STRICT POSITIVSTELLENSÄTZE FOR MATRIX POLYNOMIALS WITH SCALAR CONSTRAINTS

J. CIMPRIČ

ABSTRACT. We extend Krivine's strict positivstellensatz for usual (real multivariate) polynomials to symmetric matrix polynomials with scalar constraints. The proof is an elementary computation with Schur complements. Analogous extensions of Schmüdgen's and Putinar's strict positivstellensatz were recently proved by Hol and Scherer using methods from optimization theory.

1. Introduction

Let $S = \{g_1, \ldots, g_m\}$ be a finite subset of the algebra $\mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \ldots, X_d]$. Write

$$K_S = \{x \in \mathbb{R}^d \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$$

and

$$M_S = \{c_0 + \sum_{i=1}^{m} c_i g_i \mid c_0, \ldots, c_m \in \sum \mathbb{R}[\underline{X}]^2\}.$$

Write also $\widehat{S} = \{g_1^{\alpha_1} \cdots g_m^{\alpha_m} \mid \alpha_1, \ldots, \alpha_m \in \{0,1\}\}$ and $T_S = M_{\widehat{S}}$.

The following theorem summarizes the strict positivstellensätze of Krivine [9], [16], Schmüdgen [14], [1] and Putinar [12], [6] (respectively, (1) $\Leftrightarrow$ (2), (1) $\Leftrightarrow$ (2') and (1) $\Leftrightarrow$ (2")). A nice overview is [10].

Theorem 1. *Notation as above. For every $f \in \mathbb{R}[\underline{X}]$ the following are equivalent:*

(1) *$f(x) > 0$ for every $x \in K_S$,*
(2) *there exist $t, u \in T_S$ such that $(1+t)f = 1 + u$.*

If K_S is compact then (1) and (2) are equivalent to

(2') *there exists an $\varepsilon > 0$ such that $f - \varepsilon \in T_S$.*

If one of the sets $K_{\{g_i\}} = \{x \in \mathbb{R}^d \mid g_i(x) \geq 0\}$, $i = 1, \ldots, m$, is compact, then (1), (2) and (2') are equivalent to

Date: submitted August 1st 2010, revised October 31st 2010.
2010 *Mathematics Subject Classification.* 14P, 13J30, 47A56.
Key words and phrases. positive polynomials, matrix polynomials, real algebraic geometry.

(2") *there exists an* $\varepsilon > 0$ *such that* $f - \varepsilon \in M_S$.

Usually one has $tf = 1 + u$ in (2). Our version is then a consequence of $(1+t+u)f = tf + (1+u)f = 1 + u + tf^2$, see [10, p. 26].

Let $M_n(\mathbb{R}[\underline{X}])$ be the algebra of all $n \times n$ matrices with entries from the algebra $\mathbb{R}[\underline{X}]$. Write $\sum M_n(\mathbb{R}[\underline{X}])^2$ for the set of all finite sums of elements of the form $A^T A$ where $A \in M_n(\mathbb{R}[\underline{X}])$. For S as above write

$$M_S^n = \{C_0 + \sum_{i=1}^m C_i g_i \mid C_0, \ldots, C_m \in \sum M_n(\mathbb{R}[\underline{X}])^2\}$$

and $T_S^n = M_{\hat{S}}^n$. Clearly M_S^n is a quadratic module (i.e. M_S^n contains the identity matrix I_n, $M_S^n + M_S^n \subseteq M_S^n$ and $A^T M_S^n A \subseteq M_S^n$ for every $A \in M_n(\mathbb{R}[\underline{X}])$). The quadratic module T_S^n also satisfies $T_S \cdot T_S^n \subseteq T_S^n$.

The aim of this note is to prove the equivalence (1) $\Leftrightarrow$ (2) in the following theorem. The equivalences (1) $\Leftrightarrow$ (2') and (1) $\Leftrightarrow$ (2") just rephrase the Hol-Scherer theorem [13, Theorem 2] and are stated here for the sake of completeness.

Theorem 2. *Notation as above. For every element* $F \in M_n(\mathbb{R}[\underline{X}])$ *such that* $F^T = F$*, the following are equivalent:*

(1) $F(x)$ *is strictly positive definite for every* $x \in K_S$,

(2) *there exist* $t \in T_S$ *and* $V \in T_S^n$ *such that* $(1+t)F = I_n + V$.

If K_S *is compact then (1) and (2) are equivalent to*

(2') *there exists an* $\varepsilon > 0$ *such that* $F - \varepsilon I_n \in T_S^n$.

If one of the sets $K_{\{g_i\}} = \{x \in \mathbb{R}^d \mid g_i(x) \geq 0\}$, $i = 1, \ldots, m$, *is compact, then (1), (2) and (2') are equivalent to*

(2") *there exists an* $\varepsilon > 0$ *such that* $F - \varepsilon I_n \in M_S^n$.

2. The proof

We will need the following technical lemma:

Lemma 3. *For every* $B \in M_n(\mathbb{R}[\underline{X}])$ *there exists* $c \in \sum \mathbb{R}[\underline{X}]^2$ *such that* $cI_n - B^T B \in \sum M_n(\mathbb{R}[\underline{X}])^2$.

Proof. We can take c to be of the form kp^l where $p = 1 + \sum_{i=1}^d X_i^2$ and k, l are positive integers. Namely, let L be the set of all $B \in M_n(\mathbb{R}[\underline{X}])$ such that $kp^l I_n - B^T B \in \sum M_n(\mathbb{R}[\underline{X}])^2$ for some positive integers k and l. Clearly, L contains $X_1, \ldots, X_d$ and all constant matrices. To prove that $L = M_n(\mathbb{R}[\underline{X}])$ it suffices to show that L is closed for addition and multiplication. Suppose that $B_1, B_2 \in L$. There exist positive integers k_1, k_2, l_1, l_2 such that

$$k_1 p^{l_1} I_n - B_1^T B_1 \in \sum M_n(\mathbb{R}[\underline{X}])^2, \quad k_2 p^{l_2} I_n - B_2^T B_2 \in \sum M_n(\mathbb{R}[\underline{X}])^2.$$

The paralelogram identity implies that

$$2(k_1+k_2)p^{\max(l_1,l_2)}I_n-(B_1+B_2)^T(B_1+B_2)\in\sum M_n(\mathbb{R}[\underline{X}])^2$$

and by inserting terms $\pm k_2p^{l_2}B_1^TB_1$ we see that

$$k_1k_2p^{l_1+l_2}I_n-(B_1B_2)^T(B_1B_2)^T\in\sum M_n(\mathbb{R}[\underline{X}])^2.$$

□

We can now return to the proof of Theorem 2.

Proof. Clearly, (2") ⇒ (2') ⇒ (2) ⇒ (1) (with no assumptions on K_S).

The implications (2') ⇒ (1) when K_S is compact and (2") ⇒ (1) when one of $K_{\{g_i\}}$ is compact follow from Theorem 1 and Hol-Scherer theorem [13, Theorem 2].

We will now prove that (2) implies (1) (with no assumptions on K_S) by induction on n. The case $n=1$ is covered by Theorem 1. Suppose that (1) implies (2) for all symmetric matrix polynomials of size $n-1$ and pick a symmetric polynomial $F(x)$ of size n which satisfies (1). We write

$$F=\begin{bmatrix} f_{11} & g \\ g^T & H\end{bmatrix}$$

and observe that (in $M_n(\mathbb{R}(X))$)

$$\begin{bmatrix} 1 & -\frac{1}{f_{11}}g \\ 0 & I_{n-1}\end{bmatrix}^T\begin{bmatrix} f_{11} & g \\ g^T & H\end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{f_{11}}g \\ 0 & I_{n-1}\end{bmatrix}=\begin{bmatrix} f_{11} & 0 \\ 0 & \tilde{H}\end{bmatrix} \tag{1}$$

where $\tilde{H}=H-\frac{1}{f_{11}}g^Tg$ is the Schur complement of f_{11}. Since F is stricly positive definite on K_S, it follows that f_{11} is stricly positive on K_S, hence $\begin{bmatrix} 1 & -\frac{1}{f_{11}}g \\ 0 & I_{n-1}\end{bmatrix}$ is defined and invertible on K_S. It follows that $\tilde{H}$ is defined and strictly positive definite on K_S. Clearly, $f_{11}\tilde{H}$ is a matrix polynomial that is stricly positive definite on K_S. By the induction hypothesis there exist $s\in T$ and $U\in T_S^{n-1}$ such that

$$(1+s)f_{11}\tilde{H}=I_{n-1}+U. \tag{2}$$

On the other hand, there exists by $n=1$ elements $s_1,u_1\in T$ such that

$$(1+s_1)f_{11}=1+u_1. \tag{3}$$

Rearrange equation (1) and multiply it by f_{11}^3 to get (with $I=I_{n-1}$)

$$f_{11}^3\begin{bmatrix} f_{11} & g \\ g^T & H\end{bmatrix}=\begin{bmatrix} f_{11} & g \\ 0 & f_{11}I\end{bmatrix}^T\begin{bmatrix} f_{11}^2 & 0 \\ 0 & f_{11}\tilde{H}\end{bmatrix}\begin{bmatrix} f_{11} & g \\ 0 & f_{11}I\end{bmatrix} \tag{4}$$

Multiplying equation (4) by $(1+s)(1+s_1)^4$ and using equations (2) and (3), we get:

$$(1+s)(1+s_1)(1+u_1)^3 \begin{bmatrix} f_{11} & g \\ g^T & H \end{bmatrix} = \begin{bmatrix} 1+u_1 & (1+s_1)g \\ 0 & (1+u_1)I \end{bmatrix}^T \cdot$$
$$\cdot \begin{bmatrix} (1+s)(1+u_1)^2 & 0 \\ 0 & (1+s_1)^2(I+U) \end{bmatrix} \begin{bmatrix} 1+u_1 & (1+s_1)g \\ 0 & (1+u_1)I \end{bmatrix}$$

Since

$$\begin{bmatrix} (1+s)(1+u_1)^2 & 0 \\ 0 & (1+s_1)^2(I+U) \end{bmatrix} = I_n + W$$

for some $W \in T_S^n$, it follows that

$$(1+s)(1+s_1)(1+u_1)^3 \begin{bmatrix} f_{11} & g \\ g^T & H \end{bmatrix} =$$
$$= \begin{bmatrix} 1+u_1 & (1+s_1)g \\ 0 & (1+u_1)I \end{bmatrix}^T \begin{bmatrix} 1+u_1 & (1+s_1)g \\ 0 & (1+u_1)I \end{bmatrix} + W'$$

for some $W' \in T_S^n$. Write $\tilde{g} = (1+s_1)g$. By Lemma 3 there exists an element $c \in \sum \mathbb{R}[\underline{X}]^2$ such that

$$cI_{n-1} - \tilde{g}^T\tilde{g} =: \sigma \in \sum M_{n-1}(\mathbb{R}[\underline{X}])^2. \tag{5}$$

Write $v = 1+c$, $W'' = v(1+v)W'$ and note that

$$v(1+v)(1+s)(1+s_1)(1+u_1)^3 \begin{bmatrix} f_{11} & g \\ g^T & H \end{bmatrix} =$$
$$= v(1+v) \begin{bmatrix} (1+u_1)^2 & (1+u_1)\tilde{g} \\ (1+u_1)\tilde{g}^T & \tilde{g}^T\tilde{g} + (1+u_1)^2 I \end{bmatrix} + W'' =$$
$$= \begin{bmatrix} v(1+u_1)^2 & 0 \\ 0 & (v(1+u_1)^2 + v^2(2u_1+u_1^2)+1)I + (v+1)\sigma \end{bmatrix} +$$
$$+ \begin{bmatrix} v(1+u_1) & (1+v)\tilde{g} \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} v(1+u_1) & (1+v)\tilde{g} \\ 0 & 0 \end{bmatrix} + W''$$

which clearly belongs to $I_n + T_S^n$. It is also clear that $v(1+v)(1+s)(1+s_1)(1+u_1)^3$ belongs to $1 + T_S$. □

3. Open problems

(1) Extend Theorem 2 to the case of matrix constraints.
This problem is suggested in [8]. They extend Hol-Scherer theorem this way.

(2) Extend Krivine's nichtnegativstellensatz ($f \geq 0$ on K_S iff $ft = f^{2k} + u$ for some $t, u \in T_S$ and $k \in \mathbb{N}$) to matrix polynomials.

A possible approach is given in Section 4.2 of [15]. The matrix version of the Hilbert's 17th problem (i.e. the case $S = \emptyset$) was proved independently in [5] and [11]. For a constructive proof see Proposition 10 in [15].

(3) Suppose that for some S every $f \in \mathbb{R}[\underline{X}]$ such that $f \geq 0$ on K_S belongs to T_S. Does it follow that every symmetric F in $M_n(\mathbb{R}[\underline{X}])$ which is positive semidefinite on K_S belongs to T_S^n?

This is true in the following one-dimensional cases: $S = \emptyset$ by [7], [3] or [4] and $S = \{X\}$, $S = \{X, 1 - X\}$ by [2].

References

[1] R. Berr, T. Wörmann, Positive polynomials on compact sets. Manuscripta Math. **104** (2001), no. 2, 135–143.

[2] H. Dette, W. J. Studden, Matrix measures, moment spaces and Favard's theorem for the interval $[0, 1]$ and $[0, \infty)$. Linear Algebra Appl. **345** (2002), 169–193.

[3] D. Ž. Djoković, Hermitian matrices over polynomial rings. J. Algebra **43** (1976), no. 2, 359–374.

[4] I. Gohberg, P. Lancaster, L. Rodman, Matrix polynomials. Computer Science and Applied Mathematics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1982. xiv+409 pp. ISBN: 0-12-287160-X

[5] D. Gondard, P. Ribenboim, Le 17e problème de Hilbert pour les matrices. Bull. Sci. Math. (2) **98** (1974), no. 1, 49–56.

[6] T. Jacobi, A representation theorem for certain partially ordered commutative rings. Math. Z. **237** (2001), no. 2, 259–273.

[7] V. A. Jakubovič, Factorization of symmetric matrix polynomials. Dokl. Akad. Nauk SSSR **194** (1970), 532–535.

[8] I. Klep, M. Schweighofer, Pure states, positive matrix polynomials and sums of hermitian squares, arXiv:0907.2260

[9] J.-L. Krivine, Anneaux préordonnés. J. Analyse Math. **12** (1964), 307–326.

[10] M. Marshall, Positive polynomials and sums of squares. Mathematical Surveys and Monographs, 146. American Mathematical Society, Providence, RI, 2008. xii+187 pp. ISBN: 978-0-8218-4402-1; 0-8218-4402-4

[11] C. Procesi, M. Schacher, A non-commutative real Nullstellensatz and Hilbert's 17th problem. Ann. Math. **104** (1976), 395-406.

[12] M. Putinar, Positive polynomials on compact semi-algebraic sets. Indiana Univ. Math. J. **42** (1993), no. 3, 969–984.

[13] C. W. Scherer, C. W. J. Hol, Matrix sum-of-squares relaxations for robust semi-definite programs. Math. Program. **107** (2006), no. 1-2, Ser. B, 189–211.

[14] K. Schmüdgen, The K-moment problem for compact semi-algebraic sets. Math. Ann. **289** (1991), no. 2, 203–206.

[15] K. Schmüdgen, Noncommutative real algebraic geometry—some basic concepts and first ideas. Emerging applications of algebraic geometry, 325–350, IMA Vol. Math. Appl., 149, Springer, New York, 2009.

[16] G. Stengle, A Nullstellensatz and a Positivstellensatz in semialgebraic geometry. Math. Ann. **207** (1974), 87–97.

Jaka Cimprič, University of Ljubljana, Faculty of Math. and Phys., Dept. of Math., Jadranska 19, SI-1000 Ljubljana, Slovenija. E-mail: cimpric@fmf.uni-lj.si. www page: http://www.fmf.uni-lj.si/~cimpric.