. As $G[y, x, z, v, w] \neq P_3 \cup P_2$, there are edges between $\{v, w\}$ and $\{x, y, z\}$. But it means, that $G[u, y, x, z]$ is a bad C_4.

So, $z \sim y$, $z \not\sim x$. Suppose, that $z \sim v$. Then $z \sim u$ (because $G[z, y, u, v]$ is not a bad C_4). Therefore by the maximality of A there exists $w \in A$ such that $w \not\sim z$. For this vertex we have $w \sim y$ (as $G[x, y, z, v, w] \neq P_5, C_5$), and it implies, that $G[w, v, z, y]$ is a bad C_4.

So, $z \not\sim v$. But then $z \not\sim u$ (otherwise $G[z, y, u, v, x] = Bull$). Since $G[x, y, z, w, v] \neq P_3 \cup P_2$, there exist some of the edges from the set $\{wx, wy, wz\}$.

Suppose, that $w \sim x$. Then $w \sim y$ (because otherwise $G[w, x, y, u]$ is a bad C_4). It implies, that $w \sim z$ (as $G[w, y, x, z, v] \neq Bull$). But then $G[x, y, z, u, v, w] = \overline{Y}$.

Thus $w \not\sim x$. If $w \sim z$, then $w \sim y$ (since $G[w, u, y, z]$ is not bad C_4). It implies, that $G[w, z, y, v, x] = Bull$.

So, $w \not\sim z$. Then $w \sim y$ and $G[w, u, v, y, z, x] = X$. ☐

Let $B \setminus \{x, y\} = S_1 \cup S_2$, $S_1 = \{z \in B : z \sim \{x, y\}\}$, $S_2 = \{z \in B : z \not\sim \{x, y\}\}$. Since $\overline{G}$ is not bipartite, $S_2 \neq \emptyset$.

Lemma 5. *For every $r \in A \setminus \{u, v\}$ $r \sim \{x, y\}$ or $r \sim S_2$.*

Proof. Assume, that there exists $z \in S_2$ such, that $r \not\sim z$.

Let $z \sim v$. As $G[z, v, r, y, x] \neq P_3 \cup P_2$, $r \sim y$ or $r \sim x$. The situation, when $r \sim x$ and $r \not\sim y$, is impossible, because otherwise $G[r, u, y, x]$ is a bad C_4. If $r \sim y$, then $r \sim x$ (because $G[x, y, r, v, z] \neq P_5$).

It remains to consider the case, when $z \not\sim v$. Then $r \sim y$ or $r \sim x$, since $G[r, v, y, x, z] \neq \overline{W_4}$. As above, the case, when $r \sim x$, $r \not\sim y$, is impossible. So $r \sim y$. As $G[v, u, r, y, x, z] \neq Y$, $z \sim u$ or $r \sim x$. The situation, when $z \sim u$, $r \not\sim x$ contradicts the fact, that $G[r, u, y, x, r] \neq Bull$. So $r \sim x$. ☐

Let $A \setminus \{u, v\} = R_1 \cup R_2$, $R_1 = \{r \in A \setminus \{u, v\} : r \sim S_2\}$, $R_2 = (A \setminus \{u, v\}) \setminus R_1$. By the Lemma 5 $R_2 \sim \{x, y\}$.

Lemma 6. $S_2 \sim \{u, v\}$. *Moreover, S_2 is a clique.*

Proof. Let us first the first statement of the lemma. Let $z \in S_2$. Assume, that $z \not\sim v$. We will show, that it is impossible.

Suppose, that there exists $r \in A \setminus \{u, v\}$ such that $r \not\sim y$. By the Lemma 5 $r \sim z$. Then $r \sim x$, since $G[z, r, v, y, x] \neq P_3 \cup P_2$. But then $G[r, u, y, x]$ is a bad C_4.

So, it is proved that $y \sim A \setminus \{v\}$. Therefore there exists $s \in B \setminus \{x, y, z\}$ such that $s \sim v$ and $s \not\sim y$. Indeed, if, on the contrary, $N_B(v) \subseteq N_B(y)$, then $A' = (A \setminus \{v\}) \cup \{y\}$ is a maximum clique and for the subgraph, induced by the set $B' = V(G) \setminus A'$, we have $|E(G[B'])| < |E(G[B])|$. It contradicts the definition of the clique A.

As $G[x, y, u, v, s] \neq P_5, C_5$, $s \sim u$. Moreover, $s \not\sim x$, (because otherwise $G[x, y, u, s]$ is a bad C_4) and $s \sim z$ (because otherwise $G[v, s, y, x, z] = \overline{W_4}$). But then $G[z, s, v, y, x] = P_3 \cup P_2$.

So, $z \sim v$. Then $z \sim u$ (see the proof of the Lemma 5).

Now it is easy to see, that S_2 is a clique. Indeed, if there exist $s_1, s_2 \in S_2$ such that $s_1 \not\sim s_2$, then $G[s_1, v, s_2, y, x] = P_3 \cup P_2$. □

In particular, Lemma 6 and the maximality of A imply, that $R_2 \neq \emptyset$.

Lemma 7. *$R_2 \sim S_1$. Moreover, S_1 is a clique.*

Proof. Let there exist $r \in R_2$ and $s \in S_1$ such that $r \not\sim s$. By Lemma 5 $r \sim \{x, y\}$. By the definition there exists $z \in S_2$ such, that $z \not\sim r$. Lemma 6 implies, that $z \sim \{u, v\}$. Since $G[z, v, r, x, s] \neq P_5, C_5$, either $z \sim x$ or $s \sim v$. But in the first case $G[z, v, r, x]$ is a bad C_4, and in the second case $G[v, r, x, s]$ is a bad C_4. So, it is proved, that $R_2 \sim S_1$.

Let us show now, that S_1 is a clique. Suppose that there exist $z_1, z_2 \in S_1$ such, that $z_1 \not\sim z_2$. As $G[z_1, x, z_2, u, v] \neq P_3 \cup P_2$, there exists at least one edge between $\{z_1, z_2\}$ and $\{u, v\}$. At the same time, if $z_1 \sim v$ and $z_1 \not\sim u$, then $G[z_1, v, u, y]$ is a bad C_4.

So, without lost of generality $z_1 \sim u$. Then $z_2 \not\sim u$ (because otherwise $G[z_1, x, z_2, u]$ is a bad C_4). Since $G[v, u, z_1, x, z_2] \neq P_5, C_5$, $z_1 \sim v$. It implies, that $z_2 \not\sim v$ (otherwise $G[v, z_1, x, z_2]$ is a bad C_4).

The maximality of A implies the existence of $w \in A$ such, that $w \not\sim z_1$. $w \not\sim x$, as $G[w, v, z_1, x]$ is not a bad C_4. But then $G[w, v, z_1, x, z_2] = P_5$ or C_5. □

By Lemma 6 and Lemma 7 $V_1 = R_2 \cup S_1 \cup \{x, y\}$ and $V_2 = R_1 \cup S_2 \cup \{u, v\}$ are cliques, $V_1 \cup V_2 = V(G)$. The contradiction with the fact, that $\overline{G}$ is not bipartite, is obtained. The case 2) is considered.

So, it is proved, that G or $\overline{G}$ is either split or bipartite. Let (A, B) be the bipartition of G. Let us show, that the neighborhoods of vertices from $A(B)$ are ordered by inclusion.

Let us suppose the contrary, i.e. there exist $u, v \in A$, $x, y \in B$ such, that $u \sim x$, $v \not\sim x$, $u \sim y$, $v \not\sim y$.

Suppose, that G is bipartite. If $|V(G)| = 4$, then $\overline{G}$ the statement of the theorem obviously holds. Let there exists $z \in B \setminus \{x, y\}$. Since $G[u, v, x, y, z] \neq \overline{W_4}, P_3 \cup P_2$, $z \sim u, v$. But then $G[u, v, x, y, z] = P_5$. This contradiction proves the theorem for bipartite graphs.

Taking into account Observation 3, it remains to consider the case, when G is split and neither bipartite nor a complement of bipartite.

The following statements hold:

a) $N(x) \cup N(y) = A$ (since G does not contain $Bull$);

b) for every $z \in B \setminus \{x, y\}$ $|N(z) \cap \{u, v\}| \leq 1$ (by the same reason as in a));

c) $|A| \geq 3$, $|B| \geq 3$ (otherwise either G or $\overline{G}$ is bipartite).

Let $z \in B \setminus \{x, y\}$, $w \in A \setminus \{u, v\}$, $w \sim x$. As $G[u, v, x, y, w, z] \neq Y, \overline{Z}$, at least one of the edges zu, zv, zw belongs to $E(G)$. If there exists exactly one of this edges, then $G[u, v, w, z, y] = Bull$, $G[u, v, w, x, y, z] = X$, $G[u, v, w, z, y] = Bull$, respectively. Therefore, taking into account b), either $zw, zv \in E(G)$, $zu \notin E(G)$ or $zw, zu \in E(G)$, $zv \notin E(G)$.

In the first case $w \sim y$ (since $G[w, v, z, y, x] \neq Bull$), which implies, that $F = G[u, v, x, y, w, z] = \overline{Y}$. In the second case $w \sim y$ (since $F \neq \overline{X}$), which implies, that $F = \overline{Y}$.

The theorem is proved □

References

[1] F. Aurenhammer, M. Formann, R. M. Idury, A. A. Schăffer and F. Wagner, Faster isometric embedding in products of complete graphs, Discrete Appl Math 52 (1994), 17–28.

[2] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer, 2008.

[3] A. Brandstädt, V.B. Le and J. Spinrad, Graph classes: a survey, SIAM monographs on discrete mathematics and applications, Philadelphia, 1999.

[4] B. Brešar, Partial Hamming graphs and expansion procedures, Discrete Math 237 (2001), 13–27.

[5] V. Chepoi, d-Convexity and isometric subgraphs of Hamming graphs, Cybernetics 1 (1988), 6–9.

[6] T. Feder, P. Hell, S, Klein and R. Motwani, List partitions, SIAM J. Discrete Math. 16 (2003), 449-478.

[7] T. Feder, P. Hell and K. Tucker-Nally, Digraph matrix partitions and trigraph homomorphisms, Discrete Applied Math. 154 (2006), 2458–2469.

[8] S. Földes and P.L. Hammer, "Split graphs", Proc. of the 8-th South-East Conf. of Combinatorics, Graph Theory and Computing 19 (1977), pp. 311–315.

[9] R. L. Graham and P. M. Winkler, On isometric embeddings of graphs, Trans Am Math Soc 288 (1985), 527–536.

[10] P. Hell and J. Nešetřil, Graphs and Homomorphisms, Oxford University Press, 2004.

[11] W. Imrich and S. Klavžar, Product Graphs: Structure and Recognition, Wiley, 2000.

[12] S. Klavžar and I.Peterin, Characterizing subgraphs of Hamming graphs, J. Graph Theory 49 (2005), 302–312.

[13] N.V. Mahadev and U.N. Peled, Threshold graphs and related topics, Annals of Discrete Math. 56 (1995).

[14] J. Nešetřil and A. Pultr. Dushnik-Miller type dimension of graphs and its complexity, Springer Verlag Lecture Notes in Computer Science 56 (1977), 482-494.

[15] J. Nešetřil and V. Rödl, A simple proof of the Galvin-Ramsey property of graphs and a dimension of a graph, Discrete Math., 23 (1978), 49-56.

[16] L. Lovász, J. Nešetřil and A. Pultr, On a product dimension of graphs, J. Combin. Theory B 29 (1980), 47-96.

[17] G. Sabidussi, Graph multiplication, Math. Z. V. 72 (1960), 446-457

[18] P.V. Skums, S.V. Suzdal and R.I. Tyshkevich, Operator decomposition of graphs and the reconstruction conjecture, Discrete Math. 310(2010), 423–429

[19] P.V. Skums and R.I. Tyshkevich, Bipartite operator decomposition of graphs and the reconstruction conjecture, Electronic Notes in Discrete Math. 29 (2007), 201–205.

[20] R. Tyshkevich, Decomposition of graphical sequences and unigraphs, Discrete Math. 220(2000), 201–238.

[21] R.I. Tyshkevich and A.A. Chernyak, Canonical partition of graph defined by the degrees of its vertices, Izv. Akad. Nauk BSSR 5 (1979), 14–26.

[22] V.G. Vizing, The cartesian product of graphs, Vyčisl. Sistemy 9 (1963), 30–43.

[23] E. Wilkeit, Isometric embeddings in Hamming graphs, J Combin Theory Ser B 50 (1990), 179–197.

[24] P. Winkler, Isometric embeddings in products of complete graphs, Discrete Appl Math 7 (1984), 221–225.

[25] M. Yannakakis. The complexity of the partial order dimension problem, SIAM J. Alg. Discr. Math. 3 (1982), 351–358