L'Hôpital-Type Rules for Monotonicity with Application to Quantum Calculus*

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Abstract

We prove new l'Hôpital rules for monotonicity valid on an arbitrary time scale. Both delta and nabla monotonic l'Hôpital rules are obtained. As an example of application, we give some new upper and lower bounds for the exponential function of quantum calculus restricted to the q-scale.

Keywords: l'Hôpital-type rules, monotonic functions, q-calculus.

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1 Introduction

L'Hôpital-type rules for monotonicity have an important role in mathematics (Anderson, Vamanamurthy and Vuorinen, 2006; Boas, 1969; Pinelis, 2006; Pinelis, 2008), with numerous useful applications found in mathematical inequalities, statistics, probability, differential geometry, approximation and information theories, and mathematical physics (András and Baricz, 2008; Baricz, 2008; Baricz, 2010; Pan and Zhu, 2009; Pinelis, 2004; Pinelis, 2007; Zhu, 2009). Recently, Wu and Debnath proved the following l'Hôpital-type rules for monotonicity:

Theorem 1.1 ((Wu and Debnath, 2009))**.** *Let f and g be differentiable functions on $]a,b[$. Suppose that either $g' > 0$ everywhere on $]a,b[$ or $g' < 0$ everywhere on $]a,b[$.*

If $\dfrac{f'}{g'}$ is increasing (resp. decreasing) on $]a,b[$ and $f(a^+)$ and $g(a^+)$ exist, then the function

$$\frac{f(x)-f(a^+)}{g(x)-g(a^+)}$$

is also increasing (resp. decreasing) on $]a,b[$.

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If $\frac{f'}{g'}$ *is increasing (resp. decreasing) on* $]a,b[$ *and* $f(b^-)$ *and* $g(b^-)$ *exist, then the function*

$$\frac{f(x)-f(b^-)}{g(x)-g(b^-)}$$

is also increasing (resp. decreasing) on $]a,b[$.

In this note we generalize Theorem 1.1 by proving l'Hôpital-type rules for monotonicity on an arbitrary time scale (Section 2). Our results seem to be new even in the discrete-time case. As an application, we prove new inequalities on quantum calculus: we give new upper and lower bounds for the exponential function of quantum calculus when restricted to the q-scale (Section 3).

The theory of time scales was introduced in 1988 by Aulbach and Hilger in order to unify continuous and discrete-time theories (Aulbach and Hilger, 1990). It has found applications in several different fields that require simultaneous modeling of discrete and continuous data (Guseinov and Kaymakçalan, 2002; Wu and Debnath, 2009), and is now a subject of strong current research (see (Almeida and Torres, 2009; Bartosiewicz and Torres, 2008; Malinowska and Torres, 2009; Martins and Torres, 2009; Mozyrska and Torres, 2009; Sidi Ammi, Ferreira and Torres, 2008) and references therein). We claim that time scale theory is useful with respect to L'Hôpital-type rules for monotonicity, avoiding repetition of results in the continuous and discrete cases (Pinelis, 2007; Pinelis, 2008). We trust that the present study will mark the beginning of the study of L'Hôpital-type rules on time scales and its applications to dynamic inequalities (Ferreira and Torres, 2009; Sidi Ammi et al., 2008; Sidi Ammi and Torres, 2009), the calculus of variations (Bohner, Ferreira and Torres, 2010; Ferreira, Sidi Ammi and Torres, 2009), and quantum calculus (Cresson, Frederico and Torres, 2009; Kac and Cheung, 2002).

2 L'Hôpital rules for monotonicity

For definitions, notations, and results concerning the theory of delta and nabla calculus on time scales we refer the reader to the books (Bohner and Peterson, 2001; Bohner and Peterson, 2003). Similarly to (Wu and Debnath, 2009), we use $f(a^+)$ to denote $\lim_{x\to a^+} f(x)$ and $f(b^-)$ to denote $\lim_{x\to b^-} f(x)$. All the intervals in this work are time scale intervals.

Theorem 2.1 (Delta-monotonic l'Hôpital rules)**.** *Let* $\mathbb{T}$ *be a time scale with* $a,b\in\mathbb{T}$, $a<b$, *and* f *and* g *two* Δ*-differentiable functions on* $]a,b[$. *Suppose that either* $g^\Delta>0$ *everywhere on* $]a,\rho(b)[$ *or* $g^\Delta<0$ *everywhere on* $]a,\rho(b)[$.

1. *If* $\frac{f^\Delta}{g^\Delta}$ *is increasing (resp. decreasing) on* $]a,\rho(b)[$ *and* $f(a^+)$ *and* $g(a^+)$ *exist (finite), then the function*

$$\frac{f(x)-f(a^+)}{g(x)-g(a^+)}$$

is also increasing (resp. decreasing) on $]a,\rho(b)[$.

2. *If* $\frac{f^\Delta}{g^\Delta}$ *is increasing (resp. decreasing) on* $]a,\rho(b)[$ *and* $f(b^-)$ *and* $g(b^-)$ *exist (finite), then the function*

$$\frac{f(x)-f(b^-)}{g(x)-g(b^-)}$$

is also increasing (resp. decreasing) on $]a, \rho(b)[$.

Proof. We will prove the first assertion. Since $f(a^+)$ and $g(a^+)$ are finite numbers, we can define the following functions on $[a, b[$:

$$F(x) = \begin{cases} f(x) & \text{if} \quad x \in]a, b[\\ f(a^+) & \text{if} \quad x = a \end{cases}$$

and

$$G(x) = \begin{cases} g(x) & \text{if} \quad x \in]a, b[\\ g(a^+) & \text{if} \quad x = a\,. \end{cases}$$

Clearly, for any $x \in]a, b[$, F and G are continuous on $[a, x]$ and Δ-differentiable on $[a, x[$. Note also that $G(x) \neq G(a)$ for $x \neq a$.

By the Cauchy mean value theorem with delta derivatives (Guseinov and Kaymakçalan, 2002) we conclude that there exist $c_1, c_2 \in [a, x[$ such that

$$\frac{F^\Delta(c_1)}{G^\Delta(c_1)} \leq \frac{F(x) - F(a)}{G(x) - G(a)} \leq \frac{F^\Delta(c_2)}{G^\Delta(c_2)}.$$

For each $x \in]a, b[$ define

$$H(x) = \frac{F(x) - F(a)}{G(x) - G(a)}.$$

Note that, for each $x \in]a, \rho(b)[$,

$$\begin{aligned} H^\Delta(x) &= \frac{F^\Delta(x)(G(x) - G(a)) - G^\Delta(x)(F(x) - F(a))}{(G(x) - G(a))(G^\sigma(x) - G(a))} \\ &= \frac{G^\Delta(x)}{G^\sigma(x) - G(a)} \left(\frac{F^\Delta(x)}{G^\Delta(x)} - \frac{F(x) - F(a)}{G(x) - G(a)} \right). \end{aligned}$$

Case I. Suppose that $\frac{f^\Delta}{g^\Delta}$ is increasing on $]a, \rho(b)[$. For each $x \in]a, \rho(b)[$ there exists $c_2 \in [a, x[$ such that

$$\begin{aligned} H^\Delta(x) &\geq \frac{G^\Delta(x)}{G^\sigma(x) - G(a)} \left(\frac{F^\Delta(x)}{G^\Delta(x)} - \frac{F^\Delta(c_2)}{G^\Delta(c_2)} \right) \\ &= \frac{g^\Delta(x)}{g^\sigma(x) - g(a^+)} \left(\frac{f^\Delta(x)}{g^\Delta(x)} - \frac{f^\Delta(c_2)}{g^\Delta(c_2)} \right) > 0. \end{aligned}$$

Hence, H is increasing on $]a, \rho(b)[$ (Guseinov and Kaymakçalan, 2002). Since

$$H(x) = \frac{f(x) - f(a^+)}{g(x) - g(a^+)}$$

we can conclude that $\frac{f(x) - f(a^+)}{g(x) - g(a^+)}$ is increasing on $]a, \rho(b)[$.

Case II. Suppose that $\frac{f^\Delta}{g^\Delta}$ is decreasing on $]a, \rho(b)[$. For each $x \in]a, \rho(b)[$ there exists $c_1 \in [a, x[$ such that

$$\begin{aligned} H^\Delta(x) &\leq \frac{G^\Delta(x)}{G^\sigma(x) - G(a)} \left(\frac{F^\Delta(x)}{G^\Delta(x)} - \frac{F^\Delta(c_1)}{G^\Delta(c_1)} \right) \\ &= \frac{g^\Delta(x)}{g^\sigma(x) - g(a^+)} \left(\frac{f^\Delta(x)}{g^\Delta(x)} - \frac{f^\Delta(c_1)}{g^\Delta(c_1)} \right) < 0. \end{aligned}$$

Hence, H is decreasing on $]a, \rho(b)[$. This proves that $\dfrac{f(x) - f(a^+)}{g(x) - g(a^+)}$ is decreasing on $]a, \rho(b)[$.

The second assertion is proved in a similar way. □

From Theorem 2.1 we can deduce the following corollaries.

Corollary 2.2. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$, $a < b$, and $f, g : [a, b[\to \mathbb{R}$ be two continuous functions which are Δ-differentiable on $]a, b[$. Suppose that either $g^\Delta > 0$ everywhere on $]a, \rho(b)[$ or $g^\Delta < 0$ everywhere on $]a, \rho(b)[$.*

If $\dfrac{f^\Delta}{g^\Delta}$ is increasing (resp. decreasing) on $]a, \rho(b)[$, then the function

$$\frac{f(x) - f(a)}{g(x) - g(a)}$$

is also increasing (resp. decreasing) on $]a, \rho(b)[$.

Corollary 2.3. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$, $a < b$, and $f, g :]a, b] \to \mathbb{R}$ be two continuous functions which are Δ-differentiable on $]a, b[$. Suppose that either $g^\Delta > 0$ everywhere on $]a, \rho(b)[$ or $g^\Delta < 0$ everywhere on $]a, \rho(b)[$.*

If $\dfrac{f^\Delta}{g^\Delta}$ is increasing (resp. decreasing) on $]a, \rho(b)[$, then the function

$$\frac{f(x) - f(b)}{g(x) - g(b)}$$

is also increasing (resp. decreasing) on $]a, \rho(b)[$.

Corollary 2.4. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$, $a < b$, and $f, g :]a, b[\to \mathbb{R}$ be two Δ-differentiable functions on $]a, b[$. Suppose that either $g^\Delta > 0$ everywhere on $]a, \rho(b)[$ or $g^\Delta < 0$ everywhere on $]a, \rho(b)[$. Suppose also that $f(a^+) = g(a^+) = 0$ or $f(b^-) = g(b^-) = 0$.*

If $\dfrac{f^\Delta}{g^\Delta}$ is increasing (resp. decreasing) on $]a, \rho(b)[$, then $\dfrac{f}{g}$ is also increasing (resp. decreasing) on $]a, \rho(b)[$.

Remark 2.1. Theorem 2.1 and Corollaries 2.2, 2.3 and 2.4 hold true if the terms “increasing” and “decreasing” are replaced everywhere by “non-decreasing” and “non-increasing”, respectively.

Using the recent duality theory (Caputo, 2010; Malinowska and Torres, 2010; Pawłuszewicz and Torres, 2010), one can easily obtain the corresponding nabla result for Theorem 2.1:

Theorem 2.5 (Nabla-monotonic l’Hôpital rules)**.** *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$, $a < b$, and f and g be ∇-differentiable functions on $]a, b[$. Suppose that either $g^\nabla > 0$ everywhere on $]\sigma(a), b[$ or $g^\nabla < 0$ everywhere on $]\sigma(a), b[$.*

1. *If $\dfrac{f^\nabla}{g^\nabla}$ is increasing (resp. decreasing) on $]\sigma(a), b[$ and $f(a^+)$ and $g(a^+)$ exist (finite), then the function*

$$\frac{f(x) - f(a^+)}{g(x) - g(a^+)}$$

is also increasing (resp. decreasing) on $]\sigma(a), b[$.

2. *If* $\frac{f^{\nabla}}{g^{\nabla}}$ *is increasing (resp. decreasing) on* $]\sigma(a), b[$ *and* $f(b^-)$ *and* $g(b^-)$ *exist (finite), then the function*

$$\frac{f(x) - f(b^-)}{g(x) - g(b^-)}$$

is also increasing (resp. decreasing) on $]\sigma(a), b[$.

In the case $\mathbb{T} = \mathbb{R}$ Theorems 2.1 and 2.5 coincide and reduce to Theorem 1.1. The l'Hôpital rules of Theorems 2.1 and 2.5 seem to be new even for $\mathbb{T} = \mathbb{Z}$. In the next section we give an application of our results for $\mathbb{T} = \overline{q^{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$.

3 An application to quantum inequalities

In this section we assume familiarity with the definitions and results from the q-calculus (cf., e.g., the book (Kac and Cheung, 2002)). Let q be a real number, $0 < q < 1$. The *q-derivative* $D_q f$ of a function f is defined by

$$D_q f(x) = \begin{cases} \dfrac{f(qx) - f(x)}{(q-1)x} & \text{if } x \neq 0 \\ f'(x) & \text{if } x = 0, \end{cases}$$

provided $f'(0)$ exists. We use the following standard notations of q-calculus:

- for any real number α, $[\alpha] := \dfrac{q^{\alpha} - 1}{q - 1}$;
- $[n]! := \begin{cases} 1 & \text{if } n = 0, \\ [n][n-1]\cdots[2][1] & \text{if } n \in \mathbb{N}; \end{cases}$
- $(x-a)_q^n := \begin{cases} 1 & \text{if } n = 0, \\ (x-a)(x-qa)\cdots(x-q^{n-1}a) & \text{if } n \in \mathbb{N}. \end{cases}$

The function e_q^x defined by

$$e_q^x = \sum_{k=0}^{+\infty} \frac{x^k}{[k]!} \tag{1}$$

is called the *q-exponential function.* The basic properties of the q-exponential function can be found, e.g., in (Kac and Cheung, 2002; Lavagno, Scarfone and Swamy, 2007). Here we obtain new upper and lower bounds for (1).

Theorem 3.1. *Let* $0 < q < 1$, $a, b \in q^{\mathbb{Z}}$, $a < b$, *and* $n \in \mathbb{N}$. *Then, for any* $x \in [q^{-1}a, b]$, *the following inequalities hold:*

$$e_q^x \geq \sum_{k=0}^{n-1} \frac{e_q^a}{[k]!}(x-a)_q^k + \frac{1}{(q^{-1}a - a)_q^n}\left(e_q^{q^{-1}a} - \sum_{k=0}^{n-1} \frac{e_q^a}{[k]!}(q^{-1}a - a)_q^k\right)(x-a)_q^n,$$

$$e_q^x \leq \sum_{k=0}^{n-1} \frac{e_q^a}{[k]!}(x-a)_q^k + \frac{1}{(b-a)_q^n}\left(e_q^b - \sum_{k=0}^{n-1} \frac{e_q^a}{[k]!}(b-a)_q^k\right)(x-a)_q^n.$$

Proof. We begin by noting that if we consider

$$\mathbb{T} = \overline{q^{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\},$$

then, for any function $f : \mathbb{T} \to \mathbb{R}$, one obtains

$$f^{\nabla}(t) = D_q f(t), \quad \forall t \in \mathbb{T},$$

and $\sigma(t) = q^{-1}t$. The result follows as a corollary of Theorem 2.5. For $x \in [a, b]$ define

$$f(x) = e_q^x - \sum_{k=0}^{n-1} \frac{e_q^a}{[k]!}(x-a)_q^k$$

and

$$g(x) = (x-a)_q^n.$$

Note that

$$D_q^k f(a) = f^{\nabla^k}(a) = 0 \quad \text{and} \quad D_q^k g(a) = g^{\nabla^k}(a) = 0 \quad \text{for all} \;\; k = 0, 1, 2, \ldots, n-1,$$

where $D_q^k f$ is defined by $D_q^0 f = f$ and $D_q^k f = D_q(D_q^{k-1})f$, $k = 1, 2, 3, \ldots$ Note also that

$$D_q^n f(x) = f^{\nabla^n}(x) = e_q^x \quad \text{and} \quad D_q^n g(x) = g^{\nabla^n}(x) = [n]!$$

Thus, we have:

$$\begin{aligned}
\frac{f(x)}{g(x)} &= \frac{f(x) - f(a)}{g(x) - g(a)} \\
\frac{f^{\nabla}(x)}{g^{\nabla}(x)} &= \frac{f^{\nabla}(x) - f^{\nabla}(a)}{g^{\nabla}(x) - g^{\nabla}(a)} \\
&\vdots \\
\frac{f^{\nabla^{n-1}}(x)}{g^{\nabla^{n-1}}(x)} &= \frac{f^{\nabla^{n-1}}(x) - f^{\nabla^{n-1}}(a)}{g^{\nabla^{n-1}}(x) - g^{\nabla^{n-1}}(a)} \\
\frac{f^{\nabla^n}(x)}{g^{\nabla^n}(x)} &= \frac{e_q^x}{[n]!}.
\end{aligned} \tag{2}$$

Since e_q^x is an increasing function, then $\dfrac{f^{\nabla^n}(x)}{g^{\nabla^n}(x)}$ is increasing on $[\sigma(a), b]$. From Theorem 2.5 and the relationships listed in (2) we deduce that each of the following functions

$$\frac{f^{\nabla^n}(x)}{g^{\nabla^n}(x)}, \frac{f^{\nabla^{n-1}}(x)}{g^{\nabla^{n-1}}(x)}, \ldots, \frac{f^{\nabla}(x)}{g^{\nabla}(x)}, \frac{f(x)}{g(x)}$$

is increasing on $[\sigma(a), b]$. Therefore, for each $x \in [\sigma(a), b]$,

$$\frac{f(\sigma(a))}{g(\sigma(a))} \le \frac{f(x)}{g(x)} \le \frac{f(b)}{g(b)},$$

proving the intended inequalities. □

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