COMPARISON OF CUBICAL AND SIMPLICIAL DERIVED FUNCTORS

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Abstract. In this note we prove that the simplicial derived functors introduced by Tierney and Vogel [TV69] are naturally isomorphic to the cubical derived functors introduced by the author in [P09]. We also explain how this result generalizes the well-known fact that the simplicial and cubical singular homologies of a topological space are naturally isomorphic.

1. Introduction

In [TV69] Tierney and Vogel for any functor $F\colon \mathscr{C} \longrightarrow \mathscr{B}$, where $\mathscr{C}$ is a category with finite limits and a projective class $\mathscr{P}$, and $\mathscr{B}$ is an abelian category, constructed simplicial derived functors and investigated relationships of their theory with other theories of derived functors. Namely, they showed that if $\mathscr{C}$ is abelian and F is additive, then their theory coincides with the classical relative theory of Eilenberg-Moore [EM65], whereas if $\mathscr{C}$ is abelian and F is an arbitrary functor, then it gives a generalization of the theory of Dold-Puppe [DP61]. Besides, they proved that their derived functors are naturally isomorphic to the cotriple derived functors of Barr-Beck ([BB66], [BB69]) if there is a cotriple in $\mathscr{C}$ that realizes the given projective class $\mathscr{P}$.

The key point in the construction of the derived functors by Tierney and Vogel is that using $\mathscr{P}$-projective objects and simplicial kernels, for every C from $\mathscr{C}$ a $\mathscr{P}$-projective pseudosimplicial resolution can be constructed, which is a C-augmented pseudosimplicial object in $\mathscr{C}$ and which for a given C is unique up to a presimplicial homotopy.

In [P09] using pseudocubical resolutions instead of pseudosimplicial ones we constructed cubical derived functors for any functor $F\colon \mathscr{C} \longrightarrow \mathscr{B}$, where $\mathscr{C}$ is a category with finite limits and a projective class $\mathscr{P}$, and $\mathscr{B}$ is an abelian category. It was shown that if $\mathscr{C}$ is an abelian category, F an additive functor, and $\mathscr{P}$ is closed, then our theory coincides with the theory of Eilenberg-Moore [P09, 4.4]. However, there remained an open question whether the Tierney-Vogel simplicial derived functors and our cubical derived functors are isomorphic in general or not. In this paper we give a positive answer to this question. More precisely, we prove the following

Theorem 1.1. *Suppose $\mathscr{C}$ is a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$ in the sense of [TV69, §2], $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. Let $\mathbf{L}_n^{\triangle} F\colon \mathscr{C} \longrightarrow \mathscr{B}$, $n \geq 0$, be the Tierney-Vogel simplicial derived functors of F, and $\mathbf{L}_n^{\square} F\colon \mathscr{C} \longrightarrow \mathscr{B}$, $n \geq 0$, the cubical derived functors of F. Then there is an isomorphism*

$$\mathbf{L}_n^{\triangle} F(C) \cong \mathbf{L}_n^{\square} F(C), \quad C \in \mathscr{C}, \ \ n \geq 0,$$

which is natural in F and in C.

2000 *Mathematics Subject Classification.* 18G10, 18E25, 18G30, 18G25, 55N10.
Key words and phrases. derived functor, normalized chain complex, presimplicial object, projective class, projective pseudocubical resolution, pseudocubical object.

The main idea of the proof goes back to Barr and Beck [BB69]. The point is that passing to the unique additive extension

$$F_{ad}\colon \mathbb{Z}\mathscr{C} \longrightarrow \mathscr{B}$$

of the functor F, where $\mathbb{Z}\mathscr{C}$ denotes the free preadditive category generated by $\mathscr{C}$, one verifies that the Eilenberg-Moore derived functors of F_{ad} (with respect to the class $\mathscr{P}$) restricted to $\mathscr{C}$ are naturally isomorphic to the simplicial derived functors of F on the one hand and to the cubical derived functors of F on the other hand.

The paper is organized as follows. In Section 2 the relative Eilenberg-Moore derived functor theory of additive functors is reviewed from [EM65]. In Section 3 we recall the theory of Tierney-Vogel and prove that the simplicial derived functors of $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ are just the Eilenberg-Moore derived functors of $F_{ad}\colon \mathbb{Z}\mathscr{C} \longrightarrow \mathscr{B}$ restricted to $\mathscr{C}$. Section 4 is devoted to the definition and properties of pseudocubical normalization functor for an idempotent complete preadditive category. Note that the pseudocubical normalization is the main technical tool used in Section 5 to prove that the cubical derived functors of $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ are naturally isomorphic to the Eilenberg-Moore derived functors of $F_{ad}\colon \mathbb{Z}\mathscr{C} \longrightarrow \mathscr{B}$ restricted to $\mathscr{C}$. In the final section we briefly indicate that Theorem 1.1 generalizes the classical fact that the simplicial and cubical singular homologies of a topological space are naturally isomorphic.

2. Partially defined Eilenberg-Moore derived functors

The following definitions are well-known.

Definition 2.1. A preadditive category is a category $\mathscr{A}$ together with the following data:

(i) For any objects X, Y in $\mathscr{A}$, the set of morphisms $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ is an abelian group;

(ii) For any morphisms $f, g\colon X \longrightarrow Y$, $h\colon W \longrightarrow X$ and $u\colon Y \longrightarrow Z$ in $\mathscr{A}$, the following hold

$$(f+g)h = fh + gh, \quad u(f+g) = uf + ug.$$

In other words, a preadditive category is just a ring with several objects in the sense of [M72].

Definition 2.2. Let $\mathscr{A}$ be a preadditive category. An augmented chain complex over an object $A \in \mathscr{A}$ (or just a complex over A) is a sequence

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} A$$

such that $\partial_n \partial_{n+1} = 0$, $n \geq 0$.

Definition 2.3. Let $\mathscr{A}$ be a preadditive category and $\mathscr{P}$ a class of objects in $\mathscr{A}$ (which need not be a "projective class" in any sense). A complex

$$\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} A$$

over $A \in \mathscr{A}$ is said to be $\mathscr{P}$-acyclic if for any $Q \in \mathscr{P}$ the sequence of abelian groups

$$\cdots \longrightarrow \mathrm{Hom}_{\mathscr{A}}(Q, C_1) \xrightarrow{\partial_{1*}} \mathrm{Hom}_{\mathscr{A}}(Q, C_0) \xrightarrow{\partial_{0*}} \mathrm{Hom}_{\mathscr{A}}(Q, A) \longrightarrow 0$$

is exact.

Definition 2.4. Let $\mathscr{A}$ be a preadditive category and $\mathscr{P}$ a class of objects in $\mathscr{A}$. A $\mathscr{P}$-resolution of an object $A \in \mathscr{A}$ is a $\mathscr{P}$-acyclic complex

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A$$

over A with $P_n \in \mathscr{P}$, $n \geq 0$.

Note that an object $A \in \mathscr{A}$ need not necessarily possess a $\mathscr{P}$-resolution.

There is a comparison theorem for $\mathscr{P}$-resolutions which can be proved using the standard homological algebra arguments (see e.g. [W94, 2.2.7]). More precisely, the following is valid.

Proposition 2.5 (Comparison theorem)**.** *Let $P_* \longrightarrow A$ be a complex over $A \in \mathscr{A}$ consisting of objects of $\mathscr{P}$, and let $S_* \longrightarrow B$ be a $\mathscr{P}$-acyclic complex. Then any morphism $f\colon A \longrightarrow B$ can be extended to a morphism of augmented chain complexes*

$$\begin{array}{ccc} P_* & \longrightarrow & A \\ {\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle f} \\ S_* & \longrightarrow & B. \end{array}$$

Moreover, any two such extensions are chain homotopic.

Suppose $\mathscr{A}$ is a preadditive category, $\mathscr{P}$ a class of objects in $\mathscr{A}$, $\mathscr{B}$ an abelian category, $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ an additive functor, and $\mathscr{A}'$ the full subcategory of those objects in $\mathscr{A}$ which possess $\mathscr{P}$-resolutions. Recall that Proposition 2.5 allows one to construct the left derived functors $\mathbf{L}_n^{\mathscr{P}}F\colon \mathscr{A}' \longrightarrow \mathscr{B}$, $n \geq 0$, of F with respect to the class $\mathscr{P}$ as follows. If $A \in \mathscr{A}'$, choose (once and for all) a $\mathscr{P}$-resolution $P_* \longrightarrow A$ and define

$$\mathbf{L}_n^{\mathscr{P}}F(A) = H_n(F(P_*)), \quad n \geq 0.$$

Remark 2.6. If $\mathscr{P}$ is a projective class in the sense of [EM65], then $\mathbf{L}_n^{\mathscr{P}}F$, $n \geq 0$, are exactly the derived functors introduced in [EM65, I.3]. Note that in this case $\mathscr{A}' = \mathscr{A}$, i.e., the functors $\mathbf{L}_n^{\mathscr{P}}F$ are defined everywhere.

Further we recall

Definition 2.7 ([EM65, I.2])**.** Let $\mathscr{A}$ be a preadditive category and $\mathscr{P}$ a class of objects of $\mathscr{A}$. A sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathscr{A}$ is said to be $\mathscr{P}$-exact if $gf = 0$ and the sequence of abelian groups

$$\mathrm{Hom}_{\mathscr{A}}(P, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathscr{A}}(P, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathscr{A}}(P, Z)$$

is exact for any $P \in \mathscr{P}$.

Definition 2.8 ([EM65, I.2])**.** A closure of a class $\mathscr{P}$, denoted by $\overline{\mathscr{P}}$, is the class of all those objects $Q \in \mathscr{A}$ for which

$$\mathrm{Hom}_{\mathscr{A}}(Q, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathscr{A}}(Q, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathscr{A}}(Q, Z)$$

is exact whenever $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $\mathscr{P}$-exact.

Clearly, $\mathscr{P} \subseteq \overline{\mathscr{P}}$ and $\mathscr{P}$-exactness is equivalent to $\overline{\mathscr{P}}$-exactness. In particular, $\overline{\overline{\mathscr{P}}} = \overline{\mathscr{P}}$.

Note that if a preadditive category $\mathscr{A}$ has a terminal object, then any $\mathscr{P}$-resolution is a $\overline{\mathscr{P}}$-resolution as well. This together with 2.5 implies the following

Proposition 2.9. *Let $\mathscr{A}$ be a preadditive category with a terminal object, $\mathscr{P}$ a class of objects in $\mathscr{A}$, $\mathscr{B}$ an abelian category, $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ an additive functor, and A an object in $\mathscr{A}$ which possesses a $\mathscr{P}$-resolution. Then there is a natural isomorphism*

$$\mathbf{L}_n^{\mathscr{P}} F(A) \cong \mathbf{L}_n^{\overline{\mathscr{P}}} F(A), \quad n \geq 0.$$

3. Simplicial derived functors and Eilenberg-Moore derived functors

In this section we briefly review the construction of simplicial derived functors from [TV69, §2] and show that they can be obtained as derived functors of an additive functor.

Let us recall the following definitions.

Definition 3.1. A presimplicial object S in a category $\mathscr{C}$ is a family of objects $(S_n \in \mathscr{C})_{n\geq 0}$ together with morphisms

$$\partial_i\colon S_n \longrightarrow S_{n-1}, \quad n \geq 1, \;\; 0 \leq i \leq n,$$

in $\mathscr{C}$ satisfying the presimplicial identities

$$\partial_i \partial_j = \partial_{j-1}\partial_i, \quad i < j.$$

Definition 3.2. Let S be a presimplicial object in a preadditive category $\mathscr{A}$. The unnormalized chain complex $K(S)$ associated to S is defined by

$$K(S)_n = S_n, \quad n \geq 0,$$

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i \colon K(S)_n \longrightarrow K(S)_{n-1}, \quad n > 0.$$

The presimplicial identities imply that $\partial^2 = 0$.

Now let $\mathscr{C}$ be a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$ in the sense of [TV69, §2], $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. The simplicial derived functors $\mathbf{L}_n^{\Delta} F$ of F with respect to the class $\mathscr{P}$ are defined as follows. For any object $C \in \mathscr{C}$, choose (once and for all) a $\mathscr{P}$-projective presimplicial resolution

$$S \longrightarrow C$$

of C (i.e., a $\mathscr{P}$-exact presimplicial object S augmented over C with $S_n \in \mathscr{P}, \;\; n \geq 0$) and define

$$\mathbf{L}_n^{\Delta} F(C) = H_n(K(F(S))), \quad n \geq 0.$$

By the comparison theorem for projective presimplicial resolutions [TV69, (2.4) Theorem], the objects $\mathbf{L}_n^{\Delta} F(C)$ are well-defined and functorial in F and C.

We will now show that the derived functors $\mathbf{L}_n^{\Delta} F$ can be obtained as derived functors of some additive functor. First recall

Lemma 3.3. *Let $S \longrightarrow S_{-1}$ be an augmented presimplicial set. Suppose that $\partial_0 : S_0 \longrightarrow S_{-1}$ is surjective and the following extension condition holds: For any $n \geq 0$ and any collection of $n+2$ elements $x_i \in S_n$, $0 \leq i \leq n+1$, satisfying*

$$\partial_i x_j = \partial_{j-1} x_i, \ \ 0 \leq i < j \leq n+1,$$

there exists $x \in S_{n+1}$ such that $\partial_i x = x_i$, $0 \leq i \leq n+1$. Then the augmented chain complex

$$K(\mathbb{Z}[S]) \xrightarrow{\partial_0} \mathbb{Z}[S_{-1}]$$

is chain contractible ($\mathbb{Z}[X]$ denotes the free abelian group generated by X). In particular, it has trivial homology in each dimension.

The proof is standard (one constructs inductively a presimplicial contraction).

Example 3.4. Let $S \longrightarrow C$ be a $\mathscr{P}$-projective presimplicial resolution of C and suppose $Q \in \mathscr{P}$. Then the augmented presimplicial set

$$\mathrm{Hom}_{\mathscr{C}}(Q, S) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(Q, C)$$

satisfies the conditions of 3.3. In particular, the homologies of the augmented chain complex

$$K(\mathbb{Z}[\mathrm{Hom}_{\mathscr{C}}(Q, S)]) \longrightarrow \mathbb{Z}[\mathrm{Hom}_{\mathscr{C}}(Q, C)]$$

vanish.

Now suppose again that $\mathscr{C}$ is a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$, $\mathscr{B}$ an abelian category, and $F \colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. Let $\mathbb{Z}\mathscr{C}$ denote the free preadditive category generated by $\mathscr{C}$ [M72, §1], i.e., the objects of $\mathbb{Z}\mathscr{C}$ are those of $\mathscr{C}$, and for any objects C and D in $\mathscr{C}$, $\mathrm{Hom}_{\mathbb{Z}\mathscr{C}}(C, D)$ is the free abelian group generated by $\mathrm{Hom}_{\mathscr{C}}(C, D)$. The composition of morphisms in $\mathbb{Z}\mathscr{C}$ is induced by that in $\mathscr{C}$. Clearly, $\mathscr{C}$ is a subcategory of $\mathbb{Z}\mathscr{C}$. Further, since the category $\mathscr{B}$ is abelian (and therefore additive), the functor $F \colon \mathscr{C} \longrightarrow \mathscr{B}$ can be uniquely extended to an additive functor

$$F_{ad} \colon \mathbb{Z}\mathscr{C} \longrightarrow \mathscr{B}.$$

The following proposition relates the simplicial derived functors of F to the Eilenberg-Moore derived functors of F_{ad}.

Proposition 3.5. *Let $\mathscr{C}$ be a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$, $\mathscr{B}$ an abelian category, and $F \colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. Then:*

(i) *For any $\mathscr{P}$-projective presimplicial resolution $S \longrightarrow C$, the augmented chain complex*

$$K(S) \longrightarrow C$$

in $\mathbb{Z}\mathscr{C}$ is a $\mathscr{P}$-resolution of C in the sense of Definition 2.4.

(ii) *For any $C \in \mathscr{C}$, there is a natural isomorphism*

$$\mathbf{L}_n^{\Delta} F(C) \cong \mathbf{L}_n^{\mathscr{P}} F_{ad}(C), \quad n \geq 0.$$

Proof. The first claim immediately follows from 3.4 and the definition of $\mathbb{Z}\mathscr{C}$. The second claim is a consequence of the first one and the definition of F_{ad}. Indeed, if $S \longrightarrow C$ is a $\mathscr{P}$-projective presimplicial resolution of C, then we have

$$\mathbf{L}_n^{\triangle}F(C) = H_n(K(F(S))) =$$
$$H_n(F_{ad}(K(S))) = L_n^{\mathscr{P}}F_{ad}(C).$$

□

Remark 3.6. Proposition 3.5 is essentially due to Barr and Beck [BB69, §5]. More precisely, in the case when the projective class $\mathscr{P}$ comes from a cotriple (see [TV69, §3]) the above statement is proved in [BB69, §5]. (The cotriple derived functor theory of Barr-Beck is a special case of the Tierney-Vogel theory [TV69 §3].) Thus 3.5 is a simple generalization of the result of Barr and Beck.

4. Pseudocubical objects in idempotent complete preadditive categories

Definition 4.1 ([P09, 2.2])**.** A pseudocubical object X in a category $\mathscr{C}$ is a family of objects $(X_n \in \mathscr{C})_{n\geq 0}$ together with face operators

$$\partial_i^0, \partial_i^1 \colon X_n \longrightarrow X_{n-1}, \quad n \geq 1, \;\; 1 \leq i \leq n,$$

and pseudodegeneracy operators

$$s_i \colon X_{n-1} \longrightarrow X_n, \quad n \geq 1, \;\; 1 \leq i \leq n,$$

satisfying the pseudocubical identities

$$\partial_i^\alpha \partial_j^\varepsilon = \partial_{j-1}^\varepsilon \partial_i^\alpha \qquad i < j, \quad \alpha, \varepsilon \in \{0,1\},$$

and

$$\partial_i^\alpha s_j = \begin{cases} s_{j-1}\partial_i^\alpha & i < j, \\ \mathrm{id} & i = j, \\ s_j \partial_{i-1}^\alpha & i > j, \end{cases}$$

for $\alpha \in \{0,1\}$.

Important examples of pseudocubical objects appear in a natural way: Let $\mathscr{C}$ be a category with finite limits and $\mathscr{P}$ a projective class in $\mathscr{C}$. Then for any object $C \in \mathscr{C}$, there is a $\mathscr{P}$-exact augmented pseudocubical object

$$X \longrightarrow C$$

with $X_n \in \mathscr{P}$, $n \geq 0$, called $\mathscr{P}$-projective pseudocubical resolution of C (see [P09, §3] for details).

In [P09] we use the normalized chain complex of a pseudocubical object in an abelian category to define the cubical derived functors. (Note that the normalized chain complex of a cubical object in an abelian category was introduced by Światek in [Ś75].) Below we recall the definition and some properties of the normalized chain complex of a pseudocubical object in the general setting of idempotent complete preadditive categories. These are needed to prove a cubical analog of Proposition 3.5 in the next section.

Definition 4.2. A preadditive category $\mathscr{A}$ is said to be idempotent complete if any idempotent $p : E \longrightarrow E$ in $\mathscr{A}$ (i.e., $p^2 = p$) has a kernel. That is, there is a morphism

$$i\colon \operatorname{Ker}(p) \longrightarrow E$$

with $pi = 0$, and for any morphism $f\colon F \longrightarrow E$, satisfying $pf = 0$, there is a unique morphism $g\colon F \longrightarrow \operatorname{Ker}(p)$ such that $ig = f$.

The following two propositions are well known (see e.g. [K78]).

Proposition 4.3. *Let $\mathscr{A}$ be an idempotent complete preadditive category and $p\colon E \longrightarrow E$ an idempotent in $\mathscr{A}$. Then there is a diagram*

$$\operatorname{Ker}(p) \underset{i_1}{\overset{\pi_1}{\leftrightarrows}} E \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} \operatorname{Ker}(1-p)$$

such that

$$\pi_1 i_1 = 1, \;\; \pi_2 i_2 = 1,$$

$$\pi_1 i_2 = 0, \;\; \pi_2 i_1 = 0,$$

$$i_1 \pi_1 = 1 - p, \;\; i_2 \pi_2 = p.$$

In particular, the coproduct $\operatorname{Ker}(p) \oplus \operatorname{Ker}(1-p)$ *exists in $\mathscr{A}$ and is isomorphic to E.*

Proposition 4.4. *Let $\mathscr{A}$ be a preadditive category. Then there exists an idempotent complete preadditive category $\widetilde{\mathscr{A}}$ and a full additive embedding*

$$\varphi\colon \mathscr{A} \longrightarrow \widetilde{\mathscr{A}}$$

satisfying the following universal property: For any idempotent complete preadditive category $\mathscr{D}$ and an additive functor $\psi : \mathscr{A} \longrightarrow \mathscr{D}$, there is an additive functor $\psi' : \widetilde{\mathscr{A}} \longrightarrow \mathscr{D}$ which makes the diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\;\varphi\;} & \widetilde{\mathscr{A}} \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\psi'} & \\ & \mathscr{D} & \end{array}$$

commute up to a natural equivalence, and which is unique up to a natural isomorphism.

Let X be a pseudocubical object in an idempotent complete preadditive category $\mathscr{D}$.

Definition 4.5. The unnormalized chain complex $C(X)$ associated to X is defined by

$$C(X)_n = X_n, \quad n \geq 0,$$

$$\partial = \sum_{i=1}^{n} (-1)^i (\partial_i^1 - \partial_i^0)\colon C(X)_n \longrightarrow C(X)_{n-1}, \quad n > 0.$$

The pseudocubical identities show that $\partial^2 = 0$. Moreover, they imply that the morphisms

$$\sigma_n^X = (1 - s_1 \partial_1^1)(1 - s_2 \partial_2^1) \cdots (1 - s_n \partial_n^1)\colon X_n \longrightarrow X_n, \quad n \geq 0, \quad (\sigma_0 = 1)$$

are idempotents and form an endomorphism of the chain complex $C(X)$. We denote this endomorphism by

$$\sigma^X : C(X) \longrightarrow C(X).$$

Since $(\sigma^X)^2 = \sigma^X$ and the category $\mathscr{D}$ is idempotent complete, the chain map σ^X has a kernel $\operatorname{Ker}\sigma^X$ in the category of non-negative chain complexes in $\mathscr{D}$. Furthermore, by 4.3, there is a diagram in the category of chain complexes

$$\operatorname{Ker}(\sigma^X) \underset{i_1}{\overset{\pi_1}{\leftrightarrows}} C(X) \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} \operatorname{Ker}(1-\sigma^X)$$

such that

$$\pi_1 i_1 = 1, \quad \pi_2 i_2 = 1,$$
$$\pi_1 i_2 = 0, \quad \pi_2 i_1 = 0,$$
$$i_1\pi_1 = 1-\sigma^X, \quad i_2\pi_2 = \sigma^X.$$

Definition 4.6. Let X be a pseudocubical object in an idempotent complete preadditive category $\mathscr{D}$. The chain complex $\operatorname{Ker}(1-\sigma^X)$, denoted by $N(X)$, is called the normalized chain complex of X.

Remark 4.7. If $\mathscr{D}$ is an abelian category, then $N(X)$ admits the following description:

$$N(X)_0 = X_0, \quad N(X)_n = \bigcap_{i=1}^{n} \operatorname{Ker}(\partial_i^1), \quad n>0,$$
$$\partial = \sum_{i=1}^{n} (-1)^{i+1}\partial_i^0 : N(X)_n \to N(X)_{n-1}, \quad n>0.$$

Thus in the abelian case one does not need pseudodegeneracies to define $N(X)$.

Next, we recall the construction of cubical derived functors from [P09, §3]. Let $\mathscr{C}$ be a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$, $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. Then the cubical derived functors $\mathbf{L}_n^{\square}F$ of F with respect to the class $\mathscr{P}$ are defined as follows. For any object $C \in \mathscr{C}$, choose (once and for all) a $\mathscr{P}$-projective pseudocubical resolution

$$X \longrightarrow C$$

of C and define

$$\mathbf{L}_n^{\square}F(C) = H_n(N(F(X))), \quad n \geq 0.$$

The comparison theorem for precubical resolutions [P09, 3.3] and the homotopy invariance of the functor N [P09, 3.6] imply that the objects $\mathbf{L}_n^{\square}F(C)$ are well-defined and functorial in F and C.

Note that one cannot use the unnormalized chain complex $C(X)$ instead of $N(X)$ to define the cubical derived functors [P09, 3.8].

The following lemma is the main technical tool for proving a cubical analog of Proposition 3.5.

Lemma 4.8. *Let $F\colon \mathscr{D} \longrightarrow \mathscr{D}'$ be an additive functor between idempotent complete preadditive categories. Then for any pseudocubical object X in $\mathscr{D}$, there is a natural isomorphism*

$$F(N(X)) \cong N(F(X))$$

of chain complexes in $\mathscr{D}'$.

Proof. Applying the additive functor F to the diagram

$$\operatorname{Ker}(\sigma^X) \underset{i_1}{\overset{\pi_1}{\leftrightarrows}} C(X) \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} \operatorname{Ker}(1-\sigma^X) = N(X),$$

we get a digram in $\mathscr{D}'$

$$F(\mathrm{Ker}(\sigma^X)) \underset{F(i_1)}{\overset{F(\pi_1)}{\leftrightarrows}} C(F(X)) \underset{F(i_2)}{\overset{F(\pi_2)}{\rightleftarrows}} F(N(X))$$

whose morphisms satisfy the following identities:

$$\begin{aligned} F(\pi_1)F(i_1) = 1, &\ \ F(\pi_2)F(i_2) = 1, \\ F(\pi_1)F(i_2) = 0, &\ \ F(\pi_2)F(i_1) = 0, \\ F(i_1)F(\pi_1) &= 1 - F(\sigma^X), \\ F(i_2)F(\pi_2) &= F(\sigma^X). \end{aligned}$$

Besides, it follows from the additivity of F that $F(\sigma^X) = \sigma^{F(X)}$, and hence we obtain

$$F(i_2)F(\pi_2) = \sigma^{F(X)}.$$

This finally implies that

$$F(N(X)) \cong \mathrm{Ker}(1 - \sigma^{F(X)}) = N(F(X)).$$

□

5. Cubical derived functors and Eilenberg-Moore derived functors

Let $\mathscr{C}$ be a category with finite limits, $\mathscr{P}$ a projective class, $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. In this section we prove that for any object $C \in \mathscr{C}$, there is a natural isomorphism

$$\mathbf{L}_n^{\Box} F(C) \cong \mathbf{L}_n^{\mathscr{P}} F_{ad}(C), \quad n \geq 0.$$

This together with 3.5 obviously implies Theorem 1.1.

The proof of this isomorphism is similar to that of 3.5. However, things become a little bit complicated in the cubical setting as we have to consider normalized chain complexes in order to get the "right" homology.

Proposition 5.1. *Suppose $\mathscr{A}$ is a preadditive category, $\mathscr{P}$ a class of objects in $\mathscr{A}$, $\mathscr{B}$ an abelian category, and $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ an additive functor. Suppose further that $\overline{\mathscr{P}}$ is the closure of the class $\mathscr{P}$ in the idempotent completion $\widetilde{\mathscr{A}}$, and $\widetilde{F}\colon \widetilde{\mathscr{A}} \longrightarrow \mathscr{B}$ the extension of F. Then for any $A \in \mathscr{A}$ which possesses a $\mathscr{P}$-resolution, there is a natural isomorphism*

$$\mathbf{L}_n^{\mathscr{P}} F(A) \cong \mathbf{L}_n^{\overline{\mathscr{P}}} \widetilde{F}(A), \quad n \geq 0.$$

Proof. Since $\widetilde{\mathscr{A}}$ has a zero object, any $\mathscr{P}$-resolution in $\mathscr{A}$ is a $\overline{\mathscr{P}}$-resolution in $\widetilde{\mathscr{A}}$. The rest follows from 2.9. □

Corollary 5.2. *Assume that $\mathscr{C}$ ia a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$, $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. Assume further that $\widetilde{F_{ad}}\colon \widetilde{\mathbb{Z}\mathscr{C}} \longrightarrow \mathscr{B}$ is the extension of $F_{ad}\colon \mathbb{Z}\mathscr{C} \longrightarrow \mathscr{B}$ to the idempotent completion $\widetilde{\mathbb{Z}\mathscr{C}}$, and $\overline{\mathscr{P}}$ the closure of $\mathscr{P}$ in $\widetilde{\mathbb{Z}\mathscr{C}}$. Then for any object $C \in \mathscr{C}$, there is a natural isomorphism*

$$\mathbf{L}_n^{\mathscr{P}} F_{ad}(C) \cong \mathbf{L}_n^{\overline{\mathscr{P}}} \widetilde{F_{ad}}(C), \quad n \geq 0.$$

Next we state the following technical

Lemma 5.3. *Let $X \longrightarrow X_{-1}$ be an augmented pseudocubical set. Suppose that $\partial : X_0 \longrightarrow X_{-1}$ is surjective and the following conditions hold:*

(i) *For any $x, y \in X_0$, satisfying $\partial x = \partial y$, there exists $z \in X_1$ such that $\partial_1^0 z = x$ and $\partial_1^1 z = y$.*

(ii) *For any $n \geq 1$ and any collection of $2n+2$ elements $x_i^\varepsilon \in X_n$, $1 \leq i \leq n+1$, $\varepsilon \in \{0,1\}$, satisfying*

$$\partial_i^\alpha x_j^\varepsilon = \partial_{j-1}^\varepsilon x_i^\alpha, \quad 1 \leq i < j \leq n+1, \quad \alpha, \varepsilon \in \{0,1\},$$

there exists $x \in X_{n+1}$, such that

$$\partial_i^\varepsilon x = x_i^\varepsilon, \quad 1 \leq i \leq n+1, \quad \varepsilon \in \{0,1\}.$$

Then the augmented normalized chain complex

$$N(\mathbb{Z}[X]) \longrightarrow \mathbb{Z}[X_{-1}]$$

is chain contractible. In particular, it has trivial homology in each dimension.

We omit the routine details of the proof here. Note only that the main idea is to construct inductively a precubical homotopy equivalence between X and the constant cubical object determined by X_{-1} and then use the homotopy invariance of the functor N [P09, 3.6].

Example 5.4. Let $X \longrightarrow C$ be a $\mathscr{P}$-projective pseudocubical resolution of C and suppose $Q \in \mathscr{P}$. Then the augmented pseudocubical set

$$\mathrm{Hom}_{\mathscr{C}}(Q, X) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(Q, C)$$

satisfies the conditions of 5.3. In particular, the homologies of the augmented chain complex

$$N(\mathbb{Z}[\mathrm{Hom}_{\mathscr{C}}(Q, X)]) \longrightarrow \mathbb{Z}[\mathrm{Hom}_{\mathscr{C}}(Q, C)]$$

vanish.

We are now ready to prove the main result of this section.

Proposition 5.5. *Let $\mathscr{C}$ be a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$, $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. Then:*

(i) *For any $\mathscr{P}$-projective pseudocubical resolution $X \longrightarrow C$, the augmented chain complex*

$$N(X) \longrightarrow C$$

in the category $\widetilde{\mathbb{Z}\mathscr{C}}$ is a $\overline{\mathscr{P}}$-resolution of C in the sense of 2.4. ($\overline{\mathscr{P}}$ is the closure of $\mathscr{P}$ in $\widetilde{\mathbb{Z}\mathscr{C}}$.)

(ii) *For any $C \in \mathscr{C}$, there is a natural isomorphism*

$$\mathbf{L}_n^{\square} F(C) \cong \mathbf{L}_n^{\mathscr{P}} F_{ad}(C), \quad n \geq 0.$$

Proof. For all $n \geq 0$, $N(X)_n \in \overline{\mathscr{P}}$ since $N(X)_n$ is a retract of X_n and $\overline{\mathscr{P}}$ is closed under retracts. Further, by 4.8, one has a natural isomorphism of augmented chain complexes

$$\begin{array}{ccc} \mathrm{Hom}_{\widetilde{\mathbb{Z}\mathscr{C}}}(Q, N(X)) & \longrightarrow & \mathrm{Hom}_{\widetilde{\mathbb{Z}\mathscr{C}}}(Q, C) \\ \big\downarrow \cong & & \big\downarrow \mathrm{id} \\ N(\mathbb{Z}[\mathrm{Hom}_{\mathscr{C}}(Q, X)]) & \longrightarrow & \mathbb{Z}[\mathrm{Hom}_{\mathscr{C}}(Q, C)] \end{array}$$

for any $Q \in \mathscr{P}$. It follows from 5.4 that the lower chain complex is acyclic and thus so is the upper one. Consequently, the augmented chain complex $N(X) \longrightarrow C$ in $\widetilde{\mathbb{Z}\mathscr{C}}$ is $\mathscr{P}$-acyclic or, equivalently, $\overline{\mathscr{P}}$-acyclic. This completes the proof of the first claim.

Let us prove the second claim. By 5.2, it suffices to get a natural isomorphism

$$\mathbf{L}_n^{\square} F(C) \cong \mathbf{L}_n^{\overline{\mathscr{P}}} \widetilde{F_{ad}}(C).$$

Choose any $\mathscr{P}$-projective pseudocubical resolution $X \longrightarrow C$. The first claim together with 4.8 gives

$$\mathbf{L}_n^{\square} F(C) = H_n(N(F(X)) = H_n(N(\widetilde{F_{ad}}(X))) \cong$$
$$H_n(\widetilde{F_{ad}}(N(X))) = \mathbf{L}_n^{\overline{\mathscr{P}}} \widetilde{F_{ad}}(C).$$

□

Clearly, Theorem 1.1 is an immediate consequence of 3.5 and 5.5.

6. Connection with topology

In this section we briefly explain that Theorem 1.1 generalizes the well-known fact that the cubical and simplicial singular homologies of a topological space are naturally isomorphic. For the definition and basic properties of the cubical singular homology see [M80].

Let **Top** denote the category of topological spaces, and let Δ^n, $n \geq 0$, be the standard n-simplex. The class $\mathscr{P}_\Delta$ of all possible disjoint unions of standard simplices is a projective class in **Top** in the sense of [TV69, §2]. (Moreover, in fact, it comes from a cotriple [BB69, (10.2)].) Indeed, for any space Y, the map

$$\coprod_{\substack{\Delta^n \to Y, \\ n \geq 0}} \Delta^n \longrightarrow Y,$$

where the disjoint union is taken over all possible continuous maps $\Delta^n \longrightarrow Y$, $n \geq 0$, is a $\mathscr{P}_\Delta$-epimorphism. Consider the functor

$$F\colon \mathbf{Top} \longrightarrow \mathbf{Ab}, \quad F(Y) = H_0^{\Delta}(Y, A) = \mathbb{Z}[\pi_0 Y] \otimes A,$$

where **Ab** is the category of abelian groups, $H_*^{\Delta}(Y, A)$ the simplicial singular homology of Y with coefficients in an abelian group A, and $\pi_0 Y$ the set of path components of Y. It follows from [BB69, (10.2)] and [TV69, (3.1) Theorem] that there is a natural isomorphism

$$\mathbf{L}_n^{\Delta} F(Y) \cong H_n^{\Delta}(Y, A), \quad n \geq 0,$$

where the simplicial derived functors are taken with respect to the projective class $\mathscr{P}_\Delta$ (cf. [R69], [R72]). We sketch the proof of this natural isomorphism along the lines of [BB69, (10.2)]. The standard cosimplicial object $\Delta^\bullet$ gives rise to an augmented simplicial functor

$$F_\bullet \longrightarrow F, \quad F_n(Y) = \mathbb{Z}[\mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y)] \otimes A.$$

Further, suppose $S_\bullet \longrightarrow Y$ is a $\mathscr{P}_\Delta$-projective presimplicial resolution of Y. Evaluating $F_\bullet$ on $S_\bullet$ yields a bipresimplicial abelian group. It is easily seen that both resulting spectral sequences collapse at E^2. Finally, playing these two spectral sequences against each other gives the desired isomorphism.

Similarly, one can describe the cubical singular homologies $H_n^{\square}(Y, A)$ as cubical derived functors of the functor $F(Y) = \mathbb{Z}[\pi_0 Y] \otimes A$. For this one uses the class $\mathscr{P}_{\square}$ consisting of all possible disjoint unions of standard cubes. The class $\mathscr{P}_{\square}$ is a projective class in **Top** and there is a natural isomorphism

$$\mathbf{L}_n^{\square} F(Y) \cong H_n^{\square}(Y, A), \quad n \geq 0,$$

where the cubical derived functors are taken with respect to $\mathscr{P}_{\square}$. The proof of this isomorphism is technically a little bit complicated compared to its simplicial counterpart as one has to consider spectral sequences of bipseudocubical objects and take care of the normalizations.

Note that the class $\mathscr{P} = \mathscr{P}_{\Delta} \cup \mathscr{P}_{\square}$ is also a projective class in **Top**. Obviously, the simplicial derived functors with respect to the class $\mathscr{P}_{\Delta}$ are naturally isomorphic to the simplicial derived functors with respect to $\mathscr{P}$. On the other hand, the cubical derived functors with respect to the class $\mathscr{P}_{\square}$ are naturally isomorphic to the cubical derived functors with respect to $\mathscr{P}$. Thus, by 1.1, there is a natural isomorphism

$$\mathbf{L}_n^{\Delta} F(Y) \cong \mathbf{L}_n^{\square} F(Y), \quad n \geq 0,$$

for any topological space Y, i.e.,

$$H_n^{\Delta}(Y, A) \cong H_n^{\square}(Y, A), \quad n \geq 0.$$

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