THE HARTMAN-WATSON DISTRIBUTION REVISITED: ASYMPTOTICS FOR PRICING ASIAN OPTIONS

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Abstract. Barrieu, Rouault, and Yor [*J. Appl. Probab.* 41 (2004)] determined asymptotics for the logarithm of the distribution function of the Hartman-Watson distribution. We determine the asymptotics of the density. This refinement can be applied to the pricing of Asian options in the Black-Scholes model.

1. Introduction

The distribution of the integral of geometric Brownian motion has attracted a lot of interest, mainly because it is needed to calculate the price of Asian options in the Black-Scholes model. Yor [11] found the formula

$$\mathbf{P}[A_t^{(\nu)} \in \mathrm{d}u \mid W_t + \nu t = x] = \frac{\sqrt{2\pi t}}{u} \exp\left(\frac{x^2}{2t} - \frac{1+\mathrm{e}^{2x}}{2u}\right) \mathrm{I}_0(\mathrm{e}^x/u) f_{\mathrm{e}^x/u}(t)\,\mathrm{d}u, \tag{1}$$

where I_ν denotes, as usual, the modified Bessel function of the first kind, and

$$A_t^{(\nu)} = \int_0^t \exp(2(W_h + \nu h))\,\mathrm{d}h,$$

where W is a standard Brownian motion. The present note focuses on the function $f_r(t)$ in (1), which is the density of the Hartman-Watson distribution [6]. Small time asymptotics of (1) correspond to left tail asymptotics of $f_r(t)$, which is defined by the Laplace transform

$$\int_0^\infty \mathrm{e}^{-ut} f_r(t)\,\mathrm{d}t = \frac{\mathrm{I}_{\sqrt{2u}}(r)}{\mathrm{I}_0(r)}, \qquad r \geq 0.$$

Numerical problems in the evaluation of (1) for small t prompted Barrieu, Rouault, and Yor [1] to analyze the left tail of the Hartman-Watson distribution asymptotically. Using the Gärtner-Ellis theorem from large deviations theory, they obtained the asymptotics

$$F_r(t) = \exp\left(-\frac{\log(1/t)^2}{2t} + \mathrm{o}\Big(\frac{\log(1/t)^2}{t}\Big)\right), \qquad t \to 0, \tag{2}$$

for the distribution function. However, this result is not immediately applicable to the calculation of (1). Barrieu et al. [1] write that "the standard asymptotic methods (e.g. the saddle point method) do not seem to be suitable for this study", and that they "are not able to refine these results for the Hartman-Watson density itself".

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In fact the saddle point method is applicable to the Laplace inversion integral for the density $f_r(t)$, but needs some care. First, replacing $\mathrm{I}_{\sqrt{2u}}(r)$ by an asymptotic approximation relieves us from studying potentially difficult monotonicity properties of the modified Bessel function, and allows to formulate the result in a way that avoids roots of equations involving the Bessel function. Second, it turns out that elementary approximations of the integrand's saddle point lead to integration contours that are too far away from the saddle to make the method work. An approach based on a contour through the *exact* saddle point establishes the following asymptotics for the density $f_r(t)$. For brevity, we write

$$\rho = \log \frac{r}{2\sqrt{2}}.$$

Theorem 1. *For $t > 0$, denote by $u_0(t)$ the largest solution of the equation*

$$t = \frac{\sqrt{2}\log u}{4\sqrt{u}} + \frac{\rho\sqrt{2}}{2\sqrt{u}} - \frac{1}{4u}, \tag{3}$$

which exists for all sufficiently small t. Then the Hartman-Watson density satisfies

$$f_r(t) = \frac{1}{\pi\sqrt{\mathrm{e}}\,\mathrm{I}_0(r)}\sqrt{\frac{u_0(t)}{\log u_0(t) - 2 - 2\rho}} \times \mathrm{e}^{-tu_0(t)+\sqrt{2}(1+2\rho)\sqrt{u_0(t)}} \times \left(1 + \mathrm{O}(\sqrt{t}\log(1/t)^2)\right) \tag{4}$$

$$= \frac{1}{2\pi\sqrt{\mathrm{e}}\,\mathrm{I}_0(r)}\frac{\log(1/t)^{1/2}}{t}\mathrm{e}^{-tu_0(t)+\sqrt{2}(1+2\rho)\sqrt{u_0(t)}}\left(1 + \mathrm{O}\Big(\frac{\log\log(1/t)}{\log(1/t)}\Big)\right) \tag{5}$$

as $t \to 0$.

Formula (4) gives a much better approximation; the simplification in (5) is of little use, since $u_0(t)$ has to be computed anyway to evaluate (4) or (5) numerically.

To get a feel for the growth of the exponential in (4), we expand $u_0(t)$ by bootstrapping (cf. de Bruijn [2, Section 2.4]):

$$u_0(t) = \frac{\log(1/t)^2}{2t^2}\left(1 + \frac{2\log\log(1/t)}{\log(1/t)} + \frac{2\rho}{\log(1/t)} + \mathrm{o}\Big(\frac{1}{\log(1/t)}\Big)\right). \tag{6}$$

Therefore the exponent in (4) has the expansion

$$\begin{aligned} & -tu_0(t) + \sqrt{2}(1+2\rho)\sqrt{u_0(t)} \\ &= -\frac{\log(1/t)^2}{2t} - \frac{\log(1/t)\log\log(1/t)}{t} + (1+\rho)\frac{\log(1/t)}{t} + \mathrm{o}\Big(\frac{\log(1/t)}{t}\Big). \end{aligned} \tag{7}$$

This shows in particular that the formula

$$f_r(t) = \exp\left(-\frac{\log(1/t)^2}{2t} + \mathrm{o}\Big(\frac{\log(1/t)^2}{t}\Big)\right),$$

obtained from (2) by formal differentiation, is correct.

For numerical accuracy, it is certainly preferable to use (4) as it is, without replacing the exponent by (7); still, the expansion (6) can serve as good initial guess when computing the root of (3). In this way, the leading term of $f_r(t)$ can be calculated effortlessly even for extremely small values of t, say $t = 10^{-50}$.

2. Analysis of the Laplace inversion integral

The Laplace inversion formula yields the representation

$$f_r(t) = \frac{1}{2\mathrm{i}\pi} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{e}^{ut} \frac{\mathrm{I}_{\sqrt{2u}}(r)}{\mathrm{I}_0(r)}\, \mathrm{d}u,$$

where the integration contour lies in the right half-plane. To estimate the growth of $f_r(t)$ near $t = 0$, we have to investigate the singularity at infinity of the integrand. For large index, the modified Bessel function admits the expansion [7, 9]

$$\mathrm{I}_\nu(r) \sim \left(\frac{r}{2}\right)^\nu \mathrm{e}^\nu \nu^{-\nu-1/2} \left(c_0 + \frac{c_1}{\nu} + \dots\right), \tag{8}$$

where the c_i are constants, with $c_0 = 1/\sqrt{2\pi}$. This holds for $\nu \to \infty$, uniformly w.r.t. $\arg(\nu)$, as long as $\arg(\nu)$ is bounded away from $\pm\pi$. Horn [7] shows (8) for $\mathrm{J}_\nu(r)$, the Bessel function of the first kind, but from the relation $\mathrm{I}_\nu(r) = \mathrm{e}^{-\nu\pi\mathrm{i}/2}\mathrm{J}_\nu(r\mathrm{e}^{\pi\mathrm{i}/2})$ one easily sees that replacing $\mathrm{J}_\nu(r)$ by $\mathrm{I}_\nu(r)$ only affects the constants $c_1, c_2, \dots$ in this expansion. If we let the real part R of our integration contour tend to infinity as $t \to 0$, we therefore have

$$\begin{aligned} f_r(t) &= \frac{1}{(2\pi)^{3/2}\,\mathrm{i}\,\mathrm{I}_0(r)} \int_{R-\mathrm{i}\infty}^{R+\mathrm{i}\infty} \left(\frac{r}{2}\right)^{\sqrt{2u}} \mathrm{e}^{ut+\sqrt{2u}} (\sqrt{2u})^{-\sqrt{2u}-1/2} \mathrm{d}u \times (1 + \mathrm{O}(R^{-1/2})) \\ &= \frac{2^{-7/4}}{\pi^{3/2}\,\mathrm{i}\,\mathrm{I}_0(r)} \int_{R-\mathrm{i}\infty}^{R+\mathrm{i}\infty} \exp\Big(ut - \tfrac{1}{2}\sqrt{2}\sqrt{u}\log u + \sqrt{2}(1+\rho)\sqrt{u} \\ &\qquad - \tfrac{1}{4}\log u\Big)\, \mathrm{d}u \times (1 + \mathrm{O}(R^{-1/2})). \end{aligned}$$

The integrand of the latter integral has a saddle point, let us call it $u_0 = u_0(t)$, which is found by equating the derivative to zero. This yields equation (3). In many instances of the saddle point method, it suffices to choose a contour that passes through an approximation of the saddle point. Note that Wong [10] calls this the "steepest descent method", and reserves the term "saddle point method" for integration paths that traverse an *exact* saddle point. In our example, one might try to use a contour based on (6). However, painful calculations reveal that the concentration of the integrand around the approximate saddle point is insufficient, no matter how many terms of (6) are taken. We therefore set the real part of the integration contour to the exact saddle point, so that $R = u_0$:

$$f_r(t) \sim \frac{2^{-7/4}}{\pi^{3/2}\,\mathrm{i}\,\mathrm{I}_0(r)} \int_{u_0-\mathrm{i}\infty}^{u_0+\mathrm{i}\infty} \exp\Big(ut - \tfrac{1}{2}\sqrt{2}\sqrt{u}\log u + \sqrt{2}(1+\rho)\sqrt{u} - \tfrac{1}{4}\log u\Big)\, \mathrm{d}u. \tag{9}$$

Thus

$$u = u_0 + \mathrm{i}y, \qquad -\infty < y < \infty.$$

Close to the saddle point, i.e. for small values of the new integration variable y, we have the uniform expansions

$$\begin{aligned} \sqrt{u} &= \sqrt{u_0} + \frac{\mathrm{i}y}{2\sqrt{u_0}} + \frac{y^2}{8u_0^{3/2}} + \mathrm{O}\Big(\frac{y^3}{u_0^{3/2}}\Big), \\ \log u &= \log u_0 + \frac{\mathrm{i}y}{u_0} + \frac{y^2}{2u_0^2} + \mathrm{O}\Big(\frac{y^3}{u_0^3}\Big), \end{aligned}$$

and

$$\sqrt{u}\log u = \sqrt{u_0}\log u_0 + \frac{\mathrm{i}y}{\sqrt{u_0}} + \frac{\mathrm{i}\log(u_0)y}{2\sqrt{u_0}} + \frac{\log(u_0)y^2}{8u_0^{3/2}} + \mathrm{O}\Big(\frac{\log(u_0)y^3}{u_0^{5/2}}\Big).$$

We insert these into the exponent of (9) and obtain

$$\begin{aligned} &ut - \tfrac{1}{2}\sqrt{2}\sqrt{u}\log u + \sqrt{2}(1+\rho)\sqrt{u} - \tfrac{1}{4}\log u \\ &\qquad = u_0 t - \tfrac{1}{2}\sqrt{2}\sqrt{u_0}\log u_0 + \sqrt{2}(1+\rho)\sqrt{u_0} - \tfrac{1}{4}\log u_0 \\ &\qquad\quad - My^2 + \mathrm{O}\Big(\frac{\log(u_0)y^3}{u_0^{5/2}}\Big), \end{aligned} \tag{10}$$

where

$$M := \frac{\sqrt{2}\log u_0}{16u_0^{3/2}} - \frac{\sqrt{2}(1+\rho)}{8u_0^{3/2}} \tag{11}$$

$$= \frac{t^3}{2\log(1/t)^2}\left(1 + \mathrm{O}\Big(\frac{\log\log(1/t)}{\log(1/t)}\Big)\right). \tag{12}$$

Note that the y-terms in (10) vanish, because we integrate through a saddle point. We now have to identify a range

$$-h < y < h$$

for $y = \Im(u)$ that captures the main contribution to the integral (9). A good choice is

$$h = \frac{\log(1/t)^2}{t^{3/2}},$$

because it satisfies $h\sqrt{M} \to \infty$, so that the integral of the local expansion (10) can be completed to a full Gaussian integral:

$$\begin{aligned} \int_{-h}^{h} \mathrm{e}^{-My^2}\mathrm{d}y &= \frac{1}{\sqrt{2M}}\int_{-h\sqrt{2M}}^{h\sqrt{2M}} \mathrm{e}^{-w^2/2}\mathrm{d}w \\ &\sim \frac{1}{\sqrt{2M}}\int_{-\infty}^{\infty} \mathrm{e}^{-w^2/2}\mathrm{d}w \\ &= \sqrt{\frac{\pi}{M}} \sim \frac{\sqrt{2\pi}\log(1/t)}{t^{3/2}}. \end{aligned} \tag{13}$$

Moreover, the error from the local expansion (10) at the saddle point is o(1), since

$$\frac{\log(u_0)y^3}{u_0^{5/2}} = \mathrm{O}(\sqrt{t}\log(1/t)^2). \tag{14}$$

We can thus determine the asymptotics of the portion $|\Im(u)| \le h$ of the integral (9):

$$\begin{aligned} &\frac{2^{-7/4}}{\pi^{3/2}\,\mathrm{i}\,\mathrm{I}_0(r)}\int_{u_0-\mathrm{i}h}^{u_0+\mathrm{i}h} \exp\Big(ut - \tfrac{1}{2}\sqrt{2}\sqrt{u}\log u + \sqrt{2}(1+\rho)\sqrt{u} - \tfrac{1}{4}\log u\Big)\,\mathrm{d}u \\ &\quad\sim \frac{2^{-7/4}}{\pi^{3/2}\,\mathrm{I}_0(r)}\mathrm{e}^{u_0 t - \frac{1}{2}\sqrt{2}\sqrt{u_0}\log u_0 + \sqrt{2}(1+\rho)\sqrt{u_0} - \frac{1}{4}\log u_0}\int_{-h}^{h}\mathrm{e}^{-My^2}\mathrm{d}y \\ &\quad\sim \frac{2^{-7/4}}{\pi\,\mathrm{I}_0(r)}M^{-1/2}u_0^{-1/4}\mathrm{e}^{u_0 t - \frac{1}{2}\sqrt{2}\sqrt{u_0}\log u_0 + \sqrt{2}(1+\rho)\sqrt{u_0}}. \end{aligned}$$

This gives the right-hand side of (4), after expressing $\sqrt{u_0}\log u_0$ via the saddle point equation (3), which yields

$$-\tfrac{1}{2}\sqrt{2}\sqrt{u_0}\log u_0 = -2u_0t + \rho\sqrt{2u_0} - \tfrac{1}{2}. \tag{15}$$

Expanding u_0 by (6) gives the expression in (5). Note that we have not yet proved (4) and (5); it remains to show that the tails of (9), i.e. the parts where $|\Im(u)| \geq h$, are asymptotically negligible.

3. Tail estimate

It suffices to consider the case $y = \Im(u) \geq h$, since the lower half of the tail can be handled by symmetry.

We first deal with the part of the contour in (9) where the imaginary part of the integration variable is very large, say $y \geq \mathrm{e}^{\log(1/t)^2/4}$. Then y clearly dominates u_0, and it follows from

$$\Re(\log u) \sim \log y, \quad \Re(\sqrt{u}) \sim \tfrac{1}{2}\sqrt{2y}, \quad \text{and} \quad \Re(\sqrt{u}\log u) \sim \tfrac{1}{2}\sqrt{2y}\log y$$

that the absolute value of the integrand is bounded by

$$\mathrm{e}^{u_0t-\sqrt{y}}$$

for small t. Hence we obtain the bound

$$\mathrm{e}^{u_0t}\int_{\mathrm{e}^{\log(1/t)^2/4}}^{\infty} \mathrm{e}^{-\sqrt{y}}\mathrm{d}y \sim 2\exp\left(u_0t + \tfrac{1}{8}\log(1/t)^2 - \mathrm{e}^{\log(1/t)^2/8}\right). \tag{16}$$

Finally, we bound the portion of the integral (9) that is close to the central part, i.e.,

$$h \leq y < \mathrm{e}^{\log(1/t)^2/4}. \tag{17}$$

The following lemma shows that, for small t, the absolute value of the integrand decreases as y increases.

Lemma 2. *Let B be a real number. Then, for $\Re(u) > 0$ and $|u|$ sufficiently large, the real part of $\sqrt{u}\log u + B\sqrt{u}$ decreases w.r.t. $|\Im(u)|$.*

Proof. We write $u = x + \mathrm{i}y$. By symmetry, it suffices to consider the case $y > 0$, so that $\arg(u) > 0$. Straightforward calculations show that

$$\frac{\partial}{\partial y}\Re(\sqrt{u}) = \tfrac{1}{2}|u|^{3/2}\left(y\cos\tfrac{\arg(u)}{2} - x\sin\tfrac{\arg(u)}{2}\right)$$

and

$$\begin{aligned}\frac{\partial}{\partial y}\Re(\sqrt{u}\log u) =& \tfrac{1}{2}|u|^{3/2}\Bigg((\log|u| + 2)\left(y\cos\tfrac{\arg(u)}{2} - x\sin\tfrac{\arg(u)}{2}\right)\\ &- \arg(u)\left(x\cos\tfrac{\arg(u)}{2} + y\sin\tfrac{\arg(u)}{2}\right)\Bigg).\end{aligned}$$

Hence we are led to investigate the sign of

$$y\left((\log|u| + B + 2)\left(\cos\tfrac{\arg(u)}{2} - \tfrac{x}{y}\sin\tfrac{\arg(u)}{2}\right) - \arg(u)\left(\tfrac{x}{y}\cos\tfrac{\arg(u)}{2} + \sin\tfrac{\arg(u)}{2}\right)\right).$$

Suppose that $|u|$ is so large that $\log|u| + B + 2 \geq 12$. In the preceding formula, we estimate the trigonometric functions by the first term of their Taylor series at zero, except the first cos, where we use two terms. This yields the lower bound

$$\begin{aligned} &y\left(12\left(1-\tfrac{1}{8}\arg(u)^2-\tfrac{x}{2y}\arg(u)\right)-\tfrac{x}{y}\arg(u)-\tfrac{1}{2}\arg(u)^2\right)\\ &=y\left(12-2\arg(u)^2-\tfrac{7x}{y}\arg(u)\right)\\ &=y\left(12-2\arctan(w)^2-\tfrac{7}{w}\arctan(w)\right)\Big|_{w=y/x}. \end{aligned}$$

Now observe that $\arctan(w)^2 < \pi^2/4$ and $\arctan(w) < w$ for $w > 0$, so that

$$12-2\arctan(w)^2-\tfrac{7}{w}\arctan(w) > 12-\pi^2/2-7 > 0.$$

This shows that $\Re(\sqrt{u}\log u + B\sqrt{u})$ has a positive derivative w.r.t. y. □

Therefore, we can bound the part (17) of the integral (9) by the value of the integrand at $y = h$ times the length of the path. By (10), (15), and

$$My^2\big|_{y=h} \sim \tfrac{1}{2}\log(1/t)^2,$$

this amounts to a bound of the form

$$\begin{aligned} \text{(18)}\quad &\mathrm{e}^{-tu_0(t)+\sqrt{2}(1+2\rho)\sqrt{u_0(t)}-\frac{1}{2}\log(1/t)^2+\mathrm{o}(\log(1/t)^2)}\times \mathrm{e}^{\frac{1}{4}\log(1/t)^2}\\ &\qquad= \mathrm{e}^{-tu_0(t)+\sqrt{2}(1+2\rho)\sqrt{u_0(t)}-\frac{1}{4}\log(1/t)^2+\mathrm{o}(\log(1/t)^2)}. \end{aligned}$$

To complete the proof of Theorem 1, let us now compare the six error terms that arose in the analysis. If M is not expanded, i.e., (11) is used, then the error

Source of error	*Relative error*
Replace I_ν by (8)	$\mathrm{O}(t/\log(1/t))$
Local expansion (see (10) and (14))	$\mathrm{O}(\sqrt{t}\log(1/t)^2)$
Gaussian tails (see (13))	$\exp(-\frac{1}{2}\log(1/t)^2+\mathrm{o}(\log(1/t)^2))$
Relative error of M (see (12) and (13))	$\mathrm{O}(\frac{\log\log(1/t)}{\log(1/t)})$
Outer tail (see (16))	$\exp(-\mathrm{e}^{\log(1/t)^2/8}+\mathrm{o}(\mathrm{e}^{\log(1/t)^2/8}))$
Inner tail (see (18))	$\exp(-\frac{1}{4}\log(1/t)^2+\mathrm{o}(\log(1/t)^2))$

from the local expansion dominates, which leads to (4). If, on the other hand, the expansion (12) of M is taken, then it is the relative error of M that prevails.

4. Conclusion

A natural way to refine asymptotic results like Theorem 1 is to look for a full asymptotic expansion. Technically speaking, continuing the expansion (12) refines (5) to a full expansion. A better expansion, respecting the asymptotic scale of the problem, can be obtained by retaining the explicit formula (11) for M, and taking more terms in (8) and (10). This should pose no conceptual problems; note, however, that each term in the expansion (8) gives rise to a new saddle point, as the coefficient of $1/u$ in (3) changes.

Concerning applications, our results can be used as a substitute for the Hartman-Watson density for small arguments, in particular, for evaluating the density of $A_t^{(\nu)}$

numerically for small t (after integrating (1) w.r.t. the law of $W_t + \nu t$). The related problem of determining small time asymptotics for the density of $A_t^{(\nu)}$ is left to future research. Note that tail asymptotics [4, 5] and large time asymptotics [3, 8] of $A_t^{(\nu)}$ are known.

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