

The Universal Arrow of Time

Oleg Kupervasser^{1*}, Hrvoje Nikolić^{2†}, Vinko Zlatic^{2‡}

¹ Moscow State University, Scientific Research Computer Center

² Theoretical Physics Division, Rudjer Bošković Institute,
P.O.B. 180, HR-10002 Zagreb, Croatia

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Abstract

Statistical physics cannot explain why a thermodynamic arrow of time exists, unless one postulates very special and unnatural initial conditions. Yet, we argue that statistical physics can explain why the thermodynamic arrow of time is universal, i.e., why the arrow points in the same direction everywhere. Namely, if two subsystems have opposite arrow-directions initially, the interaction between them makes the configuration statistically unstable and causes a decay towards a system with a universal direction of the arrow of time. We present general qualitative arguments for that claim and support them by a detailed analysis of a toy model based on the baker's map.

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1 Introduction

The origin of the arrow of time is one of the greatest unsolved puzzles in physics [1, 2, 3, 4, 5]. It is well established that most arrows of time can be reduced to the thermodynamic arrow, but the origin of the thermodynamic arrow of time remains a mystery. Namely, the existence of the thermodynamic arrow of time means that the system is not in the state with the highest possible entropy. But this means that the system is not in the highest-probable state, which lacks any statistical explanation. The fact that entropy increases with time means that the system was in an even less probable state in the past, which makes the problem even harder. Of course, the phenomenological fact that entropy increases with time can be described by assuming that the universe was in a state with a

*olegkup@yahoo.com

†hrvoje@thphys.irb.hr

‡Vinko.Zlatic@irb.hr

very low entropy at the beginning, but one cannot explain why the universe started with such a very special and unnatural initial condition in the first place.

Recently, Maccone [6] argued that the problem of the origin of the arrow of time can be solved by quantum mechanics. He has shown that in quantum mechanics all phenomena which leave a trail behind (and hence can be studied by physics) are those the entropy of which increases. (The observer’s memory erasing argument and the corresponding thought experiments discussed in [6] was also used previously for a resolution of entropy increase and the quantum wave-packet reduction paradoxes [7, 8, 9].) From this he argued that the second law of thermodynamics is reduced to a mere tautology, suggesting that it solves the problem of the arrow of time in physics. However, several weaknesses on specific arguments used in [6] have been reported [10, 11, 12]. As a response to one of these objections, in a later publication [13] Maccone himself realized that his approach does not completely solve the origin of the arrow of time because the quantum mechanism he studied also requires highly improbable initial conditions which cannot be explained.

Yet, as Maccone argued in [13], we believe that some ideas presented in [6] and [13] do help to better understand the puzzle of the arrow of time. The purpose of this paper is to further develop, refine, clarify, and extend some of the ideas which were presented in [6, 13, 11], and also in a somewhat different context in [8, 9, 14]. In particular, unlike Maccone in [6, 13], we argue that quantum mechanics is not essential at all. Indeed, in this paper we consider only classical statistical mechanics.

The idea is the following. Even though statistical physics cannot explain why a thermodynamic arrow of time exists, we argue that at least it can explain why the thermodynamic arrow of time is universal, i.e., why the arrow points in the same direction everywhere. Namely, if two subsystems have opposite arrow-directions initially, we argue that the interaction between them makes the configuration statistically unstable and causes a decay towards a system with a universal direction of the arrow of time. This, of course, does not completely resolve the problem of the origin of the arrow of time. Yet, at least, we believe that this alleviates the problem.

The paper is organized as follows. In the next section we present our main ideas in an intuitive non-technical form. After that, in Sec. 3 we study the statistical properties of the baker’s map (some basic properties of which are presented in the Appendix), which serves as a toy model for studying generic features of reversible chaotic Hamiltonian systems. As a byproduct, in this section we also clarify the differences between various notions of “entropy”. Then, in Sec. 4 we study the effects of weak interactions between subsystems which, without interactions, evolve according to the baker’s map. In particular, we explain how weak interactions destroy the opposite time arrows of the subsystems, by making them much more improbable than without interactions. Finally, in Sec. 5 we present a qualitative discussion of our results, including the consistency with strongly-interacting systems in which the entropy of a subsystem may decrease with time.

2 Main ideas

A priori, the probability of having a thermodynamic arrow of time is very low. However, our idea is to think in terms of *conditional* probabilities. Given that a thermodynamic arrow exists, what can we, by statistical arguments, infer from that?

To answer this question, let us start from the laws of an underlying microscopic theory. We assume that dynamics of microscopic degrees of freedom is described by a set of second-order differential equations (with derivatives with respect to time) which are invariant under the time inversion $t \rightarrow -t$. Thus, both directions of time have an *a priori* equal roles. To specify a unique solution of the dynamical equations of motion, one also needs to choose some “initial” time t_0 on which initial conditions are to be specified. (The “initial” time does not necessarily need to be the earliest time at which the universe came into the existence. For any t_0 at which the initial conditions are specified, the dynamical equations of motion uniquely determine the state of the universe for both $t > t_0$ and $t < t_0$.) It is a purely conventional particular instant on time, which may be even in the “future”. Indeed, in this paper we adopt the “block-universe” picture of the world (see, e.g., [4, 15, 16, 17] and references therein), according to which time does not “flow”. Instead, the universe is a “static” object extended in 4 spacetime dimensions.

Of course, the *a priori* probability of small entropy at t_0 is very low. But *given that* entropy at t_0 is small, what is the conditional probability that there is a thermodynamic arrow of time? It is, of course, very high. However, given that entropy at t_0 is low, the most probable option is that entropy increases in *both* directions with a minimum at t_0 . On the other hand, in practice, at times at which we make measurements, the entropy is indeed low, but entropy does not increase in both directions. Instead, it increases in only one direction. This is because, on a typical t_0 , not only the “initial” entropy is specified, but a particular direction of the entropy increase is specified as well. At the microscopic level, this is related to the fact that on t_0 one does not only need to specify the initial particle positions, but also their initial velocities.

Given that insight, next we ask the following question. Given that at t_0 the entropy is low and increases in the positive time direction, what can be statistically inferred from that? In this case, the most probable option is that entropy will continue to increase with t for $t > t_0$, but also that it will *decrease* in the negative time direction for $t < t_0$. This is, indeed, what we observe in nature.

And now comes the central question of this section. Given that at t_0 the entropy is low, why entropy at t_0 increases in the same (say, positive) direction everywhere? Isn’t it more probable that the direction of entropy-increase varies from point to point at t_0 ? If so, then why don’t we observe it? In other words, why the arrow of time is *universal*, having the same direction everywhere for a given t_0 ? We refer to this problem as the problem of *universality of the arrow of time*.

In this paper we argue that *this* problem can be solved by statistical physics. In short, our solution is as follows. If we ignore the interactions between different degrees of freedom, then, given that at t_0 the entropy is low, the most probable option is, indeed, that the direction of the arrow of time varies from point to point. On the other hand, if different degrees of freedom interact with each other, then it is no longer the most probable option. Instead, even if the direction of the arrow of time varies from point to point at t_0 , the interaction provides a natural mechanism that aligns all time arrows to the same direction.

To illustrate the arrow-of-time dilemma, the thought experiments of Loschmidt (time reversal paradox) and Poincare (recurrence theorem) are also often used. The corresponding paradoxes in classical mechanics are resolved as follows. Classical mechanics allows, at least in principle, to exclude any effect of the observer on the observed system. However,

most realistic systems are *chaotic*, so a weak perturbation may lead to an exponential divergence of trajectories. In addition, there is also a non-negligible interaction. As a simple example, consider a gas expanding from a small region of space into a large volume. In this entropy-increasing process the time evolution of macroscopic parameters is stable against small external perturbations. On the other hand, if all the velocities are reversed, then the gas will end up in the initial small volume, but only in the absence of any perturbations. The latter entropy-decreasing process is clearly unstable and a small external perturbation would trigger a continuous entropy growth. Thus the entropy increasing processes are stable, but the decreasing ones are not. A natural consequence is that the time arrows (the directions of which are defined by the entropy growth) of both the observer and the observed system are aligned to the same direction, because of the inevitable non-negligible interaction between them. They can return back to the initial state only together (as a whole system) in both Loschmidt and Poincare paradoxes, so the observer's memory gets erased in the end. During this process the time arrow of the observer points in the backward direction, which has two consequences. First, an entropy growth is observed in the whole system as well as in its two parts, despite the fact that entropy decreases with coordinate time. Second, the memory of the observer is erased not only at the end but also close to that point, because the observer does not remember his "past" (defined with respect to the coordinate time), but remembers his "future".

Indeed, it may seem quite plausible that interaction will align all time arrows to the same direction. But the problem is - which direction? The forward direction, or the backward one? How can any particular direction be preferred, when both directions are *a priori* equally probable? Is the common direction chosen in an effectively random manner, such that it cannot be efficiently predicted? Or if there are two subsystems with opposite directions of time at t_0 , will the "stronger" subsystem (i.e., the one with a larger number of degrees of freedom) win, such that the joint system will take the direction of the "stronger" subsystem as their common direction?

The answer is as follows: It is all about conditional probabilities. One cannot question the facts which are already known, irrespective of whether these facts are in "future" or "past". The probabilistic reasoning is to be applied to only those facts which are not known yet. So, let us assume that the entropy is low at t_0 and that we have two subsystems with opposite time directions at t_0 . Let us also assume that the subsystems do not come into a mutual interaction before t_1 (where $t_1 > t_0$), after which they interact with each other. Given all that, we *know* that, for $t_0 < t \leq t_1$, entropy increases with time for one subsystem and decreases with time for another subsystem. But what happens for $t > t_1$? Due to the interaction, the two subsystems will have the same direction of time for $t > t_1$. But which direction? The probabilistic answer is: The direction which is more probable, *given* that we know what we already know. But we already know the situation for $t < t_1$ (or more precisely, for $t_0 < t \leq t_1$), so our probabilistic reasoning can only be applied to $t > t_1$. It is this *asymmetry in knowledge* that makes two directions of time different. (Of course, the interaction is also asymmetric, in the sense that interaction exists for $t > t_1$, but not for $t_0 < t \leq t_1$.) Thus, the probabilistic reasoning implies that entropy will increase in the *positive* time direction for $t > t_1$. Alternatively, if there was no such asymmetry in knowledge, we could not efficiently predict the direction of the arrow of time, so the joint direction would be chosen in an effectively random manner.

Now we can understand why the arrow of time appears to be universal. If there is a

subsystem which has an arrow of time opposite to the time-arrow that we are used to, then this subsystem is either observed or not observed by us. If it is not observed, then it does not violate the fact that the arrow of time appears universal to us. If it is observed then it interacts with us, which implies that it cannot have the opposite arrow of time for a long time. In each case, the effect is that *all what we observe must have the same direction of time* (except, perhaps, during a very short time interval). This is similar to the reasoning in [6], with an important difference that our reasoning does not rest on quantum mechanics.

In the remaining sections we support these intuitive ideas by a more quantitative analysis.

3 Statistical mechanics of the baker’s map

The baker’s map (for more details see Appendix A) maps any point of the unit square to another point of the same square. We study a collection of $N \gg 1$ such points (called “particles”) that move under the baker’s map. This serves as a toy model for a “gas” that shares all typical properties of classical Hamiltonian reversible deterministic chaotic systems. Indeed, due to its simplicity, the baker’s map is widely used for such purposes [18, 20, 21, 22].

3.1 Macroscopic entropy and ensemble entropy

To define a convenient set of macroscopic variables, we divide the unit square into 4 equal subsquares. Then the 4 variables N_1, N_2, N_3, N_4 , denoting the number of “particles” in the corresponding subsquares, are defined to be the macroscopic variables for our system. (There are, of course, many other convenient ways to define macroscopic variables, but general statistical conclusion are not expected to depend on this choice.) The *macroscopic entropy* S_m of a given macrostate is defined by the number of different microstates corresponding to that macrostate, as

$$S_m = -N \sum_{k=1}^4 \frac{N_k}{N} \log \left(\frac{N_k}{N} \right) = - \sum_{k=1}^4 N_k \log \left(\frac{N_k}{N} \right). \quad (1)$$

This entropy is maximal when the distribution of particles is uniform, in which case S_m is $S_m^{\max} = N \log 4$. Similarly, the entropy is minimal when all particles are in the same subsquare, in which case $S_m = 0$.

Let (x, y) denote the coordinates of a point on the unit square. In physical language, it corresponds to the particle position in the 2-dimensional phase space. For N particles, we consider a statistical ensemble with a probability density $\rho(x_1, y_1; \dots; x_N, y_N; t)$ on the $2N$ dimensional phase space. Here t is the evolution parameter, which takes discrete values $t = 0, 1, 2, \dots$ for the baker’s map. Then the *ensemble entropy* is defined as

$$S_e = - \int \rho(x_1, y_1; \dots; x_N, y_N; t) \log \rho(x_1, y_1; \dots; x_N, y_N; t) dX, \quad (2)$$

where

$$dX \equiv dx_1 dy_1 \cdots dx_N dy_N. \quad (3)$$

In general, ρ and S_e change during the evolution generated by the baker's map and depend on the initial ρ . However, if the initial probability-density function has a form

$$\rho(x_1, y_1; \dots; x_N, y_N) = \rho(x_1, y_1) \cdots \rho(x_N, y_N), \quad (4)$$

which corresponds to an uncorrelated density function, then the probability-density function remains uncorrelated during the evolution.

As an example, consider $\rho(x_l, y_l)$ which is uniform within some subregion Σ (with area $A < 1$) of the unit square, and vanishes outside of Σ . In other words, let

$$\rho(x_l, y_l, t) = \begin{cases} 1/A & \text{for } (x_l, y_l) \text{ inside } \Sigma, \\ 0 & \text{for } (x_l, y_l) \text{ outside } \Sigma. \end{cases} \quad (5)$$

In this case

$$S_e = - \left(\frac{1}{A}\right)^N \log \left(\frac{1}{A}\right)^N A^N = N \log A. \quad (6)$$

Since A does not change during the baker's map evolution, we see that S_e is constant during the baker's map evolution. This example can be used to show that S_e is, in fact, constant for *arbitrary* initial probability function. To briefly sketch the proof, let us divide the unit $2N$ -dimensional box into a large number of small regions Σ_a , on each of which the probability is equal to ρ_a . During the evolution, each region Σ_a changes the shape, but its $2N$ -dimensional "area" A_a remains the same. Moreover, the probability ρ_a on the new Σ_a also remains the same. Consequently, the ensemble entropy $S_e = - \sum_a A_a^N \rho_a \log \rho_a$ remains the same as well. This is the basic idea of a discrete version of the proof, but a continuous version can be done in a similar way.

3.2 Appropriate and inappropriate macroscopic variables

The macroscopic variables defined in the preceding subsection have the following properties:

1. For most initial microstates having the property $S_m < S_m^{\max}$, S_m increases during the baker's map.
2. For most initial microstates having the property $S_m = S_m^{\max}$, S_m remains constant during the baker's map.
3. The two properties above do not change when the baker's map is perturbed by a small noise.

We refer to macrovariables having these properties as *appropriate* macrovariables.

Naively, one might think that any seemingly reasonable choice of macrovariables is appropriate. Yet, this is not really the case. Let us demonstrate this by an example. Let us divide the unit square into 2^M equal vertical strips ($M \gg 1$). We define a new set of macrovariables as the numbers of particles inside each of these strips. Similarly to (1), the corresponding macroscopic entropy is

$$S_m = - \sum_{k=1}^{2^M} N_k \log \left(\frac{N_k}{N} \right), \quad (7)$$

where N_k is the number of particles in strip k . For the initial condition, assume that the gas is uniformly distributed inside odd vertical strips, while even strips are empty. Then $S_m < S_m^{\max}$ initially. Yet, for a long time during the baker's evolution, S_m does not increase for any initial microstate corresponding to this macrostate. However, during this evolution the number of filled strips decreases and their thickness increases, until only one thick filled vertical strip remains. After that, S_m starts to increase. We also note that the evolution towards the single strip can be easily destroyed by a small perturbation.

Thus we see that vertical strips lead to inappropriate macrovariables. By contrast, horizontal strips lead to appropriate macrovariables. (Yet, the macrovariables in (1) are even more appropriate, because they lead to much faster growth of S_m .) This asymmetry between vertical and horizontal strips is a consequence of the intrinsic asymmetry of the baker's map with respect to vertical and horizontal coordinates. This asymmetry is analogous to the asymmetry between canonical coordinates and momenta in many realistic Hamiltonian systems of classical mechanics. Namely, most realistic Hamiltonian systems contain only local interaction between particles, where locality refers to a separation in the coordinate (not momentum!) space.

Finally, we note that evolution of the macroscopic variables $N_k(t)$, $k = 1, 2, 3, 4$, is found by averaging over ensemble in the following way

$$N_k(t) = \int N_k(x_1, y_1; \dots; x_N, y_N; t) \rho(x_1, y_1; \dots; x_N, y_N; t) dX. \quad (8)$$

3.3 Coarsening

As we have already said, the ensemble entropy (unlike macroscopic entropy) is always constant during the baker's map evolution. One would like to have a modified definition of the ensemble entropy that increases similarly to the macroscopic entropy. Such a modification is provided by *coarsening*, which can be defined by introducing a coarsened probability-density function

$$\begin{aligned} \rho^{\text{coar}}(x_1, y_1; \dots; x_N, y_N) &= \int \Delta(x_1 - x'_1, y_1 - y'_1; \dots; x_N - x'_N, y_N - y'_N) \\ &\quad \times \rho(x'_1, y'_1; \dots; x'_N, y'_N) dX', \end{aligned} \quad (9)$$

where Δ is nonvanishing in some neighborhood of $X' = 0, 0; \dots; 0, 0$. In this way, the coarsened ensemble entropy is

$$S_e^{\text{coar}} = - \int \rho^{\text{coar}}(x_1, y_1; \dots; x_N, y_N) \log \rho^{\text{coar}}(x_1, y_1; \dots; x_N, y_N) dX. \quad (10)$$

Of course, the function Δ can be chosen in many ways. In the following we discuss a few examples.

One example is the Boltzmann coarsening, defined by

$$\rho^{\text{coar}}(x_1, y_1; \dots; x_N, y_N) = \rho(x_1, y_1) \cdots \rho(x_N, y_N), \quad (11)$$

where

$$\rho(x_1, y_1) = \int \rho(x_1, y_1; \dots; x_N, y_N) dx_2 dy_2 \cdots dx_N dy_N, \quad (12)$$

and similarly for other $\rho(x_l, y_l)$.

Another example is isotropic coarsening, having a form

$$\begin{aligned} &\Delta(x_1 - x'_1, y_1 - y'_1; \dots; x_N - x'_N, y_N - y'_N) = \\ &\Delta(x_1 - x'_1)\Delta(y_1 - y'_1) \cdots \Delta(x_N - x'_N)\Delta(y_N - y'_N). \end{aligned} \quad (13)$$

Yet another example is the Prigogine coarsening [18]

$$\Delta(x_1 - x'_1, y_1 - y'_1; \dots; x_N - x'_N, y_N - y'_N) = \Delta(y_1 - y'_1) \cdots \Delta(y_N - y'_N), \quad (14)$$

which is an anisotropic coarsening over the shrinking direction y .

Finally, let us mention the coarsening based on dividing the system into two smaller interacting subsystems. The coarsened ensemble entropy for the full system is defined as the sum of uncoarsened ensemble entropies of its subsystems. Such a coarsened entropy ignores the correlations between the subsystems.

All these types of coarsening have the following property: If the initial microstate is such that macroscopic entropy increases, then the coarsened ensemble entropy also increases for that initial microstate. Yet, the Prigogine coarsening has the following advantages over Boltzmann and isotropic coarsenings:

First, if the initial microstate is such that the macroscopic entropy *decreases*, then the Prigogine coarsened ensemble entropy does *not* decrease, while the Boltzmann and isotropic coarsened ensemble entropies decrease.

Second, assume that the initial microstate is such that the macroscopic entropy increases, and consider some “final” state with a large macroscopic entropy close to the maximal one. After this final state, consider the “inverted” baker’s evolution, (i.e., the baker’s evolution with exchanged x and y). Then the Prigogine coarsened ensemble entropy decreases, while the Boltzmann and isotropic coarsened ensemble entropies remain unchanged.

Thus, the Prigogine coarsening provides the most correct description of the ensemble-entropy increase law without any additional assumptions. For example, to get the same result with Boltzmann coarsening, one would need to introduce the additional “molecular chaos hypothesis” to replace $\rho(x_1, y_1; x_2, y_2)$ with $\rho(x_1, y_1)\rho(x_2, y_2)$ in the equation of motion for $\rho(x, y, t)$.

4 The effects of weak interactions

4.1 Small external perturbations

The growth of the ensemble entropy can be achieved even without coarsening, by introducing a small external perturbation of the baker’s map. The perturbation must be small enough not to destroy the growth of macroscopic entropy, but at the same time, it must be strong enough to destroy the reverse processes and Poincare returns. For most such perturbations, the qualitative features of the evolution do not depend much on details of the perturbation.

There are two ways how the external perturbation can be introduced. One way is to introduce a small external random noise. The macroscopic processes with the increase of macroscopic entropy are stable under such a noise. However, the area of a region is no

longer invariant under the perturbed baker's map. In this way the ensemble entropy can decrease.

The other way is to introduce a weak interaction with the environment (which can be thought of as an "observer"). Again, the macroscopic processes with the increase of macroscopic entropy are stable, but the area of a region is no longer invariant under the perturbed baker's map. Consequently, the ensemble entropy can decrease. However, such a system is no longer isolated. Instead, it is a part of a larger system divided into two subsystems. Hence, as we have already explained in Sec. 3.3, the coarsened ensemble entropy for the full system can be defined as the sum of uncoarsened ensemble entropies of its subsystems. In the next subsection we study the weak interactions with the environment in more detail.

4.2 Weak interaction and the destruction of opposite time arrows

To proceed, one needs to choose some specific interaction between two gases. In the absence of interaction, each of them evolves according to the baker's map. We put the two unit squares one above another and specify the interaction with distance σ such that, between two steps of the baker's map, all closest pairs of particles (with distance smaller than σ between them) exchange their positions. (More precisely, we first find the pair of closest particles (with distance smaller than σ between them) and exchange their positions. After that, we find the second pair of closest particles (with distance smaller than σ between them and different from previously chosen particles) and exchange their positions too. We repeat this procedure until we exhaust all particles.) Such interaction does not affect the motion of the particles, but gives rise to the mixing between the two subsystems when two particles of the pair belong to different subsystems. When they belong to the same system, we interpret them as trivial irrelevant exchanges, and consequently think of them as exchanges that have not happened at all. In this sense, the interactions happen *only* between the particles in different subsystems. Note also that such mixing by itself does not lead to the Gibbs paradox, as long as we consider the two unit squares as separate objects. The macroscopic entropy is defined as the sum of macroscopic entropies of the two subsystems.

Now let us consider the case in which the time arrows of the two subsystems have the same direction. The processes in which the macroscopic entropies of the two subsystems increase are stable under the interaction. Thus, most low-entropy initial conditions lead to a growth of macroscopic entropy of both subsystems, as well as of the full system.

Similarly, if we inverse a process above with increasing macroscopic entropy, we obtain a system in which macroscopic entropy of both subsystems, as well as of the full system - *decreases*. In this sense, the interaction does not ruin the symmetry between the two directions of time.

Now let us consider the most interesting case, in which entropy increases in the first subsystem and decreases in the second. The initial state of the first subsystem has a low entropy (for example, all particles are in some small square near the point $(0,0)$ of the unit square). Likewise, the second system has a low entropy (for example, all particles are in some small square near the point $(1,1)$ of the unit square) in the *final* state

If there was no interaction, the final state of the first subsystem would be a high-entropy state corresponding to a nearly uniform distribution of particles. Likewise, the initial state of the second system would be a high-entropy state of the same form.

However, the solutions above with two opposite arrows of time are no longer solutions when the interaction is present. In most cases, the interaction mixes the particles between the subsystems. The number of solutions with interaction which have the initial-final conditions prescribed above is very small, in fact *much smaller than the number of such solutions in the absence of interaction*.

Let us make the last assertion more quantitative. After an odd number of (non-trivial) exchanges, the particle transits to the other subsystem. Likewise, after an even number of such exchanges, it remains in the same subsystem. The probabilities for these two events are equal to $p = 1/2$ and do not depend on other particles, at least approximately. Further, we can argue that the mixing between the two subsystems is negligible in the initial and final states, as the entropies of the two subsystems are very different. We want to calculate the probability of a small mixing in the final state, given that the mixing is small in the initial state. For definiteness, we shall say that the mixing is small if the number N_t of transited particles is either $N_t < N/4$ or $N_t > 3N/4$. Thus, the probability is given by the cumulative binomial distribution $F(N_t; N, 1/2)$, given by

$$F(k; n, p) = \sum_{i=0}^{\lfloor k \rfloor} \binom{n}{i} p^i (1-p)^{n-i} \quad (15)$$

where $\lfloor k \rfloor$ is the greatest integer less than or equal to k . The function $F(k; n, p)$ satisfies the bound

$$F(k; n, p) \leq \exp\left(-2\frac{(np - k)^2}{n}\right). \quad (16)$$

Thus, since the opposite time arrows of subsystems are not destroyed when $N_t < N/4$ or $N_t > 3N/4$, we see that the probability of this is equal to

$$2F(N/4; N, 1/2) \leq 2e^{-N/8}. \quad (17)$$

Clearly, it decreases exponentially with N , which means that such a probability is negligibly small for large N . Hence, it is almost certain that processes with opposite time arrows will be destroyed.

In the model above, we need a nearly equal number of particles in the two subsystems to destroy the opposite time arrows. This is because one particle can influence the motion of only one close particle. For more realistic interactions, one particle can influence the motion of a large number of particles in its neighborhood, which means that even a very small number of particles in one system can destroy the entropy decreasing processes of the other system.

4.3 Decorrelation in the interacting system

Hamiltonian systems are described not only by a macrostate, but also by complex nonlinear correlations between microstates. These correlations are responsible for reversibility. The interaction between two subsystems destroys these correlations inside the subsystems,

but the full system remains reversible, i.e., the correlations appear in the full system. Thus, the decorrelation in the subsystems expands the correlations over the full system. (This process is a classical analogue of decoherence in quantum mechanics.)

Let us put these qualitative ideas into a more quantitative form. Linear (Pearson) correlations have a behavior very similar to the nonlinear correlations described above. The only difference is that these linear correlations decrease with time. The interaction we proposed can be approximated by a random noise with amplitude corresponding to a distance σ of the interaction between the particles. Therefore, we expect that the interaction not only causes the alignment of the time arrows, but also a decay of correlation which is even stronger than that without the interactions (Sec. A.5). During this process the evolution of subsystems is irreversible, but the full system remains reversible.

We can quantify this decay of correlations by calculating the Pearson correlation for our subsystems, given by

$$r(m) = \frac{C(m)}{\sqrt{C(0)\langle C^m(0) \rangle}}, \quad (18)$$

where $\langle C^m(0) \rangle$ is the expected variance of the random variable x calculated after m iterations of the map. The variance $C^m(0)$ can be calculated as

$$C^m(0) = \sum_{j=0}^{2^m-1} \int_{j2^{-m}}^{(j+1)2^{-m}} (2^m x - j - \langle x \rangle + S)^2 dx, \quad (19)$$

where S is a random number defined as $S = \sum_{k=0}^{m-1} 2^k \zeta_k$. Here ζ_k is an i.i.d. random number with zero mean and variance σ^2 , which models the influence of interactions on the evolution of the system. After a short calculation we get

$$\langle C^m(0) \rangle = C(0) + \langle S^2 \rangle = C(0) + \sum_{k,k'=0}^{m-1} 2^{k+k'} \langle \zeta_k \zeta_{k'} \rangle. \quad (20)$$

Using the properties of i.i.d. variables $\langle \zeta_k \zeta_{k'} \rangle = \delta_{kk'} \sigma^2$, it follows that

$$\langle C^m(0) \rangle = C(0) + \frac{2^{2m} - 1}{3} \sigma^2. \quad (21)$$

It is clear that the interactions will enhance the decay of correlations of at least linear dependencies, because

$$r(m) = \frac{2^{-m}}{\sqrt{1 + 4(2^{2m} - 1)\sigma^2}}. \quad (22)$$

Yet, for the full system the Pearson correlation $r(m) = 2^{-m}$ remains the same. Since $\langle S^2 \rangle^{1/2}$ must be much smaller than the system size (unit square), we can conclude that our assumptions resulting in (22) are correct only for $\langle S^2 \rangle = [(2^{2m} - 1)/3]\sigma^2 \ll 1$ and $\sigma^2/2^{-2m} \ll 1$.

4.4 Numerical simulation

So far, we have been using general abstract arguments. In this subsection we support these arguments by a concrete numerical simulation. We consider two subsystems (labeled as 1

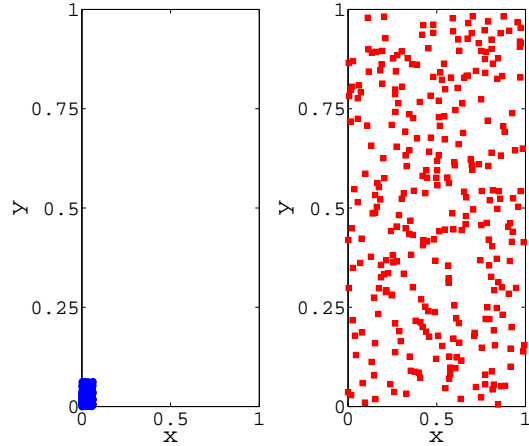


Figure 1: The initial particle configuration at $t = 1$.

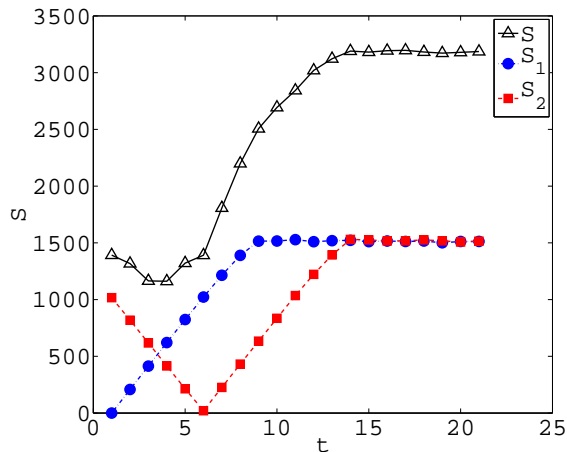


Figure 2: Evolution of entropy without interaction.

and 2), each with $N_1 = N_2 = 300$ particles. The two subsystems occupy two unit squares. To define the coarsened entropy, each unit square is divided into $16 \times 16 = 256$ small squares. Thus, the entropy in the two subsystems is given by

$$S_i = -N_i \sum_{k=1}^{512} f_{k,i} \log f_{k,i}, \quad (23)$$

where $i = 1, 2$, $f_{k,i} = n_{k,i}/N_i$, and $n_{k,i}$ is the number of particles in the corresponding small square. Similarly, the total entropy is defined as

$$S = -(N_1 + N_2) \sum_{k=1}^{512} f_k \log f_k, \quad (24)$$

where $f_k = (n_{k,1} + n_{k,2})/(N_1 + N_2)$

For the system 1 we choose a zero-entropy initial state at $t = 1$ (see Fig. 1). Similarly, for the system 2 we choose a zero-entropy “initial” state at $t = 6$. Such initial conditions

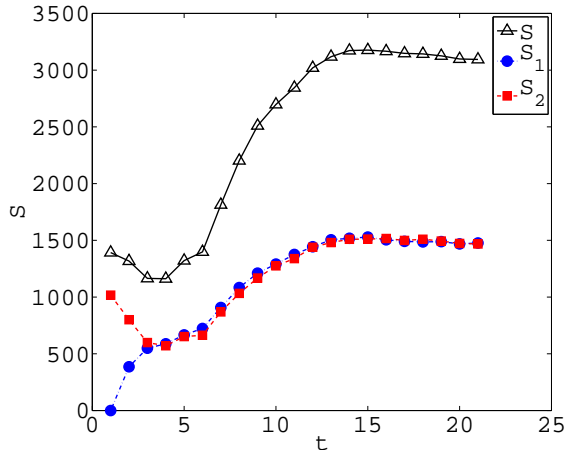


Figure 3: Evolution of entropy with interaction.

provide that, in the absence of interactions, S_1 increases with time, while S_2 decreases with time for $t < 6$. To avoid numerical problems arising from the finite precision of computer representation of rational numbers, (27) is replaced by $x' = ax - \lfloor ax \rfloor$, $y' = (y + \lfloor ax \rfloor)/2$, with $a = 1.999999$. The results of a numerical simulation are presented in Fig. 1 and Fig. 2.

To include the effects of interaction, we define interaction in the following way. (For the sake of computational convenience, it is defined slightly differently than in Sec. 4.2.) We take a small range of interaction $r_y = 0.01$ in the y -direction, which can be thought of as a parameter that measures the weakness of interaction. (Recall that y and x are analogous to a canonical coordinate and a canonical momentum, respectively, in a Hamiltonian phase space.) The interaction exchanges the closest pairs similarly as in Sec. 4.2, but now “the closest” refers to the distance in the y -direction, and there is no exchange if the closest distance is larger than r_y . In addition, now interaction is defined such that only the x -coordinates of the particles are exchanged. By choosing the same initial conditions at $t = 1$ as in the non-interacting case (Fig. 1), the results of a numerical simulation with the interaction are presented in Fig. 3. We see that with interaction (Fig. 3) S_2 starts to increase at earlier times than without interaction (Fig. 2).

5 Discussion

In this paper, we have used the toy model based on the baker’s map to demonstrate features which seem to be valid for general systems described by reversible Hamiltonian mechanics. Clearly, for such systems one can freely choose either final or initial conditions, but one cannot freely choose them both. For most mixed initial-final conditions, an appropriate solution (of the Hamiltonian equations of motion) does not exist. Similarly, our toy model suggests that for most Hamiltonians with weak interactions, the number of solutions with given coarse-grained initial-final conditions is much smaller than the number of solutions with only coarse-grained initial or only coarse-grained final conditions. This explains why, in practice, we never observe subsystems with opposite arrows of time, i.e.,

why the the arrow of time is universal.

In a sense, this destruction of opposite arrows of time is similar to ergodicity. Both properties are valid for all practical purposes only, they are not exact laws. They are true for most real systems, but counterexamples can always be found. Also, they both may seem intuitively evident, but to prove them rigorously is very difficult. For ergodicity the relevant rigorous result is the KAM (Kolmogorov–Arnold–Moser) theorem, while for the destruction of the opposite time arrows a rigorous theorem is still lacking.

Our results also resolve the “contradiction” between the Prigogine’s “New Dynamics” [18] (discussed in Sec. 3.3 of the present paper) and Bricmont’s comments [23]. Dynamics of interacting systems we can be divided into two types of dynamics:

1. Reversible ideal dynamics is considered with respect to the coordinate time, in which case entropy can either decrease or increase.
2. Irreversible observable dynamics is considered with respect to the intrinsic time arrow of interacting systems, in which case entropy increases.

In the framework of this terminology, the Prigogine’s “New Dynamics” [18] is one of the forms of the observable dynamics, while the Bricmont’s paper [23] considers ideal dynamics. In particular, the observable dynamics does not include Poincare’s returns and reversibility, that are indeed unobservable by a real observer, which makes it simpler than ideal dynamics. Yet, in principle, both types of dynamics are correct.

It should also be noted that our results are not in contradiction with the existence of dissipative systems [24] (such as certain self-organizing biological systems) in which entropy of a subsystem can decrease with time, despite the fact that entropy of the environment increases. The full-system entropy (including the entropies of both the dissipative system and the environment) increases, which is consistent with the entropy-increase law. For such systems, it is typical that the interaction with the environment is *strong*, while results of our paper refer to *weak* interactions between the subsystems. For example, for existence of living organisms, a strong energy flow from the Sun is needed. The small flow from the stars is not sufficient for life, but is sufficient for the decorrelation and for the alignment of the time arrows. To quote from [6]: “However, an observer is macroscopic by definition, and all remotely interacting macroscopic systems become correlated very rapidly (e.g. Borel famously calculated that moving a gram of material on the star Sirius by 1 m can influence the trajectories of the particles in a gas on earth on a time scale of μs [19]).”

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A Basic properties of the baker’s map

In this appendix we present some basic properties of the baker’s map. More details can be found, e.g., in [25].

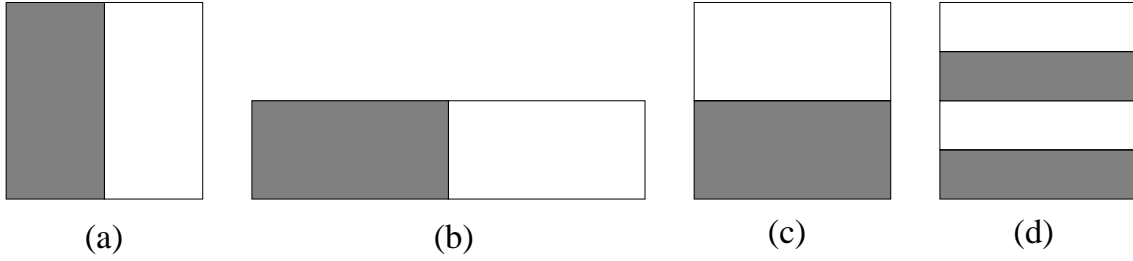


Figure 4: Geometric interpretation of the baker's map. (a) Initial configuration. (b) Uniform squeezing in vertical direction and stretching in horizontal direction by a factor of 2. (c) The final configuration after cutting the right half and putting it over the left one. (d) The final configuration after two iterations.

A.1 Definition of the baker's map

Consider a binary symbolic sequence

$$\dots S_{-2}, S_{-1}, S_0; S_1, S_2, S_3 \dots \quad (25)$$

infinite on both sides. Such a sequence defines two real numbers

$$x = 0.S_1S_2S_3\dots, \quad y = 0.S_0S_{-1}S_{-2}\dots \quad (26)$$

The sequence can be moved reversibly with respect to the semicolon in both directions. After the left shift we get new real numbers

$$x' = 2x - [2x], \quad y' = \frac{1}{2}(y + [2x]), \quad (27)$$

where $[x]$ is the greatest integer less than or equal to x . This map of unit square into itself is called the *baker's map*.

The baker's map has a simple geometrical interpretation presented in Fig. 4. There (a) is the initial configuration and (c) is the final configuration after one baker's iteration, with an intermediate step presented in (b). The (d) part represents the final configuration after two iterations.

A.2 Unstable periodic orbits

The periodic symbolic sequences (0) and (1) correspond to fixed points $(x, y) = (0, 0)$ and $(x, y) = (1, 1)$, respectively. The periodic sequence (10) corresponds to the period-2 orbit $\{(1/3, 2/3), (2/3, 1/3)\}$. From periodic sequence $\dots 001; 001 \dots$ we get $\{(1/7, 4/7), (2/7, 2/7), (4/7, 1/7)\}$. Similarly, from $\dots 011; 011 \dots$ we get $\{(3/7, 6/7), (6/7, 3/7), (5/7, 5/7)\}$.

Any x and y can be approximated arbitrarily well by $0.X_0\dots X_n$ and $0.Y_0\dots Y_m$, respectively, provided that n and m are sufficiently large. Therefore the periodic sequence $(Y_m\dots Y_0X_0\dots X_n)$ can approach any point of the unit square arbitrarily close. Thus, the set of all periodic orbits makes a dense set on the unit square.

A.3 Ergodicity, mixing, and area conservation

Due to stretching in the horizontal direction, all close points diverge exponentially under the baker's iterations. In these iterations, a random symbolic sequence approaches any point of the square arbitrarily close. In general, such an ergodic property can be used to replace the "time" average $\langle A \rangle$ by the "ensemble" average

$$\langle A \rangle = \sum_n A(x_n, y_n) = \int A(x, y) d\mu(x, y) = \int A(x, y) \rho(x, y) dx dy, \quad (28)$$

where $d\mu(x, y)$ is the invariant measure and $\rho(x, y)$ is the invariant density for the map. For the baker's map, $\rho(x, y) = 1$.

Under the baker's iterations, any region maps into a set of narrow horizontal strips. Eventually, it fills uniformly the whole unit square, which corresponds to mixing. Similarly, reverse iterations map the region into narrow vertical strips, which also corresponds to mixing.

During these iterations, the area of the region does not change. This property is the area conservation law for the baker's map.

A.4 Lyapunov exponent, shrinking and stretching directions

If $x_0^{(1)}$ and $x_0^{(2)}$ have equal k first binary digits, then, for $n < k$,

$$x_n^{(2)} - x_n^{(1)} = 2^n(x_0^{(2)} - x_0^{(1)}) = (x_0^{(2)} - x_0^{(1)})e^{n \log 2}, \quad (29)$$

where $\Lambda = \log 2$ is the first positive Lyapunov exponent for the baker's map. Consequently, the distance between two close orbits increases exponentially with increasing n , and after k iterations becomes of the order of 1. This property is called sensitivity to initial conditions. Due to this property, all periodic orbits are unstable.

Since the area is conserved, the stretching in the horizontal direction discussed above implies that that some shrinking direction must also exist. Indeed, the evolution in the vertical y direction is opposite to that of the horizontal x direction. If $(x_0^{(1)}, y_0^{(1)})$ and $(x_0^{(2)}, y_0^{(2)})$ are two points with $x_0^{(1)} = x_0^{(2)}$, then

$$y_n^{(2)} - y_n^{(1)} = 2^{-n}(y_0^{(2)} - y_0^{(1)}) = (y_0^{(2)} - y_0^{(1)})e^{n(-\log 2)}. \quad (30)$$

Hence $\Lambda = -\log 2$ is the second negative Lyapunov exponent for the baker's map.

A.5 Decay of correlations

Since x -direction is the unstable direction, the evolution in that direction exhibits a decay of correlations. The average correlation function $C(m)$ for a sequence x_k is usually defined as

$$C(m) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (x_k - \langle x \rangle) (x_{k+m} - \langle x \rangle), \quad (31)$$

where $\langle x \rangle = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k / n$. Correlations can be more easily calculated if one knows the invariant measure $\mu(x)$, in which case

$$C(m) = \int (x - \langle x \rangle) (f^m(x) - \langle x \rangle) d\mu(x), \quad (32)$$

where $f^m(x) = x_m$ is the function that maps the variable x to its image after m iterations of the map. For the baker's map $d\mu(x) = dx$, so we can write

$$C(m) = \sum_{j=0}^{2^m-1} \int_{j2^{-m}}^{(j+1)2^{-m}} (x - \langle x \rangle) (2^m x - j - \langle x \rangle) dx, \quad (33)$$

which yields

$$C(m) = \sum_{j=0}^{2^m-1} \left[2^m \frac{x^3}{3} - (2^m \langle x \rangle + \langle x \rangle) \frac{x^2}{2} + \langle x \rangle^2 x - j \left(\frac{x^2}{2} - \langle x \rangle x \right) \right]_{j2^{-m}}^{(j+1)2^{-m}}. \quad (34)$$

For the baker's map $\langle x \rangle = 1/2$, so the sum above can be calculated explicitly

$$C(m) = \frac{2^{-m}}{12}. \quad (35)$$

This shows that the correlations decay exponentially with m . The Pearson correlation for the system is given by

$$r(m) = C(m)/C(0) = 2^{-m}. \quad (36)$$

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