

BIFURCATION VALUES OF MIXED POLYNOMIALS

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ABSTRACT. We study the bifurcation locus $B(f)$ of real polynomials $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^2$. We find a semialgebraic approximation of $B(f)$ by using the ρ -regularity condition and we compare it to the Sard type theorem by Kurdyka, Orro and Simon. We introduce the Newton boundary at infinity for mixed polynomials and we extend structure results by Kushnirenko and by Némethi and Zaharia, under the Newton non-degeneracy assumption.

1. INTRODUCTION

For a complex polynomial function $f : \mathbb{C}^n \rightarrow \mathbb{C}$, it is well known that there is a locally trivial fibration $f|_{\mathbb{C}^n \setminus f^{-1}(\Lambda)} : \mathbb{C}^n \setminus f^{-1}(\Lambda) \rightarrow \mathbb{C} \setminus \Lambda$ over the complement of some finite subset $\Lambda \subset \mathbb{C}$, see e.g. [Va], [Ve]. The minimal such Λ is called the set of *bifurcation values*, or the set of *atypical values*, and shall be denoted by $B(f)$. It was studied in several papers, such as [Br1], [Br2], [Ne1], [NZ1], [ST], [Pa] etc. The difficulty to apprehend it comes from the fact that besides the critical values of f , $B(f)$ may contain other values due to the asymptotical “bad” behaviour at infinity. In some special cases f has no atypical values at infinity, for instance: “convenient polynomials with non-degenerate Newton principal part at infinity” (Kouchnirenko [Ku]), see §3.1, “tame polynomials” (Broughton [Br1], [Br2]), “M-tame” (Némethi [Ne1], [Ne2]), “cohomologically tame” (Sabbah, Némethi [NS], [Sa]). In two variables one has several characterisations of the atypical values at infinity, see e.g. [Du], [Til]. In higher dimensions the problem is still open and one looks for some significant set $A \supset B(f)$ which bounds $B(f)$ reasonably well. For instance, in case of non-convenient polynomials but still Newton non-degenerate, Némethi and Zaharia [NZ1] found an interesting approximation $A \supset B(f)$ in terms of certain faces of the support of f , see Theorem 3.4. This provides a large class of polynomials for which we control rather well the bifurcation locus.

In the real setting one has a similar notion of *bifurcation locus*, namely, for a polynomial map $F : \mathbb{R}^m \rightarrow \mathbb{R}^p$, $m > p$, this is the minimal set $B(F)$ such that F is a locally trivial fibration over $\mathbb{R}^p \setminus B(F)$. For $m = 2$ and $p = 1$ one has a characterisation of $B(F)$, cf [TZ], which is more complicated than in the corresponding complex setting, cf [HL]. More recently, Kurdyka, Orro and Simon [KOS] found a certain closed semi-algebraic set $K(F)$ which includes $B(F)$, see Theorem 2.7.

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In this paper we study the bifurcation locus of real polynomial maps $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^2$. We work with the ρ -regularity, a condition which extends Milnor's condition to maps and allows us to exhibit a certain closed semi-algebraic set $S(f)$. We use it to improve Kurdyka, Orro and Simon's result [KOS] by providing a sharper approximation of the set of atypical values at infinity: $B(f) \subset S(f) \subset K(f)$, where the inclusion $S(f) \subset K(f)$ is strict in general. This is the object of Proposition 2.8 and of the Fibration Theorem 2.10.

In the second part of this paper we introduce the Newton boundary at infinity $\Gamma^+(f)$ and the notion of "bad faces" of the support $\text{supp}(f)$ in order to prove a real counterpart of Némethi and Zaharia's main result cited above: Theorem 3.5, and its corollaries in §4. In the same time, our proof yields a refinement of the result [NZ1] for holomorphic polynomials.

Along the way, we discuss Newton non-degeneracy in §3 and we prove several preliminary statements, such as that the Newton non-degeneracy is an open dense condition, §3.3.

In our study we view f in the following way. If $f = (g, h) : \mathbb{R}^{2n} \rightarrow \mathbb{R}^2$, where $g(x_1, \dots, y_n)$ and $h(x_1, \dots, y_n)$ are real polynomial functions then, by writing $\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathbb{C}^n$, where $z_k = x_k + iy_k$ for $k = 1, \dots, n$, one gets a polynomial function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ in variables \mathbf{z} and $\bar{\mathbf{z}}$, namely $f(\mathbf{z}, \bar{\mathbf{z}}) := g\left(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2}, \frac{\mathbf{z}-\bar{\mathbf{z}}}{2i}\right) + ih\left(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2}, \frac{\mathbf{z}-\bar{\mathbf{z}}}{2i}\right)$. Oka has called such functions *mixed polynomials*¹ and has studied in a recent series of papers [Oka1, Oka2, Oka3] the topology of the germs at the origin of mixed polynomials and mixed hypersurfaces. Reciprocally, a mixed polynomial function $f(\mathbf{z}, \bar{\mathbf{z}}) = g(\mathbf{x}, \mathbf{y}) + ih(\mathbf{x}, \mathbf{y})$ defines a real polynomial map $(g, h) : \mathbb{R}^{2n} \rightarrow \mathbb{R}^2$ as above. Working with mixed functions instead of real maps (g, h) has the advantage to allow the use of some tools from holomorphic setting, such as the "curve selection lemma at infinity", cf Lemma 2.5.

2. ATYPICAL VALUES OF MIXED POLYNOMIALS

We show here that the fibres of a mixed polynomial f which are asymptotically tangent to the spheres may cause atypical behaviour at infinity and that the ρ -regularity, defined in [Til] for polynomial functions, is more general than other regularity conditions at infinity. We first set some notations and definitions.

We shall use the following notations: $df := \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$, $\bar{d}f := \left(\frac{\partial f}{\partial \bar{z}_1}, \dots, \frac{\partial f}{\partial \bar{z}_n}\right)$, and $\overline{df} := \left(\frac{\partial \bar{f}}{\partial \bar{z}_1}, \dots, \frac{\partial \bar{f}}{\partial \bar{z}_n}\right)$ is the conjugate of df .

Lemma 2.1. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a mixed polynomial. The intersection of the fibre $f^{-1}(f(\mathbf{z}, \bar{\mathbf{z}}))$ with the sphere S_r^{2n-1} of radius $r = \|\mathbf{z}\|$ is not transversal at $\mathbf{z} \in \mathbb{C}^n \setminus \{0\}$ if and only if there exist $\mu \in \mathbb{C}^*$, $\lambda \in \mathbb{R}$ such that:*

$$\lambda \mathbf{z} = \mu \overline{df}(\mathbf{z}, \bar{\mathbf{z}}) + \bar{\mu} df(\mathbf{z}, \bar{\mathbf{z}}).$$

Proof. Let us write f as the map:

$$f : \mathbb{C}^n = \mathbb{R}^{2n} \rightarrow \mathbb{R}^2, f(z_1, \dots, z_n) = (\text{Re} f, \text{Im} f)$$

where $z_k = x_k + iy_k = (x_k, y_k)$, for $k = 1, \dots, n$, and let us denote $\mathbf{v} := (x_1, y_1, \dots, x_n, y_n)$.

¹the name "mixed polynomial" was introduced by Oka [Oka1] but the concept has been used before, notably by Ruas, Seade and Verjovsky. The historical references may be found in [Oka1, Oka2, Oka3].

If $f^{-1}(f(\mathbf{z}, \bar{\mathbf{z}}))$ does not intersect transversely the sphere S_r^{2n-1} at \mathbf{z} , then there exist $\alpha, \beta, \gamma \in \mathbb{R}$, $|\alpha| + |\beta| + |\gamma| \neq 0$, such that

$$(1) \quad \gamma \mathbf{v} = \alpha d\text{Re}f(\mathbf{v}) + \beta d\text{Im}f(\mathbf{v}).$$

Since $\text{Re}f = \frac{f+\bar{f}}{2}$, $\text{Im}f = \frac{f-\bar{f}}{2i}$ and $\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial z_k} + \frac{\partial f}{\partial \bar{z}_k}$, $\frac{\partial f}{\partial y_k} = i\frac{\partial f}{\partial z_k} - i\frac{\partial f}{\partial \bar{z}_k}$, $k = 1, \dots, n$, we get: $\gamma x_k = \frac{\alpha}{2}(\frac{\partial f}{\partial z_k} + \frac{\partial f}{\partial \bar{z}_k} + \frac{\partial \bar{f}}{\partial z_k} + \frac{\partial \bar{f}}{\partial \bar{z}_k}) + \frac{\beta}{2i}(\frac{\partial f}{\partial z_k} + \frac{\partial f}{\partial \bar{z}_k} - \frac{\partial \bar{f}}{\partial z_k} - \frac{\partial \bar{f}}{\partial \bar{z}_k})$ and $\gamma y_k = \frac{\alpha i}{2}(\frac{\partial f}{\partial z_k} - \frac{\partial f}{\partial \bar{z}_k} + \frac{\partial \bar{f}}{\partial z_k} - \frac{\partial \bar{f}}{\partial \bar{z}_k}) + \frac{\beta}{2}(\frac{\partial f}{\partial z_k} - \frac{\partial f}{\partial \bar{z}_k} - \frac{\partial \bar{f}}{\partial z_k} + \frac{\partial \bar{f}}{\partial \bar{z}_k})$.

Therefore, $\gamma z_k = (\alpha + \beta i)\frac{\partial f}{\partial \bar{z}_k} + (\alpha - \beta i)\frac{\partial f}{\partial z_k}$ for every $k \in \{1, \dots, n\}$. We get our claim by taking $\lambda = \gamma$ and $\mu = \alpha + \beta i$. \square

The *singular locus* $\text{Sing}f$ of a mixed polynomial f is by definition the set of critical points of f as a real-valued map. From Lemma 2.1, by taking $\lambda = 0$ and dividing by μ , we get the following characterisation:

Lemma 2.2. [Oka1, Proposition 1] $\mathbf{z} \in \text{Sing}f$ if and only if there exists $\mu \in \mathbb{C}$, $|\mu| = 1$, such that $\bar{d}f(\mathbf{z}, \bar{\mathbf{z}}) = \mu \bar{d}f(\mathbf{z}, \bar{\mathbf{z}})$. \square

2.1. ρ -regularity. Using Lemma 2.1 we obtain the following useful display of the critical locus of the map (f, ρ) , where $\rho : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ is the Euclidean distance function. In case of holomorphic f , this was called *Milnor set* in [NZ1].

Definition 2.3. The *Milnor set* of a mixed polynomial f is

$$M(f) = \{\mathbf{z} \in \mathbb{C}^n \mid \exists \lambda \in \mathbb{R}, \mu \in \mathbb{C}^*, \text{ such that } \lambda \mathbf{z} = \mu \bar{d}f(\mathbf{z}, \bar{\mathbf{z}}) + \bar{\mu} d\bar{f}(\mathbf{z}, \bar{\mathbf{z}})\}.$$

By its definition, $M(f)$ is a closed algebraic subset of \mathbb{C}^n . We may now introduce:

Definition 2.4. The set of *asymptotic ρ -nonregular values* of a mixed polynomial f is

$$S(f) = \{c \in \mathbb{C} \mid \exists \{\mathbf{z}_k\}_{k \in \mathbb{N}} \subset M(f), \lim_{k \rightarrow \infty} \|\mathbf{z}_k\| = \infty \text{ and } \lim_{k \rightarrow \infty} f(\mathbf{z}_k, \bar{\mathbf{z}}_k) = c\}.$$

A value $c \notin S(f)$ will be called an *asymptotic ρ -regular value*. This definition implicitly refers to the ρ -regularity condition, which was previously used only in the setting of polynomial functions (complex and real), see [Ti1, Ti2].

To investigate the properties of $S(f)$ we need a version of the *Curve Selection Lemma* at infinity. Milnor [Mi] has proved this lemma at points of the closure of a semi-analytic set. Némethi and Zaharia [NZ1], [NZ2], showed how to extend the result at infinity at some fibre of a holomorphic polynomial function. We give here a more general statement including the case when the value of $|f|$ tends to infinity. Since the proof is similar to the one in [NZ2] and uses Milnor's result, we may safely leave it to the reader.

Lemma 2.5. Curve Selection Lemma at infinity

Let $U \subseteq \mathbb{R}^n$ be a semi-analytic set. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial function. If there is $\{\mathbf{x}_k\}_{k \in \mathbb{N}} \subset U$ such that $\lim_{k \rightarrow \infty} \|\mathbf{x}_k\| = \infty$ and $\lim_{k \rightarrow \infty} g(\mathbf{x}_k) = c$, where $c \in \mathbb{R}$, $c = \infty$ or $c = -\infty$, then there exist a real analytic path $\mathbf{x}(t) = \mathbf{x}_0 t^\alpha + \mathbf{x}_1 t^{\alpha+1} + \text{h.o.t.}$ defined on some small enough interval $]0, \varepsilon[$, such that $\mathbf{x}_0 \neq 0$, $\alpha < 0$, $\alpha \in \mathbb{Z}$, and that $\lim_{t \rightarrow 0} g(\mathbf{x}(t)) = c$. \square

We have the following structure result:

Proposition 2.6. *If $f : \mathbb{C}^n \rightarrow \mathbb{C}$ is a mixed polynomial, then $S(f)$ and $f(\text{Sing } f) \cup S(f)$ are closed semi-algebraic sets.*

Proof. $S(f)$ may be presented as the projection of a semi-algebraic set. Indeed, consider the embedding of \mathbb{C}^n into $\mathbb{C}^{n+1} \times \mathbb{C}$ given by the semi-algebraic map:

$$\varphi : (z_1, \dots, z_n) \mapsto \left(\frac{z_1}{\sqrt{1 + \|z\|^2}}, \dots, \frac{z_n}{\sqrt{1 + \|z\|^2}}, \frac{1}{\sqrt{1 + \|z\|^2}}, f(z, \bar{z}) \right).$$

Then $U_1 := \overline{\varphi(M(f))} \cap \{(x_1, \dots, x_{n+1}, c) \in \mathbb{C}^{n+1} \times \mathbb{C} \mid x_{n+1} = 0\}$ is a semi-algebraic set and $S(f) = \pi(U_1)$, where $\pi : \mathbb{C}^{n+1} \times \mathbb{C} \rightarrow \mathbb{C}$ is the projection. Therefore $S(f)$ is semi-algebraic, by the Tarski-Seidenberg theorem.

Let now $c \in S(f)$. There exists a sequence $\{c_i\}_i \subset S(f)$ such that $\lim_{i \rightarrow \infty} c_i = c$. For any i , we have by definition a sequence $\{\mathbf{z}_{i,n}\}_n \subset M(f)$ such that $\lim_{n \rightarrow \infty} \mathbf{z}_{i,n} = \infty$ and $\lim_{n \rightarrow \infty} f(\mathbf{z}_{i,n}, \bar{\mathbf{z}}_{i,n}) = c_i$. Take a sequence $\{r_i\}_i \subset \mathbb{R}_+$ such that $\lim_{i \rightarrow \infty} r_i = \infty$. For each i there exists $n(i) \in \mathbb{N}$ such that $\mathbf{z}_{i,n} > r_i$ implies $|f(\mathbf{z}_{i,n}, \bar{\mathbf{z}}_{i,n}) - c_i| < \frac{1}{r_i}$, $\forall n \geq n(i)$. Setting $\mathbf{z}_k := \mathbf{z}_{k,n(k)}$ we get a sequence $\{\mathbf{z}_k\}_k$ such that $\lim_{k \rightarrow \infty} \|\mathbf{z}_k\| = \infty$ and $\lim_{k \rightarrow \infty} f(\mathbf{z}_k, \bar{\mathbf{z}}_k) = c$, which shows that $c \in S(f)$.

Let now $a \in \overline{f(\text{Sing } f) \cup S(f)}$. Since we have proved that $S(f)$ is closed, we may assume that $a \in \overline{f(\text{Sing } f)}$. Then there exists a sequence $\{\mathbf{z}_n\}_n \subset \text{Sing } f$, such that $\lim_{n \rightarrow \infty} f(\mathbf{z}_n, \bar{\mathbf{z}}_n) = a$. If $\{\mathbf{z}_n\}_n$ is not bounded, then we may choose a subsequence $\{\mathbf{z}_{n_k}\}_k$ such that $\lim_{k \rightarrow \infty} \mathbf{z}_{n_k} = \infty$ and $\lim_{k \rightarrow \infty} f(\mathbf{z}_{n_k}, \bar{\mathbf{z}}_{n_k}) = a$. Since $\text{Sing } f \subset M(f)$, it follows that $a \in S(f)$, see also Remark 4.2. In the other case, if $\{\mathbf{z}_n\}_n$ is bounded, then we may choose a subsequence $\{\mathbf{z}_{n_k}\}_k$ such that $\lim_{k \rightarrow \infty} \mathbf{z}_{n_k} = \mathbf{z}_0$ and $\lim_{k \rightarrow \infty} f(\mathbf{z}_{n_k}, \bar{\mathbf{z}}_{n_k}) = a$. Since $\text{Sing } f$ is a closed algebraic set, this implies $\mathbf{z}_0 \in \text{Sing } f$, so $a = f(\mathbf{z}_0, \bar{\mathbf{z}}_0) \in f(\text{Sing } f)$. \square

2.2. KOS-regularity. For holomorphic polynomials one has the *Malgrange regularity condition*, mentioned by Pham and used in many papers, see e.g. [Pa, ST, Ti1, Ti2]. This is known to be more general than “tame” or “quasi-tame”. It was extended to real maps by Kurdyka, Orro and Simon. These authors define in [KOS] *the set of generalized critical values* $K(F) = F(\text{Sing } F) \cup K_\infty(F)$ of a differentiable semi-algebraic map $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$, where

$$K_\infty(F) := \{y \in \mathbb{R}^k \mid \exists \{x_l\}_l \subset \mathbb{R}^n, \|x_l\| \rightarrow \infty \\ F(x_l) \rightarrow y \text{ and } (1 + \|x_l\|)\nu(dF(x_l)) \rightarrow 0\}$$

is the set of *asymptotic critical values* of F . In this definition they use the following distance function:

$$(2) \quad \nu(A) := \inf_{\|\varphi\|=1} \|A^* \varphi\|$$

for $A \in L(\mathbb{R}^n, \mathbb{R}^k)$. In the holomorphic setting, $\nu(df(x)) = \|\text{grad } f(x)\|$. Their main result is the following:

Theorem 2.7. [KOS, Theorem 3.1]

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a differentiable semi-algebraic map. Then $K(F)$ is a closed semi-algebraic set of dimension less than k .

Moreover, if F is of class C^2 , then $F : \mathbb{R}^n \setminus F^{-1}(K(F)) \rightarrow \mathbb{R}^k \setminus K(F)$ is a locally trivial fibration over each connected component of $\mathbb{R}^k \setminus K(F)$. In particular, the set $B(F)$ of bifurcation values of F is included in $K(F)$. \square

2.3. The fibration theorem. By the next two results we prove that $S(f)$ contains the atypical values due to the asymptotical behaviour and that it is contained in $K^\infty(f)$.

Proposition 2.8. *Let f be a mixed polynomial. Then $S(f) \subset K_\infty(f)$.*

REMARK 2.9. The above inclusion is *strict* in general. This holds already in the holomorphic setting; to prove it, we may use the examples constructed by Păunescu and Zaharia in [PZ], as follows. Let $f_{n,q} : \mathbb{C}^3 \rightarrow \mathbb{C}$, $f_{n,q}(x, y, z) := x - 3x^{2n+1}y^{2q} + 2x^{3n+1}y^{3q} + yz$, where $n, q \in \mathbb{N} \setminus \{0\}$. These polynomials are ρ -regular at infinity and therefore we have $S(f_{n,q}) = \emptyset$. It was also shown in [PZ] that $f_{n,q}$ satisfies Malgrange's condition for any $t \in \mathbb{C}$ if and only if $n \leq q$. Therefore, in case $n > q$, we have $\emptyset = S(f_{n,q}) \subsetneq K_\infty(f_{n,q}) \neq \emptyset$.

Proof of Proposition 2.8. Let (g, h) be the corresponding real map of the mixed polynomial f and denote $\nu(\mathbf{x}) := \nu(d(g, h)(\mathbf{x}))$. Let us first prove:

$$(3) \quad \nu(\mathbf{x}) = \inf_{\mu \in S^1} \|\mu \overline{d}f(\mathbf{z}, \overline{\mathbf{z}}) + \overline{\mu} d f(\mathbf{z}, \overline{\mathbf{z}})\|.$$

By the definition (2) of $\nu(\mathbf{x})$, we have:

$$\nu(\mathbf{x}) = \inf_{(a,b) \in S^1} \|adg(\mathbf{x}) + bdh(\mathbf{x})\|.$$

But the proof of Lemma 2.1 shows:

$$\|adg(\mathbf{x}) + bdh(\mathbf{x})\| = \|\mu \overline{d}f(\mathbf{z}, \overline{\mathbf{z}}) + \overline{\mu} d f(\mathbf{z}, \overline{\mathbf{z}})\|$$

for $\mu = a + ib \in S^1$. Our claim is proved.

Let then $c \in S(f)$. By Definition 2.4 and Lemma 2.5, there exist real analytic paths, $\mathbf{z}(t)$ in $M(f)$, $\lambda(t)$ in \mathbb{R} and $\mu(t)$ in \mathbb{C}^* , defined on a small enough interval $]0, \varepsilon[$, such that $\lim_{t \rightarrow 0} \|\mathbf{z}(t)\| = \infty$ and $\lim_{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) = c$ and that:

$$(4) \quad \lambda(t)\mathbf{z}(t) = \mu(t)\overline{d}f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) + \overline{\mu}(t)d f(\mathbf{z}(t), \overline{\mathbf{z}}(t)).$$

Let us assume that $\lambda(t) \neq 0$. Dividing (4) by $\|\mu(t)\|$ yields:

$$(5) \quad \lambda_0(t)\mathbf{z}(t) = \mu_0(t)\overline{d}f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) + \overline{\mu}_0(t)d f(\mathbf{z}(t), \overline{\mathbf{z}}(t))$$

where $\lambda_0(t) := \frac{\lambda(t)}{\|\mu(t)\|}$ and $\mu_0(t) := \frac{\mu(t)}{\|\mu(t)\|}$; therefore $\beta := \text{ord}_t(\mu_0(t)) = 0$.

Since $\lim_{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) = c$, we have $\alpha := \text{ord}_t \frac{d}{dt} f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \geq 0$. Then the following computation:

$$\begin{aligned} \overline{\mu}_0(t) \frac{d}{dt} f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) + \mu_0(t) \frac{d}{dt} \overline{f}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) &= \left\langle \mu_0(t) \overline{d}f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) + \overline{\mu}_0(t) d f(\mathbf{z}(t), \overline{\mathbf{z}}(t)), \frac{d}{dt} \mathbf{z}(t) \right\rangle \\ &\quad + \left\langle \frac{d}{dt} \mathbf{z}(t), \mu_0(t) \overline{d}f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) + \overline{\mu}_0(t) d f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \right\rangle \\ &\stackrel{\text{by (5)}}{=} \lambda_0(t) \left(\left\langle \mathbf{z}(t), \frac{d}{dt} \mathbf{z}(t) \right\rangle + \left\langle \frac{d}{dt} \mathbf{z}(t), \mathbf{z}(t) \right\rangle \right) \\ &= \lambda_0(t) \frac{d}{dt} \|\mathbf{z}(t)\|^2 \end{aligned}$$

shows that $\text{ord}_t(\lambda_0(t) \frac{d}{dt} \|\mathbf{z}(t)\|^2) \geq \alpha + \beta \geq 0$. But since $\text{ord}_t(\mathbf{z}(t)) < 0$, this implies that $\lim_{t \rightarrow 0} |\lambda_0(t)| \|\mathbf{z}(t)\|^2 = 0$. Note that this limit holds true for $\lambda(t) \equiv 0$ too.

From the last limit, by using (5), we get:

$$(6) \quad \lim_{t \rightarrow 0} \|\mathbf{z}(t)\| \|\mu_0(t) \overline{df}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) + \overline{\mu}_0(t) \overline{df}(\mathbf{z}(t), \overline{\mathbf{z}}(t))\| = 0$$

which, by (3), implies

$$\lim_{t \rightarrow 0} \|\mathbf{x}(t)\| \|\nu(\mathbf{x}(t))\| = 0,$$

showing that $c \in K_\infty(f)$. □

Theorem 2.10. *Fibration Theorem*

Let f be a mixed polynomial. Then the restriction:

$$f|_1 : \mathbb{C}^n \setminus f^{-1}(f(\text{Sing } f) \cup S(f)) \rightarrow \mathbb{C} \setminus f(\text{Sing } f) \cup S(f)$$

is a locally trivial C^∞ fibration. In particular $B(f) \subset f(\text{Sing } f) \cup S(f)$.

REMARK 2.11. In the setting of mixed functions, our Theorem 2.10 extends [KOS, Theorem 3.1] since, by our Proposition 2.8, $S(f) \subsetneq K_\infty(f)$, and therefore we get a sharper approximation of the bifurcation set $B(f)$. While our proof does not explicitly bound the dimension of $S(f)$, it follows from the preceding inclusion and from [KOS, Theorem 3.1] that $S(f)$ has real dimension less than 2.

Proof of the Fibration Theorem. Let $c \notin f(\text{Sing } f) \cup S(f)$. Then there is a closed disk D centered at c such that $D \subset \mathbb{C} \setminus f(\text{Sing } f) \cup S(f)$, since the latter is an open set by Proposition 2.6. Let us first remark that there exists $R_0 \gg 0$ such that $M(f) \cap f^{-1}(D) \setminus B_{R_0}^{2n} = \emptyset$. Indeed, if this were not true, then there exists a sequence $\{\mathbf{z}_k\}_{k \in \mathbb{N}} \subset f^{-1}(D) \cap M(f)$ such that $\lim_{k \rightarrow \infty} \|\mathbf{z}_k\| = \infty$. Since D is compact, there is a sub-sequence $\{\mathbf{z}_{k_i}\}_{i \in \mathbb{N}} \subset M(f)$ and $c_0 \in D$ such that $\lim_{i \rightarrow \infty} \|\mathbf{z}_{k_i}\| = \infty$ and $\lim_{i \rightarrow \infty} f(\mathbf{z}_{k_i}) = c_0$, which contradicts $D \subset \mathbb{C} \setminus S(f)$.

We claim that the map:

$$(7) \quad f|_1 : f^{-1}(D) \setminus B_{R_0}^{2n} \rightarrow D$$

is a trivial fibration on the manifold with boundary $(f^{-1}(D) \setminus B_{R_0}^{2n}, f^{-1}(D) \cap S_R^{2n-1})$, for any $R \geq R_0$. Indeed, this is a submersion by hypothesis but it is not proper, so one cannot apply Ehresmann's theorem directly. Instead, we consider the map $(f, \rho) : f^{-1}(D) \setminus B_{R_0}^{2n} \rightarrow D \times [R_0, \infty[$. As a direct consequence of its definition, this is a proper map. It is moreover a submersion since $\text{Sing}(f, \rho) \cap f^{-1}(D) \setminus B_{R_0}^{2n} = \emptyset$ by the above remark concerning the set $M(f)$, which is nothing else but $\text{Sing}(f, \rho)$. We then apply to (f, ρ) Ehresmann's theorem to conclude that it is a locally trivial, hence a trivial fibration over $D \times [R_0, \infty[$. Take now the projection $\pi : D \times [R_0, \infty[\rightarrow D$ which is a trivial fibration by definition and remark that our map (7) is the composition $\pi \circ (f, \rho)$ of two trivial fibrations, hence a trivial fibration too.

Next remark that, since $D \cap f(\text{Sing } f) = \emptyset$, the restriction:

$$(8) \quad f|_1 : f^{-1}(D) \cap \overline{B}_{R_0}^{2n} \rightarrow D$$

is a proper submersion on the manifold with boundary $(f^{-1}(D) \cap \overline{B}_{R_0}^{2n}, f^{-1}(D) \cap S_{R_0}^{2n-1})$ and therefore a locally trivial fibration by Ehresmann's theorem, hence a trivial fibration over D .

Finally we glue the two trivial fibrations (8) and (7) by using an isotopy and the trivial fibration from the following commuting diagram, for some $R > R_0$:

$$(9) \quad \begin{array}{ccc} (\bar{B}_R \setminus B_{R_0}^\circ) \cap f^{-1}(D) & \xrightarrow{(f,\rho)} & D \times [R_0, R] \\ \downarrow \simeq & \nearrow & \downarrow \text{pr} \\ \hat{F} \times D \times [R_0, R] & \xrightarrow{\text{pr}} & D \end{array}$$

where \hat{F} denotes the fibre of the trivial fibration $f|_1 : S_R \cap f^{-1}(D) \rightarrow D$ and does not depend on the radius $R \geq R_0$. \square

3. BIFURCATION VALUES OF NEWTON NON-DEGENERATE MIXED POLYNOMIALS

We prove an estimation for the set of ρ -nonregular values at infinity under the condition of Newton non-degeneracy of the mixed polynomial. We first introduce the necessary notions, then state the result.

3.1. Newton boundary at infinity and non-degeneracy. Let f be a mixed polynomial:

$$f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu, \mu} c_{\nu, \mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$$

where $\mathbf{z}^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ for $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$ and $\bar{\mathbf{z}}^\mu = \bar{z}_1^{\mu_1} \cdots \bar{z}_n^{\mu_n}$ for $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$.

Definition 3.1. We call $\text{supp}(f) = \{\nu + \mu \in \mathbb{N}^n \mid c_{\nu, \mu} \neq 0\}$ the *support* of f . We say that f is *convenient* if the intersection of $\text{supp}(f)$ with each coordinate axis is non-empty. We denote by $\overline{\text{supp}(f)}$ the convex hull of the set $\text{supp}(f) \setminus \{0\}$. The *Newton polyhedron* of a mixed polynomial f , denoted by $\Gamma_0(f)$, is the convex hull of the set $\{0\} \cup \text{supp}(f)$. The *Newton boundary at infinity*, denoted by $\Gamma^+(f)$, is the union of the faces of the polyhedron $\Gamma_0(f)$ which do not contain the origin. By ‘‘face’’ we mean face of any dimension.

Definition 3.2. For any face Δ of $\overline{\text{supp}(f)}$, we denote the restriction of f to $\Delta \cap \text{supp}(f)$ by $f_\Delta := \sum_{\nu + \mu \in \Delta \cap \text{supp}(f)} c_{\nu, \mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$. The mixed polynomial f is called *non-degenerate* if $\text{Sing} f_\Delta \cap f_\Delta^{-1}(0) \cap (\mathbb{C}^*)^n = \emptyset$, for each face Δ of $\Gamma^+(f)$. Following Oka’s terminology [Oka2], we say that f is *Newton strongly non-degenerate* if the stronger condition $\text{Sing} f_\Delta \cap (\mathbb{C}^*)^n = \emptyset$ is satisfied for any face Δ of $\Gamma^+(f)$.

Kushnirenko [Ku] had first introduced the Newton boundary of holomorphic germs, which we denote by Γ_- and which is different from Γ^+ . Recently, Mutsuo Oka took over the program in the setting of mixed function germs and proved, among other results, the following *local fibration theorem* :

Theorem 3.3. [Oka2, Lemma 28, Theorem 29]

Let $f(z, \bar{z})$ be the germ at 0 of a mixed polynomial which has a strongly non-degenerate and convenient Newton boundary $\Gamma_-(f)$. Then f has an isolated singularity at 0 and the map:

$$f|_1 : B_\varepsilon^{2n} \cap f^{-1}(D_\delta^*) \rightarrow D_\delta^*.$$

is a locally trivial fibration. \square

In the setting of holomorphic polynomials (i.e. polynomial in the variables \mathbf{z} only), similar objects have been studied by Broughton [Br2]. He proved for instance that if f is a complex polynomial with Newton non-degenerate and convenient polyhedron $\Gamma^+(f)$, then $S(f) = \emptyset$. Later, Némethi and Zaharia [NZ1] dropped the convenience condition, defined the set \mathfrak{B} of “bad faces” of $\overline{\text{supp}f}$ and proved the following result.

Theorem 3.4. [NZ1, Theorem 2] *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a complex polynomial, Newton non-degenerate and $f(0) = 0$. Then:*

$$B(f) \subset f(\text{Sing}f) \cup \{0\} \cup \bigcup_{\Delta \in \mathfrak{B}} f_{\Delta}(\text{Sing}f_{\Delta}).$$

□

We prove here:

Theorem 3.5. *Let f be a mixed polynomial which depends effectively on all the variables and let $f(0) = 0$. If f is Newton non-degenerate then:*

- (a) $S(f) \subset \{0\} \cup \bigcup_{\Delta \in \mathfrak{B}} f_{\Delta}(\text{Sing}f_{\Delta})$, where \mathfrak{B} is the set of bad faces of $\overline{\text{supp}f}$.
- (b) If f is moreover Newton strongly non-degenerate then $f(\text{Sing}f)$ and $S(f)$ are bounded.

It appears that our theorem improves [NZ1, Theorem 2] even in the holomorphic setting, since it bounds the atypical values at infinity by the set $S(f)$ and not the whole bifurcation set $B(f)$ which contains in addition the critical values of f . Theorem 3.5 gives also more details, see for instance Remark 4.2 on the values $c \neq 0$ such that the fibre $f^{-1}(c)$ contains unbounded branches of $M(f)$ (or of $\text{Sing}f$): they are detected by the critical values due to the bad faces.

REMARK 3.6. If f satisfies the conditions of Theorem 3.5 except of $f(0) = 0$, then we replace f by $h = f - f(0)$ and apply to it Theorem 3.5. Since $\overline{df}(\mathbf{z}, \overline{\mathbf{z}}) = \overline{dh}(\mathbf{z}, \overline{\mathbf{z}})$ and $\overline{df}(\mathbf{z}, \overline{\mathbf{z}}) = \overline{dh}(\mathbf{z}, \overline{\mathbf{z}})$, we get $M(f) = M(h)$ and $c \in S(f) \Leftrightarrow c - f(0) \in S(h)$.

Before giving the proof in §4, we need to define the ingredients and prove several preliminary facts.

3.2. The “bad” faces of the support. We consider a mixed polynomial $f : \mathbb{C}^n \rightarrow \mathbb{C}$, $f \not\equiv 0$.

Definition 3.7. A face $\Delta \subseteq \overline{\text{supp}(f)}$ is called *bad* if:

- (i) the affine subspace of the same dimension spanned by Δ contains the origin,
- (ii) there exists a hyperplane $H \subset \mathbb{R}^n$ of the equation $a_1x_1 + \dots + a_nx_n = 0$ (where x_1, \dots, x_n are the coordinates in \mathbb{R}^n) such that:
 - (a) there exist i and j with $a_i < 0$ and $a_j > 0$,
 - (b) $H \cap \overline{\text{supp}(f)} = \Delta$.

Let \mathfrak{B} denote the set of bad faces of $\overline{\text{supp}(f)}$.

The following lemma will be used in the proof of our theorem.

Lemma 3.8. *Let $l_{\mathbf{p}}(x) = \sum_{i=1}^n p_i x_i$ be a linear function such that $p = \min_{1 \leq i \leq n} \{p_i\} < 0$. We consider the restriction of $l_{\mathbf{p}}(x)$ on $\overline{\text{supp}(f)}$ and denote by $\Delta_{\mathbf{p}}$ the unique maximal face of $\overline{\text{supp}(f)}$ (with respect to the inclusion of faces) where $l_{\mathbf{p}}(x)$ takes its minimal value $d_{\mathbf{p}}$. Suppose $d_{\mathbf{p}} \leq 0$. Then:*

- (a) *If $d_{\mathbf{p}} < 0$, then $\Delta_{\mathbf{p}}$ is a face of $\Gamma^+(f)$.*
- (b) *If $d_{\mathbf{p}} = 0$, then either $\Delta_{\mathbf{p}}$ is a face of $\Gamma^+(f)$ or $\Delta_{\mathbf{p}}$ satisfies condition (i) of Definition 3.7.*

Proof. Let us first remark that from Definition 3.1 we have $\Gamma_0(f) = \text{cone}_0(\Gamma^+(f))$, where $\text{cone}_0(A)$ denotes the compact cone over the set A with vertex the origin. For each face Δ of $\Gamma_0(f)$ we have that either Δ is a face of $\Gamma^+(f)$ or $\Delta \ni 0$ and in this case we have $\Delta = \text{cone}_0(\Delta \cap \overline{\text{supp}(f)}) = \text{cone}_0(\Delta \cap \Gamma^+(f))$.

Next, considering the restriction of $l_{\mathbf{p}}(x)$ to $\Gamma_0(f)$, we denote by Δ_1 the maximal face of $\Gamma_0(f)$ where $l_{\mathbf{p}}(x)$ takes its minimal value d . Note that $l_{\mathbf{p}}(x)$ can not attain its minimal value d at interior points of $\Gamma_0(f)$. Since $\Gamma^+(f) \subset \overline{\text{supp}(f)} \subset \Gamma_0(f)$, we have $d \leq d_{\mathbf{p}}$.

(a). If $d_{\mathbf{p}} < 0$ then it follows by our initial remark that Δ_1 is a face of $\Gamma^+(f)$, since otherwise we have $0 \in \Delta_1$ and $d = 0$. We therefore get $\Delta_{\mathbf{p}} = \Delta_1 \subset \Gamma^+(f)$ and $d = d_{\mathbf{p}}$.

(b). If $d_{\mathbf{p}} = 0$ and Δ_1 is not a face of $\Gamma^+(f)$, then by the same initial remark we have $\Delta_1 \ni 0$ and therefore $d = 0$. Since Δ_1 is the maximal face of $\Gamma_0(f)$ where $l_{\mathbf{p}}(x)$ takes its minimal value d , we get $\Delta_{\mathbf{p}} \subset \Delta_1$. Let us denote the hyperplane $\{x \in \mathbb{R}^n \mid l_{\mathbf{p}}(x) = 0\}$ by H . We then have $\Delta_{\mathbf{p}} = \overline{\text{supp}(f)} \cap H$, $\Delta_1 = \Gamma_0(f) \cap H$, and therefore $\Delta_{\mathbf{p}} = \Delta_1 \cap \overline{\text{supp}(f)}$. Let us assume that $\Delta_{\mathbf{p}}$ does not verify condition (i) of Definition 3.7, namely that we have $\dim \text{cone}_0(\Delta_{\mathbf{p}}) > \dim \Delta_{\mathbf{p}}$. This implies that $\Delta_{\mathbf{p}}$ does not contain any interior point of $\text{cone}_0(\Delta_{\mathbf{p}})$. By the initial remark, $\Delta_1 = \text{cone}_0(\Delta_1 \cap \Gamma^+(f)) = \text{cone}_0(\Delta_{\mathbf{p}})$. Then $\Delta_{\mathbf{p}}$ is a face of $\Gamma^+(f)$, which contradicts our assumption. \square

Let $I \subset \{1, \dots, n\}$. We shall use the following notations:

$$\mathbb{C}^I = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_j = 0, j \notin I\}, \text{ and similarly } \mathbb{R}_{\geq 0}^I,$$

$$\mathbb{C}^{*n} := (\mathbb{C}^*)^n, \mathbb{C}^{*I} := \mathbb{C}^I \cap \mathbb{C}^{*n}, f^I := f|_{\mathbb{C}^I}.$$

From Definition 3.1, the faces of f^I are among the faces of f , so we have the following:

REMARK 3.9. Let f be a mixed Newton non-degenerate polynomial. If $I \subset \{1, 2, \dots, n\}$ such that f^I is not identically zero then:

- (1) f^I is a mixed Newton non-degenerated polynomial.
- (2) $\Gamma(f^I) = \Gamma^+(f) \cap \mathbb{R}_{\geq 0}^I$.

We shall use the following fact for the restriction of f to its bad faces.

REMARK 3.10. If a mixed polynomial f is Newton non-degenerate then, for any bad face $\Delta \subset \overline{\text{supp}(f)}$, f_{Δ} is Newton non-degenerate. Indeed, any face Δ' of $\Gamma^+(f_{\Delta})$ is also a subface of Δ , hence a subface of $\Gamma^+(f)$. The Newton non-degeneracy of f implies that the restriction f_{Δ} is also Newton non-degenerate.

3.3. Newton non-degeneracy is an open dense condition. For a fixed polyhedron Γ which is the Newton boundary at infinity of some mixed polynomial, we may define the subset of all mixed polynomials having the same Newton boundary at infinity,

$U_\Gamma := \{[c_1, c_2, \dots, c_m] \in \mathbb{P}^{m-1} \mid \text{the polynomial } f(\mathbf{z}, \bar{\mathbf{z}}, c) = \sum_{j=1}^m c_j \mathbf{z}^{\mu_j} \bar{\mathbf{z}}^{\nu_j} \text{ is Newton non-degenerate and } \Gamma^+(f) = \Gamma\}$. Then:

Proposition 3.11. *The subset $U_\Gamma \subset \mathbb{P}^{m-1}$ of Newton non-degenerate mixed polynomials with fixed Newton boundary Γ at infinity is a semi-algebraic open dense set.*

REMARK 3.12. In the holomorphic setting such a result was proved by Kushnirenko [Ku] and is a consequence of the Bertini-Sard theorem. Note that in this setting “strongly non-degenerate” is equivalent to “non-degenerate”.

Unlike the holomorphic setting, one does not have the connexity in general. Let us also point out that, in the mixed setting, the Newton strong non-degeneracy does not insure anymore the density. We may see this by a simple example. Consider $f : \mathbb{C} \rightarrow \mathbb{C}$, $f(z, \bar{z}) = az^2 + bz\bar{z} + c\bar{z}^2$, where $a, b, c \in \mathbb{C}$. By direct computations using the homogeneity of f , we get that f is Newton non-degenerate if and only if $(|a|^2 - |c|^2)^2 > |\bar{a}b - c\bar{b}|^2$. This inequality describes an open set in \mathbb{C}^3 which is not dense. Note also that $\text{supp}(f)$ is a single point.

Proof of Proposition 3.11. Let H denote the hypersurface $\{f = 0 \mid f \in U_\Gamma\} \subset \mathbb{C}^n \times \mathbb{P}^{m-1}$ and observe that $\text{Sing}H \cap \mathbb{C}^{*n} \times \mathbb{P}^{m-1} = \emptyset$. We apply the semi-algebraic Sard theorem to the manifold $H^* := H \cap \mathbb{C}^{*n} \times \mathbb{P}^{m-1}$ and its projection $\pi : H^* \rightarrow \mathbb{P}^{m-1}$. Since $U_\Gamma = \mathbb{P}^{m-1} \setminus \pi(H^*)$, it follows that U_Γ is the complement of a semi-algebraic set of dimension $< m - 1$.

Let us show now that it is open. In the holomorphic setting this follows from the proper mapping theorem but in the real setting this is no more true. We need another proof; we took some hints from Oka’s holomorphic proof in [Oka3, Appendix].

For every face $\Delta \subset \Gamma$ we define:

$$V(\Delta) := \{(\mathbf{z}, c) \in \mathbb{C}^n \times \mathbb{P}^{m-1} \mid \exists \lambda \in S_1^1, \text{ such that } \overline{df}_\Delta(\mathbf{z}, \bar{\mathbf{z}}, c) = \lambda \overline{df}_\Delta(\mathbf{z}, \bar{\mathbf{z}}, c)\}$$

$$V(\Delta)^* := V(\Delta) \cap \{(\mathbf{z}, c) \in \mathbb{C}^n \times \mathbb{P}^{m-1} \mid z_1 z_2 \dots z_n \neq 0\}.$$

Note that $\overline{V(\Delta)^*} = V(\Delta)$. Let us consider the union $V^* = \cup_{\Delta \subset \Gamma} V(\Delta)^*$ and the projection $\pi : \mathbb{C}^n \times \mathbb{P}^{m-1} \rightarrow \mathbb{P}^{m-1}$. To show that U is an open set, it is enough to prove that the image $W = \pi(V^*)$ is a closed set. Let us remark that W is a semi-algebraic set, since the projection of a semi-algebraic set.

Let $c_0 \in \overline{W}$. Then, by the Curve Selection Lemma, there exists a face Δ_0 of $\Gamma(f)$ and a real analytic path $(\mathbf{z}(t), c(t)) \subset V(\Delta_0)^*$ defined on a small enough interval $]0, \varepsilon[$ such that $\lim_{t \rightarrow 0} c(t) = c_0$ and either $\lim_{t \rightarrow 0} \|\mathbf{z}(t)\| = \infty$ or $\lim_{t \rightarrow 0} \mathbf{z}(t) = \mathbf{z}_0 \in V(\Delta_0) \setminus V(\Delta_0)^*$.

Let then $z_i(t) = a_i t^{p_i} + \text{h.o.t.}$ for $1 \leq i \leq n$ where $a_i \neq 0, p_i \in \mathbb{Z}$ and $\lambda(t) = \lambda_0 + \lambda_1 t + \text{h.o.t.}$, where $\lambda_0 \in S_1^1$. Let $\mathbf{a} := (a_1, \dots, a_n) \in \mathbb{C}^{*n}$, $\mathbf{P} := (p_1, \dots, p_n) \in \mathbb{Z}^n$ and consider the linear function $l_{\mathbf{P}} = \sum_{i=1}^n p_i x_i$ defined on Δ_0 . Let Δ_1 be the maximal face of Δ_0 where $l_{\mathbf{P}}$ takes its minimal value, say this value is $d_{\mathbf{P}}$. We have:

$$\frac{\overline{\partial f_{\Delta_1}}}{\partial z_i}(\mathbf{a}, \bar{\mathbf{a}}, c(t)) t^{d_{\mathbf{P}} - p_i} + \text{h.o.t.} = \lambda_0 \frac{\partial f_{\Delta_1}}{\partial \bar{z}_i}(\mathbf{a}, \bar{\mathbf{a}}, c(t)) t^{d_{\mathbf{P}} - p_i} + \text{h.o.t.}$$

and by taking the limit $c(t) \rightarrow c_0$ and focussing on the first terms of the expansions:

$$\overline{df}_{\Delta_1}(\mathbf{a}, \bar{\mathbf{a}}, c_0) = \lambda_0 \overline{df}_{\Delta_1}(\mathbf{a}, \bar{\mathbf{a}}, c_0)$$

which implies that $(\mathbf{a}, c_0) \in V^*$, since $\mathbf{a} \in \mathbb{C}^{*n}$. Thus $c_0 \in W$ and we may conclude that $W = \overline{W}$. \square

4. PROOF OF THEOREM 3.5 AND SOME CONSEQUENCES

4.1. Proof of Theorem 3.5(a). Let $c \in S(f)$. By Definition 2.4 and Lemma 2.5, there exist real analytic paths, $\mathbf{z}(t)$ in $M(f)$, $\lambda(t)$ in \mathbb{R} and $\mu(t)$ in \mathbb{C}^* , defined on a small enough interval $]0, \varepsilon[$, such that $\lim_{t \rightarrow 0} \|\mathbf{z}(t)\| = \infty$ and $\lim_{t \rightarrow 0} f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) = c$ and that:

$$(10) \quad \lambda(t)\mathbf{z}(t) = \mu(t)\overline{df}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) + \overline{\mu}(t)\overline{df}(\mathbf{z}(t), \overline{\mathbf{z}}(t)).$$

Consider the expansion of $f(\mathbf{z}(t), \overline{\mathbf{z}}(t))$. We have two situations, either:

$$(11) \quad f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) \equiv c$$

or

$$(12) \quad f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) = c + bt^\delta + \text{h.o.t.}, \quad \text{where } c, b \in \mathbb{C}, b \neq 0, \delta \in \mathbb{N}^*.$$

Let $I = \{i \mid z_i(t) \neq 0\}$, remark that $I \neq \emptyset$ since $\lim_{t \rightarrow 0} \|\mathbf{z}(t)\| = \infty$, and write:

$$(13) \quad z_i(t) = a_i t^{p_i} + \text{h.o.t.}, \quad \text{where } a_i \neq 0, p_i \in \mathbb{Z}, i \in I.$$

By eventually transposing the coordinates, we may assume that $I = \{1, \dots, m\}$ and that $p = p_1 \leq p_2 \leq \dots \leq p_m$. Since $\lim_{t \rightarrow 0} \|\mathbf{z}(t)\| = \infty$, this implies $p = \min_{j \in I} \{p_j\} < 0$. We denote $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{C}^{*I}$, $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ and consider the linear function $l_{\mathbf{p}} = \sum_{i=1}^m p_i x_i$ defined on $\text{supp}(f^I)$.

Let us remark that since $f(0) = 0$, if $c \neq 0$ then, in both situations (11) and (12), we have that $\text{supp}(f^I)$ is not empty. Let then Δ be the *maximal face* of $\text{supp}(f^I)$ where $l_{\mathbf{p}}$ takes its minimal value, say $d_{\mathbf{p}}$. We have:

$$(14) \quad f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) = f^I(\mathbf{z}(t), \overline{\mathbf{z}}(t)) = f^I_{\Delta}(\mathbf{a}, \overline{\mathbf{a}})t^{d_{\mathbf{p}}} + \text{h.o.t.}$$

where $d_{\mathbf{p}} \leq \text{ord}_t(f(\mathbf{z}(t), \overline{\mathbf{z}}(t))) = 0$.

We assume in the following that $c \neq 0$. (For the case $c = 0$, we refer to Remark 4.1.) For $i \in I$ we have the equalities: $\frac{\partial f}{\partial z_i}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) = \frac{\partial f^I}{\partial z_i}(\mathbf{z}(t), \overline{\mathbf{z}}(t))$ and $\frac{\partial f}{\partial \overline{z}_i}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) = \frac{\partial f^I}{\partial \overline{z}_i}(\mathbf{z}(t), \overline{\mathbf{z}}(t))$. Then we may write:

$$(15) \quad \begin{aligned} \frac{\partial f}{\partial z_i}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) &= \frac{\partial f^I_{\Delta}}{\partial z_i}(\mathbf{a}, \overline{\mathbf{a}})t^{d_{\mathbf{p}}-p_i} + \text{h.o.t.} \\ \frac{\partial f}{\partial \overline{z}_i}(\mathbf{z}(t), \overline{\mathbf{z}}(t)) &= \frac{\partial f^I_{\Delta}}{\partial \overline{z}_i}(\mathbf{a}, \overline{\mathbf{a}})t^{d_{\mathbf{p}}-p_i} + \text{h.o.t.} \end{aligned}$$

Consider the expansion of $\lambda(t)$, in case $\lambda(t) \neq 0$, and that of $\mu(t)$:

$$\begin{aligned} \lambda(t) &= \lambda_0 t^\gamma + \text{h.o.t.}, & \text{where } \lambda_0 \in \mathbb{R}^*, \gamma \in \mathbb{Z}, \\ \mu(t) &= \mu_0 t^l + \text{h.o.t.}, & \text{where } \mu_0 \neq 0, l \in \mathbb{Z}. \end{aligned}$$

Using all the expansions we get from (10), for any $i \in I$:

$$\left(\mu_0 \frac{\partial f^I_{\Delta}}{\partial z_i}(\mathbf{a}, \overline{\mathbf{a}}) + \overline{\mu_0} \frac{\partial f^I_{\Delta}}{\partial \overline{z}_i}(\mathbf{a}, \overline{\mathbf{a}})\right)t^{d_{\mathbf{p}}-p_i+l} + \text{h.o.t.} = \lambda_0 a_i t^{p_i+\gamma} + \text{h.o.t.}$$

Since $\lambda_0 a_i \neq 0$, comparing the orders of the two sides in the above formula, we obtain:

$$(16) \quad \overline{\mu_0 \frac{\partial f_\Delta^I}{\partial z_i}(\mathbf{a}, \bar{\mathbf{a}})} + \bar{\mu}_0 \frac{\partial f_\Delta^I}{\partial \bar{z}_i}(\mathbf{a}, \bar{\mathbf{a}}) = \begin{cases} \lambda_0 a_i, & \text{if } d_{\mathbf{p}} - p_i + l = p_i + \gamma \\ 0, & \text{if } d_{\mathbf{p}} - p_i + l < p_i + \gamma \end{cases}$$

Let $J = \{j \in I \mid d_{\mathbf{p}} - p_j + l = p_j + \gamma\}$. If we suppose that $J \neq \emptyset$, then $J = \{j \in I \mid p_j = p = \min_{j \in I} \{p_j\} < 0\}$.

In the situation (12) we have $\frac{df(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{dt} = b\delta t^{\delta-1} + \text{h.o.t}$ and on the other hand:

$$(17) \quad \frac{df(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{dt} = \sum_{i=1}^m \left(\frac{\partial f}{\partial z_i} \cdot \frac{\partial z_i}{\partial t} + \frac{\partial f}{\partial \bar{z}_i} \cdot \frac{\partial \bar{z}_i}{\partial t} \right) = \sum_{i=1}^m \left(\frac{\partial f^I}{\partial z_i} \cdot \frac{\partial z_i}{\partial t} + \frac{\partial f^I}{\partial \bar{z}_i} \cdot \frac{\partial \bar{z}_i}{\partial t} \right) \\ = \left[\left\langle \mathbf{p}\mathbf{a}, \overline{df_\Delta^I(\mathbf{a}, \bar{\mathbf{a}})} \right\rangle + \left\langle \mathbf{p}\bar{\mathbf{a}}, \overline{df_\Delta^I(\mathbf{a}, \bar{\mathbf{a}})} \right\rangle \right] t^{d_{\mathbf{p}}-1} + \text{h.o.t.}$$

where $\mathbf{p}\mathbf{a} = (p_1 a_1, \dots, p_m a_m)$. Comparing the orders of the two expansions of $\frac{df(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{dt}$ and using the inequality $d_{\mathbf{p}} < \delta$ implied by $c \neq 0$ (see after (14)), we find:

$$(18) \quad \left\langle \mathbf{p}\mathbf{a}, \overline{df_\Delta^I(\mathbf{a}, \bar{\mathbf{a}})} \right\rangle + \left\langle \mathbf{p}\bar{\mathbf{a}}, \overline{df_\Delta^I(\mathbf{a}, \bar{\mathbf{a}})} \right\rangle = 0.$$

Let us remark here that the proof of formula (18) holds under the more general condition $d_{\mathbf{p}} < \delta$.

Let now consider the situation (11). In this case the formula (18) is true more directly, since $\frac{df(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{dt} = 0$ and after comparing this to (17).

Next, multiplying (18) by $\bar{\mu}_0$ and taking the real part, we get:

$$\begin{aligned} & \text{Re} \left\langle \mathbf{p}\mathbf{a}, \mu_0 \overline{df_\Delta^I(\mathbf{a}, \bar{\mathbf{a}})} \right\rangle + \text{Re} \left\langle \mathbf{p}\bar{\mathbf{a}}, \mu_0 \overline{df_\Delta^I(\mathbf{a}, \bar{\mathbf{a}})} \right\rangle \\ & = \text{Re} \left\langle \mathbf{p}\mathbf{a}, \mu_0 \overline{df_\Delta^I(\mathbf{a}, \bar{\mathbf{a}})} + \bar{\mu}_0 \overline{df_\Delta^I(\mathbf{a}, \bar{\mathbf{a}})} \right\rangle = 0. \end{aligned}$$

On the other hand, from (16), we have:

$$\text{Re} \left\langle \mathbf{p}\mathbf{a}, \mu_0 \overline{df_\Delta^I(\mathbf{a}, \bar{\mathbf{a}})} + \bar{\mu}_0 \overline{df_\Delta^I(\mathbf{a}, \bar{\mathbf{a}})} \right\rangle = \sum_{i \in J} \lambda_0 p \|a_j\|^2$$

which is different from zero since $\lambda_0 \neq 0$, $p < 0$ and $a_j \neq 0$. This contradicts formula (18). We have therefore proved that $J = \emptyset$.

From (16) we obtain:

$$(19) \quad \overline{\mu_0 df_\Delta^I(\mathbf{a}, \bar{\mathbf{a}})} + \bar{\mu}_0 \overline{df_\Delta^I(\mathbf{a}, \bar{\mathbf{a}})} = 0.$$

Let us remark that in case $\lambda(t) \equiv 0$ we have $J = \emptyset$ and therefore we get directly (19).

What (19) tells us is that \mathbf{a} is a singularity of f_Δ^I . Set now $\mathbf{A} = (\mathbf{a}, 1, 1, \dots, 1)$ with the i^{th} coordinate $z_i = 1$ for $i \notin I$. Since $\Delta \subset \text{supp}(f^I)$, the restriction f_Δ does not depend on the variables z_{m+1}, \dots, z_n or their conjugates. Thus for any $i \in \{1, 2, \dots, n\}$, we have $\frac{\partial f_\Delta}{\partial z_i}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \frac{\partial f_\Delta^I}{\partial z_i}(\mathbf{z}(t), \bar{\mathbf{z}}(t))$ and $\frac{\partial f_\Delta}{\partial \bar{z}_i}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \frac{\partial f_\Delta^I}{\partial \bar{z}_i}(\mathbf{z}(t), \bar{\mathbf{z}}(t))$. By replacing f_Δ^I with f_Δ in (19), we get that $\mathbf{A} \in \mathbb{C}^{*n}$ is a singularity of f_Δ .

We may now apply Lemma 3.8 to $d_{\mathbf{p}}$ and Δ . We have the following two cases:

(I). If $d_{\mathbf{p}} < 0$, then, by Lemma 3.8(a), Δ is a face of $\Gamma^+(f^I)$. Since $\mathbf{A} \in \mathbb{C}^{*n}$ is a singularity of f_{Δ} and since we have $f_{\Delta}(\mathbf{A}, \overline{\mathbf{A}}) = 0$ by (14) for $d_{\mathbf{p}} < 0$, this contradicts the Newton non-degeneracy of f (Definition 3.2) assumed in the statement of Theorem 3.5.

(II). Let $d_{\mathbf{p}} = 0$. Then Δ cannot be a face of $\Gamma^+(f^I)$ since this gives the same contradiction as in (I). Thus, by Lemma 3.8(b), Δ satisfies condition (i) of Definition 3.7. We shall prove that Δ is actually a bad face of $\overline{\text{supp}(f)}$. Let us denote by d the minimal value of the restriction of $l_{\mathbf{p}}$ to $\text{supp}(f)$. Since $\text{supp}(f^I) = \text{supp}(f) \cap \mathbb{R}_{\geq 0}^I$, we have $d \leq d_{\mathbf{p}} = 0$. Let H be the hyperplane defined by the equation $\sum_{i=1}^m p_i x_i + q \sum_{i=m+1}^n x_i = 0$, where $q > -d + 1 > 0$. Then, for any $(x_1, \dots, x_n) \in \overline{\text{supp}(f)} \setminus \overline{\text{supp}(f^I)}$, the value of $\sum_{i=1}^m p_i x_i + q \sum_{i=m+1}^n x_i$ is positive. We therefore get $\Delta = \overline{\text{supp}(f^I)} \cap H = \overline{\text{supp}(f)} \cap H$.

If Δ does not satisfy condition (ii)(a) of Definition 3.7, then we have $m = n$ and $p_i \leq 0$ for all $1 \leq i \leq n$. Since, by the hypothesis in the statement of the theorem, f depends effectively on the variable z_1 , the value $d_{\mathbf{p}}$ must be negative, which gives a contradiction with the above original assumption.

We have thus proved that Δ is a bad face of $\overline{\text{supp}(f)}$. Moreover, since $d_{\mathbf{p}} = 0$, we obtain $c = f_{\Delta}^I(\mathbf{a}, \overline{\mathbf{a}}) = f_{\Delta}(\mathbf{A}, \overline{\mathbf{A}}) \in f_{\Delta}(\text{Sing } f_{\Delta})$.

This ends our proof. \square

REMARK 4.1. The equality (19) is the key of the above proof of Theorem 3.5(a). If $c = 0$, then we have two cases in situation (12):

(1) If $d_{\mathbf{p}} = \text{ord}_t(f(\mathbf{z}(t), \overline{\mathbf{z}}(t)))$, then formula (19) might be not true.

(2) If $d_{\mathbf{p}} < \text{ord}_t(f(\mathbf{z}(t), \overline{\mathbf{z}}(t)))$, then we get the same proof of formula (19) as in Proof of (a) (see the remark after formula (18)).

REMARK 4.2. Let $\Sigma^{\infty} := \{c \in \mathbb{C} \mid f^{-1}(c) \cap M(f) \text{ is not bounded}\}$. Under the hypotheses of Theorem 3.5, the above proof also shows that if $c \in \Sigma^{\infty}$ and $c \neq 0$ then c is a critical value of f_{Δ} , for some bad face Δ . Indeed, if the path $\mathbf{z}(t) \subset M(f) \cap f^{-1}(c)$ is not bounded, then it must be included in the singular locus $\text{Sing } f^{-1}(c)$ since the fibre $f^{-1}(c)$ is an algebraic set. (An alternate argument may be extracted from the last part of the proof of Proposition 2.8). This shows the inclusion $\Sigma^{\infty} \subset S(f) \cap f(\text{Sing } f)$. By Theorem 3.5(a) we then have $\Sigma^{\infty} \setminus \{0\} \subset \bigcup_{\Delta \in \mathfrak{B}} f_{\Delta}(\text{Sing } f_{\Delta})$.

4.2. **Proof of Theorem 3.5(b).** By absurd, let us suppose $f(\text{Sing } f)$ is not bounded. Since $\text{Sing } f$ is an algebraic set, by Lemma 2.5, there exists a real analytic path $\mathbf{z}(t) \subset \text{Sing } f$ defined on a small enough interval $]0, \varepsilon[$ such that:

$$\lim_{t \rightarrow 0} \|\mathbf{z}(t)\| = \infty, \text{ and } \lim_{t \rightarrow 0} |f(\mathbf{z}(t), \overline{\mathbf{z}}(t))| = \infty$$

We use the same notations as in the proof of (a). Since $\mathbf{z}(t) \subset \text{Sing } f$, we have $\lambda(t) \equiv 0$ and we therefore obtain (19) directly (see the remark after (19)). From $\lim_{t \rightarrow 0} |f(\mathbf{z}(t), \overline{\mathbf{z}}(t))| = \infty$ it follows that $d_{\mathbf{p}} \leq \text{ord}_t(f(\mathbf{z}(t), \overline{\mathbf{z}}(t))) < 0$. We are in the situation of (I) from the proof of Theorem 3.5(a) but without being able to insure the equality $f_{\Delta}(\mathbf{A}, \overline{\mathbf{A}}) = 0$. We therefore need here the Newton strong non-degeneracy to get a contradiction.

To prove that $f_{\Delta}(\text{Sing } f_{\Delta})$ is bounded, for any bad face $\Delta \subset \overline{\text{supp}(f)}$, we use Remark 3.10 and we replace f by f_{Δ} in the above proof.

Since $\overline{\text{supp}(f)}$ has finitely many faces and since, by Theorem 3.5(a), we have the inclusion $S(f) \subset \{0\} \cup \bigcup_{\Delta \in \mathfrak{B}} f_{\Delta}(\text{Sing} f_{\Delta})$, it follows that $S(f)$ is bounded. \square

4.3. Consequences and examples. We get some sharper statements for significant particular classes of non-degenerate mixed polynomials. The following result extends the one for holomorphic polynomials proved in [Ku].

Corollary 4.3. *If f is a mixed Newton non-degenerate and convenient polynomial, then $S(f) = \emptyset$.*

Proof. Under the same notations and definitions as in the proof of Theorem 3.5(a), since $l_{\mathbf{p}}(x) = \sum_{i=1}^m p_i x_i$ has at least a coefficient $p_j < 0$ for some j and the intersection of $\text{supp}(f)$ with each positive coordinate axis is non-empty, the value of $l_{\mathbf{p}}(x)$ at a point of the intersection of $\text{supp}(f)$ with the j -axis is negative. This implies that the minimal value $d_{\mathbf{p}}$ is negative. By Lemma 3.8(a), Δ is a face of $\Gamma^+(f)$.

On the other hand, we have $d_{\mathbf{p}} < \text{ord}_t(f(\mathbf{z}(t), \bar{\mathbf{z}}(t))) = 0$. Applying Remark 4.1, we get formula (19) and a singularity $\mathbf{A} \in \mathbb{C}^{*n}$ of f_{Δ} , which contradicts the Newton non-degeneracy of f . \square

Definition 4.4. A mixed polynomial f is called *(radial) weighted-homogeneous* if there exist positive integers q_1, \dots, q_n with $\gcd(q_1, \dots, q_n) = 1$ and a positive integer m such that $\sum_{j=1}^n q_j(\nu_j + \mu_j) = m$, or, equivalently, such that $f(t \circ \mathbf{z}) = t^m f(\mathbf{z}, \bar{\mathbf{z}})$ for any $t \in \mathbb{R}^*$, where $t \circ \mathbf{z} := (t^{q_1} z_1, \dots, t^{q_n} z_n)$.

Corollary 4.5. *Let f be a mixed polynomial, weighted-homogeneous and Newton strongly non-degenerate. Then:*

- (a) $\text{Sing} f \cap \mathbb{C}^{*n} = \emptyset$,
- (b) $S(f) \cup f(\text{Sing} f) \subset \{0\}$.

Proof. Since f is weighted-homogeneous, let's say of degree m , we have $f(0) = 0$ and $\overline{\text{supp}(f)}$ is contained in a single hyperplane which does not pass through the origin. Therefore the Newton boundary $\Gamma^+(f)$ has a single maximal face and its non-degeneracy implies $\text{Sing} f \cap \mathbb{C}^{*n} = \emptyset$. Since $\overline{\text{supp}(f)}$ has no bad face and since by Theorem 3.5(a) we have $S(f) \subset \{0\} \cup \bigcup_{\Delta \in \mathfrak{B}} f_{\Delta}(\text{Sing} f_{\Delta})$, it follows that $S(f) \subset \{0\}$.

By absurd, let us suppose that $c \in f(\text{Sing} f) \cap \mathbb{C}^*$. For any $\mathbf{z} \in \text{Sing} f$ such that $f(\mathbf{z}, \bar{\mathbf{z}}) = c$, there exists $\lambda \in S_1^1$ such that $\bar{d}f(\mathbf{z}, \bar{\mathbf{z}}) = \lambda \bar{d}f(\mathbf{z}, \bar{\mathbf{z}})$. Multiplying by t^{m-q_i} the equalities $\frac{\partial f}{\partial z_i}(\mathbf{z}, \bar{\mathbf{z}}) = \lambda \frac{\partial f}{\partial z_i}(\mathbf{z}, \bar{\mathbf{z}})$ for $i = 1, 2, \dots, n$, and using that f is weighted-homogeneous, we get that $\bar{d}f(t \circ \mathbf{z}, t \circ \bar{\mathbf{z}}) = \lambda \bar{d}f(t \circ \mathbf{z}, t \circ \bar{\mathbf{z}})$. This implies that $t \circ \mathbf{z} \in \text{Sing} f$ and $t^m c \in f(\text{Sing} f)$, therefore $f(\text{Sing} f)$ is not bounded, which contradicts Theorem 3.5(b). This proves that $f(\text{Sing} f) \subset \{0\}$. \square

EXAMPLE 4.6. $f : \mathbb{C}^2 \rightarrow \mathbb{C}$, $f = z_1 z_2 + \bar{z}_1^2 \bar{z}_2^2$. This is a Newton strongly non-degenerate mixed polynomial, where $\Gamma^+(f) = (2, 2)$ and $\text{supp}(f)$ consists of just a bad face Δ . We have $df = (z_2, z_1)$ and $\bar{d}f = (2\bar{z}_1 \bar{z}_2^2, 2\bar{z}_1^2 \bar{z}_2)$. For any $\lambda \in \mathbb{C}$, $|\lambda| = 1$, the solution of $\bar{d}f(\mathbf{z}, \bar{\mathbf{z}}) = \lambda \bar{d}f(\mathbf{z}, \bar{\mathbf{z}})$ is $\{z_1 z_2 = \frac{1}{2\lambda}\}$ or $\{z_1 = z_2 = 0\}$. Thus $f(\text{Sing} f_{\Delta}) = f(\text{Sing} f) =$

$\{0\} \cup \{\frac{1}{2\lambda} + \frac{1}{4\lambda^2} \mid \lambda \in S_1^1\}$. By taking $z_1 z_2 = \frac{1}{2\lambda}$ with $z_1 \rightarrow 0$, hence $z_2 \rightarrow \infty$, we get $f(\text{Sing } f) \setminus \{0\} \subset S(f)$ and $\{0\} \notin S(f)$.

On the other hand, for any $\mathbf{z} \in M(f) \setminus \text{Sing } f$, by a straightforward computation, we obtain $\|z_1\| = \|z_2\|$. For $\{\mathbf{z}^k\}_{k \in \mathbb{N}} \subset M(f) \setminus \text{Sing } f$ such that $\lim_{k \rightarrow \infty} \|\mathbf{z}^k\| = \infty$, by using the inequality $|f(z_1, z_2)| \geq |||z_1 z_2|^2 - \|z_1 z_2|||$, we get $|f(\mathbf{z}^k)| \rightarrow \infty$. Finally, using Theorem 3.5(b), this shows that $S(f) \setminus f(\text{Sing}(f)) = \emptyset$.

Hence $S(f) = \{\frac{1}{2\lambda} + \frac{1}{4\lambda^2} \mid \lambda \in S_1^1\}$, which is diffeomorphic to a circle. This also shows that the inclusion of Theorem 3.5(a) may be strict.

EXAMPLE 4.7. The polynomial $f : \mathbb{C}^2 \rightarrow \mathbb{C}$, $f = z_1 + z_2 + \bar{z}_1^2 + \bar{z}_2^2$, is Newton strongly non-degenerate and convenient. By direct computation of $M(f)$ we obtain that $S(f) = \emptyset$, as predicted by Corollary 4.3, and $f(\text{Sing } f) = \{a + \frac{1}{2}\bar{a}^2 \mid a \in S^1\}$, which is diffeomorphic to a circle and agrees with Theorem 3.5(b).

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