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ARCHIMEDEAN OPERATOR-THEORETIC POSITIVSTELLENSÄTZE

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Abstract. We prove a general archimedean positivstellensatz for hermitian operator-valued polynomials and show that it implies the multivariate Fejer-Riesz Theorem of Dritschel-Rovnyak and Positivstellensätze of Ambrozie-Vasilescu, Scherer-Hol and Klep-Schweighofer. We also obtain several generalizations of these and related results. The proof of the main result depends on an extension of the abstract archimedean positivstellensatz for $*$-algebras that is interesting in its own right.

1. Introduction

We fix $d \in \mathbb{N}$ and write $\mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_d]$. In real algebraic geometry, a *positivstellensatz* is a theorem which for given polynomials $p_1, \ldots, p_m \in \mathbb{R}[x]$ characterizes all polynomials $p \in \mathbb{R}[x]$ which satisfy $p_1(a) \geq 0, \ldots, p_m(a) \geq 0 \Rightarrow p(a) > 0$ for every point $a \in \mathbb{R}^d$. A nice survey of them is [12]. The name *archimedean positivstellensatz* is reserved for the following result of Putinar [14] and Jacobi [8]:

Theorem A. Let $S = \{p_1, \ldots, p_m\}$ be a finite subset of $\mathbb{R}[x]$. Write $M_S := \{c_0 + \sum_{i=1}^m c_i p_i \mid c_0, \ldots, c_m$ are sums of squares of polynomials from $\mathbb{R}[x]\}$. If the set M_S is archimedean (i.e. if for every $f \in \mathbb{R}[x]$ there is $l \in \mathbb{N}$ such that $l \pm f \in M_S$, or equivalently, if the set M_S contains an element g such that the set $\{x \in \mathbb{R}^d \mid g(x) \geq 0\}$ is compact), then for every $p \in \mathbb{R}[x]$ the following are equivalent:

(1) $p(x) > 0$ on $K_S := \{x \in \mathbb{R}^d \mid p_1(x) \geq 0, \ldots, p_m(x) \geq 0\}$.
(2) There exists an $\epsilon > 0$ such that $p - \epsilon \in M_S$.

An important corollary of Theorem A is the following theorem of Putinar and Vasilescu [15, Corollary 4.4]. The case $S = \emptyset$ was first done by Reznick [16].

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Theorem B. Notation as in Theorem A. If $p_1, \ldots, p_m$ and p are homogeneous of even degree and if $p(x) > 0$ for every nonzero $x \in K_S$, then there exists $\theta \in \mathbb{N}$ such that $(x_1^2 + \ldots + x_d^2)^\theta p \in M_S$.

Another important corollary of Theorem A is the following multivariate Fejer-Riesz theorem.

Theorem C. Every element of $\mathbb{R}[\cos\phi_1, \sin\phi_1, \ldots, \cos\phi_d, \sin\phi_d]$ which is strictly positive for every $\phi_1, \ldots, \phi_d$ is equal to a sum of squares of elements from $\mathbb{R}[\cos\phi_1, \sin\phi_1, \ldots, \cos\phi_d, \sin\phi_d]$.

We get Theorem C from Theorem A for $S = \{1 - x_1^2 - y_1^2, x_1^2 + y_1^2 - 1, \ldots, 1 - x_d^2 - y_d^2, x_d^2 + y_d^2 - 1\}$, a subset of $\mathbb{R}[x_1, y_1, \ldots, x_d, y_d]$. Note that it implies neither the classical univariate Fejer-Riesz theorem nor its multivariate extension from [13] which both work for nonnegative trigonometric polynomials.

We are interested in generalizations of Theorems A, B and C to hermitian operator-valued polynomials, i.e. elements of $\mathbb{R}[x] \otimes A_h$ where A is some operator algebra with involution. Such results are of interest in control theory. They fit into the emerging field of noncommutative real algebraic geometry, see [19].

The first result in this direction was the following generalization of Theorem B which was proved by Ambrozie and Vasilescu in [1], see the last part of their Theorem 8. We say that an element a of a C^*-algebra A is *nonnegative* (i.e. $a \geq 0$) if $a = b^*b$ for some $b \in A$ and that it is *strictly positive* (i.e. $a > 0$) if $a - \epsilon \geq 0$ for some real $\epsilon > 0$.

Theorem D. Let A be a C^*-algebra and let $p \in \mathbb{R}[x] \otimes A_h$ and $p_k \in \mathbb{R}[x] \otimes M_{\nu_k}(\mathbb{C})_h$, $k = 1, \ldots, m$, $\nu_k \in \mathbb{N}$, be homogeneous polynomials of even degree. Assume that $K_0 := \{t \in S^{d-1} \mid p_1(t) \geq 0, \ldots, p_m(t) \geq 0\}$ is nonempty and $p(t) > 0$ for all $t \in K_0$. Then there are homogeneous polynomials $q_j \in \mathbb{R}[x] \otimes A$, $q_{jk} \in \mathbb{R}[x] \otimes M_{\nu_k \times 1}(A)$, $j \in J$, J finite, and an integer θ such that

$$(x_1^2 + \ldots + x_d^2)^\theta p = \sum_{j \in J} \big(q_j^* q_j + \sum_{k=1}^m q_{jk}^* p_k q_{jk}\big).$$

It is clear from the proof of Theorem D (combine Theorem 3 and Lemma 5 in [1]) that authors were aware of the following generalization of Theorem A.

Theorem E. Let A be a C^*-algebra, p an element of $\mathbb{R}[x] \otimes A_h$ and $S = \{p_1, \ldots, p_k\}$ a finite subset of $\mathbb{R}[x]$ such that the set M_S is archimedean. If the set K_S is nonempty and $p(t) > 0$ for every $t \in K_S$, then there exist polynomials $q_j, q_{jk} \in \mathbb{R}[x] \otimes A$, $j \in J$, J finite, such that

$$p = \sum_{j \in J} \big(q_j^* q_j + \sum_{k=1}^m q_{jk}^* p_k q_{jk} \big).$$

Theorem E was explicitly stated for the first time by Scherer and Hol in [17], see their Theorem 2. Because of their techniques they had to assume that A is finite-dimensional.

An interesting special case of Theorem E is the following generalization of Theorem C which was proved (for $A = B(H)$) by Dritschel and Rovnyak in [6], see their Theorem 5.1.

Theorem F. Let A be a C^*-algebra. If an element

$$p \in \mathbb{R}[\cos\phi_1, \sin\phi_1, \ldots, \cos\phi_n, \sin\phi_n] \otimes A_h$$

is strictly positive for every $\phi_1, \ldots, \phi_n$ then $p = \sum_{j \in J} q_j^* q_j$ for some finite J and $q_j \in \mathbb{R}[\cos\phi_1, \sin\phi_1, \ldots, \cos\phi_n, \sin\phi_n] \otimes A$.

Our interest in this subject stems from the following theorem of Klep and Schweighofer, see Theorem 13 in [10], which generalizes the finite-dimensional version of Theorem E (from scalar to matrix constraints) but does not seem to follow from the proof of Theorem D.

Theorem G. For a finite subset $S = \{p_1, \ldots, p_m\}$ of $M_\nu(\mathbb{R}[x])_h$, $\nu \in \mathbb{N}$, write $K_S := \{t \in \mathbb{R}^d \mid p_1(t) \geq 0, \ldots, p_m(t) \geq 0\}$ and $M_S := \{\sum_{j \in J} \big(q_j^* q_j + \sum_{k=1}^m q_{jk}^* p_k q_{jk} \big) \mid q_j, q_{jk} \in M_\nu(\mathbb{R}[x]), j \in J, J \text{ finite}\}$. If the set $M_S \cap \mathbb{R}[x]$ is archimedean (in the sense of Theorem A) then for every $p \in M_\nu(\mathbb{R}[x])_h$ such that $p(t) > 0$ on K_S we have that $p \in M$.

The aim of this paper is to prove the following very general operator-theoretic Positivstellensatz and show that it implies generalizations of Theorems D, E, F and G. (Theorems D, E and F will be extended from C^*-algebras to algebraically bounded $*$-algebras and Theorems E and G will be extended from $\mathbb{R}[x]$ to any commutative real algebra. Theorem G will also be extended from matrices to more general operators.)

Theorem H. Let R be a commutative real algebra, A a real or complex $*$-algebra and M an archimedean quadratic module in $R \otimes A$ (definitions are in section 2). Then for every $p \in R \otimes A_h$ the following are equivalent:

(1) $p \in \epsilon + M$ for some real $\epsilon > 0$.
(2) For every multiplicative state ϕ on R, there exists real $\epsilon_\phi > 0$ such that $(\phi \otimes \mathrm{id}_A)(p) \in \epsilon_\phi + (\phi \otimes \mathrm{id}_A)(M)$.

One of the main differences between the operator case and the scalar case is that in the operator case an element of A_h that is not ≤ 0 is not necessarily > 0. We would like to give an algebraic characterization of operator-valued polynomials that are not ≤ 0 in every point from a given set. Every theorem of this type is called a *nichtnegativsemidefinitheitsstellensatz*. We will prove variants of Theorems E and G that fit into this context.

Finally, we use our results and the main theorem from [9] to get a generalization of the existence result for operator-valued moment problems from [1] to algebraically bounded $*$-algebras.

2. Factorizable states

Associative unital algebras with involution will be called $*$-algebras for short. Let B be a real or complex $*$-algebra. Write $Z(B)$ for the center of B and write $B_h = \{b \in B | b^* = b\}$ for its set of hermitian elements. Note that the set B_h is a real vector space; we assume that it is equipped with the *finest locally convex topology*, i.e. every convex absorbing set in B_h is a neighbourhood of zero.

Clearly, every linear functional on B_h is continuous with respect to the finest locally convex topology. In other words, the algebraic and the topological dual of B_h are the same; we will write $(B_h)'$ for both. We assume that $(B_h)'$ is equipped with the weak*-topology, i.e. topology of pointwise convergence. We say that $\omega \in (B_h)'$ is *factorizable* if $\omega(xy) = \omega(x)\omega(y)$ for every $x \in B_h$ and $y \in Z(B)_h$. Clearly, the set of all factorizable linear functionals on B_h is closed in the weak*-topology.

We say that a subset M of B_h is a *quadratic module* if $1 \in M$, $M + M \subseteq M$ and $b^*Mb \subseteq M$ for every $b \in B$. The smallest quadratic module in B is the set $\Sigma^2(B)$ which consists of all finite sums of elements b^*b with $b \in B$. The largest quadratic module in B is the set B_h. A quadratic module is *proper* if it is different from B_h (or equivalently, if $-1 \notin M$.) We say that an element $b \in B_h$ is *bounded* w.r.t. a quadratic module M if there exists a number $l \in \mathbb{N}$ such that $l \pm b \in M$. A quadratic module M is *archimedean* if every element $b \in B_h$ is bounded w.r.t. M (or equivalently, if 1 is an interior point of M.)

For every subset M of B_h write $M^\vee$ for the set of all $f \in (B_h)'$ which satisfy $f(1) = 1$ and $f(M) \geq 0$. The set of all extreme points of $M^\vee$ will be denoted by $\operatorname{ex} M^\vee$. Elements of $M^\vee$ will be called *M-states* and elements of $\operatorname{ex} M^\vee$ *extreme M-states*. A $\Sigma^2(B)$-state is simply called a *state*. Suppose now that M is an archimedean quadratic module. Note that $M^\vee$ is non-empty iff M is proper. Applying the Banach-Alaoglu Theorem to $V = (M - 1) \cap (1 - M)$ which is a neighbourhood of zero, we see that $M^\vee$ is compact. The Krein-Milman theorem then implies, that $M^\vee$ is equal to the closure of the convex hull of the set $\operatorname{ex} M^\vee$.

Recall that a (bounded) $*$-*representation* of B is a homomorphism of unital $*$-algebras from B to the algebra of all bounded operators on some Hilbert space H_π. We say that a $*$-representation π of B is M-positive for a given subset M of B_h if $\pi(m)$ is positive semidefinite for every $m \in M$. For every such π and every $v \in H_\pi$ of norm 1, $\omega_{\pi,v}(x) := \langle \pi(x)v, v\rangle$ belongs to $M^\vee$. Conversely, if M is a quadratic module, then every $\omega \in M^\vee$ is of this form by the GNS construction.

The equivalence of (1)-(4) in the following result is sometimes referred to as archimedean positivstellensatz for $*$-algebras. It originates from the Vidav-Handelmann theory, cf. [7, Section 1] and [22]. Our aim is to add assertions (5) and (6) to this equivalence.

Proposition 1. *For every archimedean quadratic module M in B and every element $b \in B_h$ the following are equivalent:*

1. *$b \in M^\circ$ (the interior w.r.t. the finest locally convex topology),*
2. *$b \in \epsilon + M$ for some real $\epsilon > 0$,*
3. *$\pi(b)$ is strictly positive definite for every M-positive $*$-representation π of B,*
4. *$f(b) > 0$ for every $f \in M^\vee$,*
5. *$f(b) > 0$ for every $f \in \overline{\operatorname{ex} M^\vee}$,*
6. *$f(b) > 0$ for every factorizable $f \in M^\vee$.*

Proof. (1) implies (2) because the set $M - b$ is absorbing, hence $-1 \in t(M-b)$ for some $t > 0$. Clearly (2) implies (3). (3) implies (4) because the cyclic $*$-representation that belongs to f by the GNS construction clearly has the property that $\pi(m)$ is positive semidefinite for every $m \in M$. (4) implies (1) by the separation theorem for convex sets. The details can be found in [3, Theorem 12] or [19, Proposition 15] or [5, Proposition 1.4].

If (5) is true then, by the compactness of $\overline{\operatorname{ex} M^\vee}$, there exists $\epsilon > 0$ such that $f(b) \geq \epsilon$ for every $f \in \overline{\operatorname{ex} M^\vee}$, hence (4) is true by the Krein-Milman theorem. Clearly, (4) implies (6). By Proposition 3 below and the fact that the set of all factorizable M-states is closed, (6) implies (5). □

Similarly, we have the following:

Proposition 2. *For every archimedean quadratic module M in B and every element $b \in B_h$ the following are equivalent:*

1. *$b \in \overline{M}$ (the closure w.r.t. the finest locally convex topology),*
2. *$b + \epsilon \in M$ for every $\epsilon > 0$,*

(3) $\pi(b)$ *is positive semidefinite for every M-positive $*$-representation π of B,*
(4) $f(b) \geq 0$ *for every* $f \in M^\vee$,
(5) $f(b) \geq 0$ *for every* $f \in \overline{\operatorname{ex} M^\vee}$,
(6) $f(b) \geq 0$ *for every factorizable* $f \in M^\vee$.

The following proposition which extends [21, Ch. IV, Lemma 4.11] was used in the proof of equivalences (4)-(6) in Propositions 1 and 2. Its proof depends on the equivalence of (2) and (3) in Proposition 2.

Proposition 3. *If M is an archimedean quadratic module in B then all extreme M-positive states are factorizable.*

Proof. Pick any $\omega \in \operatorname{ex} M^\vee$ and $y \in Z(B)_h$. We claim that $\omega(xy) = \omega(x)\omega(y)$ for every $x \in B_h$. Since $y = \frac{1}{4}((1+y)^2 - (1-y)^2)$ and $(1 \pm y)^2 \in M$, we may assume that $y \in M$. Since M is archimedean, we may also assume that $1 - y \in M$.

Claim: If $\omega(y) = 0$, then $\omega(y^2) = 0$. (Equivalently, if $\omega(1-y) = 0$, then $\omega((1-y)^2) = 0$.)

Since $1 \pm (1-y) \in M$, it follows that $1-(1-y)^2 = \frac{1}{2}\big(y(2-y)^2 + (2-y)y^2\big) \in M$. Since ω is an M-positive state, it follows that $\omega((1-y)^2) \leq 1$. On the other hand, $\omega((1-y)^2)\omega(1^2) \geq |\,\omega((1-y)\cdot 1)|^2$ by the Cauchy-Schwartz inequality. Now, $\omega(y) = 0$ implies that $\omega((1-y)^2) = 1$, hence $\omega(y^2) = 0$.

Case 1: If $\omega(y) = 0$, then $\omega(xy) = 0$ for every $x \in B_h$. (Equivalently, if $\omega(1-y) = 0$, then $\omega(x(1-y)) = 0$ for every $x \in B_h$.) Namely, by the Cauchy-Schwartz inequality and the Claim, $|\,\omega(xy)|^2 \leq \omega(x^2)\omega(y^2) = 0$. It follows that $\omega(xy) = \omega(x)\omega(y)$ if $\omega(y) = 0$ or $\omega(y) = 1$.

Case 2 : If $0 < \omega(y) < 1$, then

$$\omega_1(x) := \frac{1}{\omega(y)}\,\omega(xy) \quad \text{ and } \quad \omega_2(x) := \frac{1}{\omega(1-y)}\,\omega(x(1-y))$$

are M-positive states on B_h. Namely, for every M-positive $*$-representation π of B and every $x \in M$, we have that $\pi(xy) = \pi(x)\pi(y)$ is a product of two commuting positive semidefinite bounded operators, hence a positive semidefinite bounded operator. By the equivalence of assertions (2) and (3) in Proposition 2, $xy + \epsilon \in M$ for every $\epsilon > 0$. Since ω is M-positive, it follows that $\omega(xy) \geq 0$ as claimed. Similarly, we prove that ω_2 is M-positive. Clearly, $\omega = \omega(y)\omega_1 + \omega(1-y)\omega_2$. Since ω is an extreme point of the set of all M-positive states on B_h, it follows that either $\omega_1 = 0$ or $\omega_2 = 0$. Eitherway, $\omega(xy) = \omega(x)\omega(y)$. $\square$

If we apply Proposition 1 or 2 to $b = 1$, we get the following corollary, parts of which were already mentioned above.

Corollary 4. *For every archimedean quadratic module M in B the following are equivalent:*

(1) $-1 \notin M$,
(2) *there exists an M-positive $*$-representation of B,*
(3) *there exists an M-state on B,*
(4) *there exists an extreme M-state on B,*
(5) *there exists a factorizable M-state on B.*

The following variant of Proposition 1 which follows easily from Corollary 4 was proved in [4, Theorem 5]. We could call it archimedean nichtnegativsemidefinitheitsstellensatz for $*$-algebras.

Proposition 5. *For every archimedean proper quadratic module M on a real or complex $*$-algebra B and for every $x \in B_h$, the following are equivalent:*

(1) *For every M-positive $*$-representation ψ of B, $\psi(x)$ is not negative semidefinite (i.e. $\langle \psi(x)v, v\rangle > 0$ for some $v \in H_\psi$).*
(2) *There exists $k \in \mathbb{N}$ and $c_1, \ldots, c_k \in B$ such that $\sum_{i=1}^k c_i x c_i^* \in 1 + M$.*

3. Theorems H and G

The aim of this section is to prove Theorem H (see Theorem 6) and show that it implies a generalization of Theorem G to compact operators. We also prove a concrete version of Proposition 5.

Theorem 6. *Let R be a commutative real algebra with trivial involution, A a $*$-algebra over $F \in \{\mathbb{R}, \mathbb{C}\}$ and M an archimedean quadratic module in $B := R \otimes A$. For every element p of $B_h = R \otimes A_h$, the following are equivalent:*

(1) *$p \in \epsilon + M$ for some real $\epsilon > 0$.*
(2) *For every multiplicative state ϕ on R, there exists real $\epsilon_\phi > 0$ such that $(\phi \otimes \mathrm{id}_A)(p) \in \epsilon_\phi + (\phi \otimes \mathrm{id}_A)(M)$.*

The following are also equivalent:

(1') *$p + \epsilon \in M$ for every real $\epsilon > 0$.*
(2') *For every multiplicative state ϕ on R and every real $\epsilon > 0$ we have that $(\phi \otimes \mathrm{id}_A)(p) + \epsilon \in (\phi \otimes \mathrm{id}_A)(M)$.*

Moreover, the following are equivalent:

(1") *There exist finitely many $c_i \in B$ such that $\sum_i c_i^* p c_i \in 1 + M$.*

(2”) *For every multiplicative state* ϕ *on* R *there exist finitely many* $d_i \in A$ *such that* $\sum_i d_i^*(\phi \otimes \mathrm{id}_A)(p)d_i \in 1 + (\phi \otimes \mathrm{id}_A)(M)$.

Proof. Clearly (1) implies (2). We will prove the converse in several steps. Note that for every multiplicative state ϕ on R, the mapping $\phi \otimes \mathrm{id}_A \colon B \to A$ is a surjective homomorphism of $*$-algebras, hence $(\phi \otimes \mathrm{id}_A)(M)$ is an archimedean quadratic module in A. Replacing B, M, f and p in Proposition 1 with A, $(\phi \otimes \mathrm{id}_A)(M)$, σ and $(\phi \otimes \mathrm{id}_A)(p)$, we see that (2) is equivalent to

(A) For every multiplicative state ϕ on R and every state σ on A_h such that $\sigma((\phi \otimes \mathrm{id}_A)(M)) \geq 0$ we have that $\sigma((\phi \otimes \mathrm{id}_A)(p)) > 0$.

Note that $(\phi \otimes \sigma)(r \otimes a) = \phi(r)\sigma(a) = \sigma(\phi(r)a) = \sigma((\phi \otimes \mathrm{id}_A)(r \otimes a))$ for every $r \in R$ and $a \in A_h$. It follows that $\phi \otimes \sigma = \sigma \circ (\phi \otimes \mathrm{id}_A)$. Thus, (A) is equivalent to

(B) for every M-positive state on $R \otimes A_h$ of the form $\omega = \phi \otimes \sigma$ where ϕ is multiplicative, we have that $\omega(p) > 0$.

Since $R \otimes 1 \subseteq Z(B)$, every factorizable state ω satisfies $\omega(r \otimes a) = \omega(r \otimes 1)\omega(1 \otimes a)$ and $\omega(rs \otimes 1) = \omega(r \otimes 1)\omega(s \otimes 1)$ for any $r, s \in R$ and $a \in A_h$. Hence $\omega = \phi \otimes \sigma$ where ϕ is a multiplicative state on R and σ is a state on A_h. Therefore, (B) implies that

(C) $\omega(p) > 0$ for every factorizable $\omega \in M^\vee$.

By Proposition 1, (C) is equivalent to (1).

The equivalence of (1’) and (2’) can be proved in a similar way using Proposition 2. It can also be easily deduced from the equivalence of (1) and (2).

Clearly (1”) implies (2”). Conversely, if (1”) is false, then $-1 \notin N$ where $N := \{m - \sum c_i^* p c_i \mid m \in M, c_i \in B\}$ is the smallest quadratic module in B which contains M and $-p$. By Corollary 4, there exists a factorizable state $\omega \in N^\vee$. From the proof of (1) $\Leftrightarrow$ (2), we know that $\omega = \phi \otimes \sigma = \sigma \circ (\phi \otimes \mathrm{id}_A)$ for a multiplicative state ϕ on R and a state σ on A. Since $\sigma((\phi \otimes \mathrm{id}_A)(N)) = \omega(N) \geq 0$, it follows by Corollary 4 that $-1 \notin (\phi \otimes \mathrm{id}_A)(N)$. Since $(\phi \otimes \mathrm{id}_A)(N) = \{(\phi \otimes \mathrm{id}_A)(m) - \sum_j d_j^*(\phi \otimes \mathrm{id}_A)(p)d_j \mid m \in M, d_j \in A\}$, we get that (2”) is false. □

For every Hilbert space H we write $B(H)$ for the set of all bounded operators on H, $P(H) = \Sigma^2(B(H))$ for the set of all positive semidefinite operators on H and $K(H)$ for the set of all compact operators on H.

Lemma 7. *Let* H *be a separable Hilbert space and* M *a quadratic module in* $B(H)$ *which is not contained in* $P(H)$. *Then* $\overline{M}$ *contains all hermitian compact operators, i.e.* $K(H)_h \subseteq \overline{M}$.

Proof. Let M be a quadratic module in $B(H)$ which is not contained in $P(H)$. Pick an arbitrary operator L in $M \setminus P(H)$ and a vector $v \in H$ such that $\langle v, Lv\rangle < 0$. Write P for the orthogonal projection of H on the span of v. Clearly, $PLP = \lambda P$ where $\lambda < 0$, hence $-P \in M$. If Q is an orthogonal projection of rank 1, then $Q = U^*PU$ for some unitary U, hence $-Q \in M$. Since also $Q = Q^*Q \in M$, M contains all hermitian operators of rank 1. Therefore, M contains all finite rank operators. Pick any $K \in K(H)_h \cap P(H)$ and note that $\sqrt{K} \in K(H)_h \cap P(H)$ as well. Clearly, $-K + \epsilon\sqrt{K} \in M$ for every $\epsilon > 0$ since it is a sum of an element from $K(H)_h \cap P(H)$ and a finite rank operator (check the eigenvalues). It follows that $-K \in \overline{M}$. It is also clear that every element of $K(H)_h$ is a difference of two elements from $K(H)_h \cap P(H)$, hence $K(H)_h \subseteq \overline{M}$. □

As the first application of Theorem 6 and Lemma 7, we prove the following generalization of Theorem G. By Lemma 11 below, Theorem G corresponds to the case $R = \mathbb{R}[x]$ and H finite-dimensional, i.e. in the finite-dimensional case we can omit the assumption on p.

Theorem 8. *Let H be a separable Hilbert space, M an archimedean quadratic module in $R \otimes B(H)$ and p an element of $R \otimes B(H)$.*

If for every multiplicative state ϕ on R there exists a real $\eta_\phi > 0$ such that $(\phi \otimes \mathrm{id}_{B(H)})(p) \in \eta_\phi + P(H) + K(H)_h$ (e.g. if $p \in \eta + R \otimes K(H)_h$ for some $\eta > 0$) then the following are equivalent:

1. *$p \in M^\circ$,*
2. *For every multiplicative state ϕ on R such that $(\phi \otimes \mathrm{id}_{B(H)})(M) \subseteq P(H)$, there exists $\epsilon_\phi > 0$ such that $(\phi \otimes \mathrm{id}_{B(H)})(p) \in \epsilon_\phi + P(H)$.*

If $(\phi \otimes \mathrm{id}_{B(H)})(p) \in P(H) + K(H)_h$ for every multiplicative state ϕ on R (e.g. if $p \in R \otimes K(H)_h$) then the following are equivalent:

(1') *$p \in \overline{M}$,*
(2') *For every multiplicative state ϕ on R such that $(\phi \otimes \mathrm{id}_{B(H)})(M) \subseteq P(H)$, we have that $(\phi \otimes \mathrm{id}_{B(H)})(p) \in P(H)$.*

Proof. Suppose that (1) is true, i.e. $p \in \epsilon + M$ for some $\epsilon > 0$. For every multiplicative state ϕ on R such that $(\phi \otimes \mathrm{id}_{B(H)})(M) \subseteq P(H)$ we have that $(\phi \otimes \mathrm{id}_{B(H)})(p) \in (\phi \otimes \mathrm{id}_{B(H)})(\epsilon + M) \subseteq \epsilon + P(H)$, hence (2) is true. Conversely, suppose that (2) is true. We claim that for every multiplicative state ϕ on R there exists $\epsilon_\phi > 0$ such that $(\phi \otimes \mathrm{id}_{B(H)})(p) \in \epsilon_\phi + (\phi \otimes \mathrm{id}_{B(H)})(M)$. Then it follows by Theorem 6 that (1) is true. If $(\phi \otimes \mathrm{id}_{B(H)})(M) \subseteq P(H)$, then $(\phi \otimes \mathrm{id}_{B(H)})(p) \in \epsilon_\phi + P(H) \subseteq \epsilon_\phi + (\phi \otimes \mathrm{id}_{B(H)})(M)$ by (2) and the fact that $(\phi \otimes \mathrm{id}_{B(H)})(M)$ is a quadratic module in $B(H)$. On the other hand, if $(\phi \otimes \mathrm{id}_{B(H)})(M) \not\subseteq$

$P(H)$, then $K(H)_h \subseteq \overline{(\phi \otimes \mathrm{id}_{B(H)})(M)}$ by Lemma 7. The assumption $(\phi \otimes \mathrm{id}_{B(H)})(p) \in \eta_\phi + P(H) + K(H)_h$ for some $\eta_\phi > 0$ then implies that $(\phi \otimes \mathrm{id}_{B(H)})(p) \in \frac{\eta_\phi}{2} + (\phi \otimes \mathrm{id}_{B(H)})(M)$ as claimed. The proof of the equivalence $(1') \Leftrightarrow (2')$ is similar. □

In the infinite-dimensional case, the assumption on p cannot be omitted as the following example shows:

Example. Let H be an infinite-dimensional separable Hilbert space, $0 \neq E \in B(H)_h$ an orthogonal projection of finite rank and T an element of $B(H)_h$ such that $T \notin P(H) + K(H)_h$. Since the quadratic module $\Sigma^2(B(H)/K(H))$ is closed, also $P(H)+K(H)_h$ is closed, hence there exists a real $\epsilon > 0$ such that $T + \epsilon \notin P(H) + K(H)_h$. Write $p_1 = -x^2 E$, $p_2 = 1 - x^2$ and $p = \epsilon + x^2 T$. Let M be the quadratic module in $\mathbb{R}[x] \otimes B(H)$ generated by p_1 and p_2. Since $p_2 \in M$, it follows from Lemma 11 below that M is archimedean. For every point $a \in \mathbb{R}$ such that $p_1(a) \geq 0$ and $p_2(a) \geq 0$ we have that $a = 0$, hence $p(a) = \epsilon$. Therefore, assertion (2) of Theorem 8 is true for our M and p. Assertion (1), however, fails for our M and p. If it was true then there would exist finitely many $q_i, u_j, v_k \in \mathbb{R}[x] \otimes B(H)$ and a real $\eta > 0$ such that $p = \eta + \sum_i q_i^* q_i + \sum_j u_j^* p_1 u_j + \sum_k v_k^* p_2 v_k$. For $x = 1$, we get $\epsilon + T = \eta + \sum_i q_i(1)^* q_i(1) - \sum_j u_j(1)^* E u_j(1)$. The first two terms belong to $P(H)$ and the last term belongs to $K(H)_h$, a contradiction with the choice of T.

We finish this section with a concrete version of Proposition 5 in the spirit of Theorem G. For $R = \mathbb{R}[x]$, we get [10, Corollary 22].

Theorem 9. *Let R be a commutative real algebra with trivial involution, $\nu \in \mathbb{N}$, and M an archimedean quadratic module in $M_\nu(R)$. For every element $p \in M_\nu(R)_h$, the following are equivalent:*

(1) *There are finitely many $c_i \in M_\nu(R)$ such that $\sum_i c_i^* p c_i \in 1 + M$.*
(2) *For every multiplicative state ϕ on R such that $(\phi \otimes \mathrm{id})(m)$ is positive semidefinite for all $m \in M$, we have that the operator $(\phi \otimes \mathrm{id})(p)$ is not negative semidefinite.*

Proof. Write $A = M_\nu(\mathbb{R})$. Clearly, a matrix $C \in A_h$ is not negative semidefinite (i.e. it has at least one strictly positive eigenvalue) iff there exist a matrices $D_i \in A$ such that $\sum_i D_i^* C D_i - I$ is positive semidefinite. It follows that a quadratic module M in A which is different from $\Sigma^2(A)$ contains $-I$, hence it is equal to A_h. (This also follows from Lemma 7.) Now we use equivalence (1") $\Leftrightarrow$ (2") of Theorem 6. □

4. Theorem D

Recall that a $*$-algebra A is *algebraically bounded* if the quadratic module $\Sigma^2(A)$ is archimedean. For an element $a \in A_h$ we say that $a \geq 0$ iff $a + \epsilon \in \Sigma^2(A)$ for all real $\epsilon > 0$ (i.e. iff $a \in \overline{\Sigma^2(A)}$) and that $a > 0$ iff $a \in \epsilon + \Sigma^2(A)$ for some real $\epsilon > 0$ (i.e. iff $a \in \Sigma^2(A)^\circ$). It is well-known that every Banach $*$-algebra is algebraically bounded.

The aim of this section is to deduce the following theorem from Theorem 6 and to show that it implies Theorem D. Other applications of Theorem 6 will be discussed in section 5.

Theorem 10. *Let R be a commutative real algebra with trivial involution and A an algebraically bounded $*$-algebra over $F \in \{\mathbb{R}, \mathbb{C}\}$. Let U be an inner product space over F, $\mathcal{L}(U)$ the $*$-algebra of all adjointable linear operators on U, $\mathcal{L}(U)_+$ its subset of positive semidefinite operators, and M an archimedean quadratic module in $R \otimes \mathcal{L}(U)$.*

Write $B := R \otimes A$ and consider the vector space $B \otimes U$ as left $R \otimes \mathcal{L}(U)$ right B bimodule which is equipped with the biadditive form $\langle \cdot, \cdot \rangle$ defined by $\langle b_1 \otimes u_1, b_2 \otimes u_2 \rangle := b_1^* b_2 \langle u_1, u_2 \rangle_U$. Write M' for the subset of B_h which consists of all finite sums of elements of the form $\langle q, mq \rangle$ where $m \in M$ and $q \in B \otimes U$.

We claim that the set M' is an archimedean quadratic module and that for every element $p \in R \otimes A_h$, the following are equivalent:

1. *$p \in \epsilon + M'$ for some real $\epsilon > 0$.*
2. *For every multiplicative state ϕ on R such that $(\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(M) \subseteq \mathcal{L}(U)_+$, we have that $(\phi \otimes \mathrm{id}_A)(p) > 0$.*

Moreover, the following are equivalent:

- (1') *$p \in \overline{M'}$.*
- (2') *For every multiplicative state ϕ on R such that $(\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(M) \subseteq \mathcal{L}(U)_+$, we have that $(\phi \otimes \mathrm{id}_A)(p) \geq 0$.*

Finally, the following are equivalent:

- (1") *There exist finitely many $c_i \in B$ such that $\sum_i c_i^* p c_i \in 1 + M'$.*
- (2") *For every multiplicative state ϕ on R such that $(\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(M) \subseteq \mathcal{L}(U)_+$, there exist finitely many elements $d_i \in A$ such that $\sum_i d_i^* (\phi \otimes \mathrm{id}_A)(p) d_i - 1 \geq 0$.*

We will need the following observation which follows from the fact that the set of bounded elements w.r.t. a given quadratic module is closed for addition and multiplication of commuting elements.

Lemma 11. *Let R be a commutative algebra with trivial involution and A an algebraically bounded $*$-algebra. A quadratic module N in $R \otimes A$ is archimedean if and only if $N \cap R$ is archimedean in R. If*

$x_1, \dots, x_d$ are generators of R, then N is archimedean if and only if it contains $K^2 - x_1^2 - \ldots - x_d^2$ for some real K.

Proof of Theorem 10. To prove that (1) implies (2), it suffices to prove:

Claim 1. For every multiplicative state ϕ on R such that $(\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(M) \subseteq \mathcal{L}(U)_+$, we have that $(\phi \otimes \mathrm{id}_A)(M') \subseteq \Sigma^2(A)$.

For every $q \in B \otimes U$ and $m \in M$ we have that $(\phi \otimes \mathrm{id}_A)(\langle q, mq \rangle) = \langle s, (\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(m)s \rangle$ where $s = (\phi \otimes \mathrm{id}_A \otimes \mathrm{id}_U)(q) \in A \otimes U$. If $s = \sum_{i=1}^k a_i \otimes u_i$, then

$$\langle s, (\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(m)s \rangle = \begin{bmatrix} a_1^* & \ldots & a_k^* \end{bmatrix} T \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

where $T = [\langle u_i, (\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(m)u_j \rangle]_{i,j=1}^k \in M_k(F)$. Since $(\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(m)$ is positive semidefinite for every $m \in M$, T is also positive semidefinite.

To prove that (2) implies (1), consider the following statement:

(3) *For every multiplicative state ϕ on R there exists a real $\epsilon_\phi > 0$ such that $(\phi \otimes \mathrm{id}_A)(p) \in \epsilon_\phi + (\phi \otimes \mathrm{id}_A)(M')$.*

We claim that (2) implies (3) and (3) implies (1).

Clearly, $b^* \langle q, mq \rangle b = \langle qb, mqb \rangle \in M'$ for every $m \in M$, $q \in B \otimes U$ and $b \in B$, hence the set M' is a quadratic module in B. Clearly, $M' \cap R$ is archimedean since it contains $M \cap R$. By Lemma 11, M' is also archimedean. Hence (3) implies (1) by Theorem 6.

Suppose that (2) is true and pick a multiplicative state ϕ on R. Clearly, $(\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(M)$ is a quadratic module in $\mathcal{L}(U)$ and $(\phi \otimes \mathrm{id}_A)(M')$ is a quadratic module in A. If $(\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(M) \subseteq \mathcal{L}(U)_+$, then (2) implies that $(\phi \otimes \mathrm{id}_A)(p) \in \epsilon_\phi + \Sigma^2(A) \subseteq \epsilon_\phi + (\phi \otimes \mathrm{id}_A)(M')$ for some real $\epsilon_\phi > 0$, hence (3) is true. If $(\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(M) \not\subseteq \mathcal{L}(U)_+$ then (3) follows from:

Claim 2. For every multiplicative state ϕ on R such that $(\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(M) \not\subseteq \mathcal{L}(U)_+$ we have that $(\phi \otimes \mathrm{id}_A)(M') = A_h$.

We could use Lemma 7 but we prefer to prove this claim from scratch. Pick any $C \in (\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(M) \setminus \mathcal{L}(U)_+$. There exists $u \in U$ of length 1 such that $\langle u, Cu \rangle < 0$. Write P for the orthogonal projection of U on the span of $\{u\}$. Clearly, $P^*CP = -\lambda P$ for some $\lambda > 0$, hence $-P \in (\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(M)$. Also, $P = P^*P \in (\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(M)$. Let $m_\pm \in M$ be such that $(\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(m_\pm) = \pm P$. Pick any $a \in A_h$ and write $q_\pm = 1_R \otimes \frac{1 \pm a}{2} \otimes u$ where $1 = 1_A$. The element $m' = \langle q_+, m_+ q_+ \rangle + \langle q_-, m_- q_- \rangle$ belongs to M' and, by the proof of Claim 1, $(\phi \otimes \mathrm{id}_A)(m') = \langle s_+, (\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(m_+) s_+ \rangle + \langle s_-, (\phi \otimes \mathrm{id}_{\mathcal{L}(U)})(m_-) s_- \rangle$

where $s_\pm = (\phi \otimes \mathrm{id}_A \otimes \mathrm{id}_U)(q_\pm) = \frac{1\pm a}{2} \otimes u$. Therefore, $(\phi \otimes \mathrm{id}_A)(m') = (\frac{1+a}{2})^2 \langle u, Pu\rangle + (\frac{1-a}{2})^2 \langle u, -Pu\rangle = (\frac{1+a}{2})^2 - (\frac{1-a}{2})^2 = a$.

Claim 1 also gives implications (1') $\Rightarrow$ (2') and (1") $\Rightarrow$ (2") and Claim 2 gives their converses. Note that assertion (3) must be replaced with suitable assertions (3') and (3") to which Theorem 6 can be applied. $\square$

For $U = F^\nu$, we have that $B \otimes U \cong B^\nu \cong M_{\nu\times 1}(B)$, $R \otimes \mathcal{L}(U) \cong M_\nu(R) \subseteq M_\nu(B)$ and $\langle q, mq\rangle = q^* mq$ in Theorem 10. However, if also $A = M_\nu(F)$ (i.e. $B = M_\nu(R)$), we do not get Theorem G. Combining both theorems, we get that archimedean quadratic modules M and M' in $M_\nu(R)$ have the same interior and the same closure.

Finally, we would like to show that Theorem 10 implies Theorem D. The proof also works for algebraically bounded $*$-algebras.

Proof of Theorem D. Write $\nu = 2+\nu_1+\ldots+\nu_m$, $\|x\| = \sqrt{x_1^2+\ldots+x_d^2}$ and $p_0 = [1-\|x\|^2]\oplus[\|x\|^2-1]\oplus p_1 \oplus \ldots \oplus p_m \in M_\nu(\mathbb{R}[x])$. Clearly, $K_0 = \{t \in \mathbb{R}^d \mid p_0(t) \geq 0\}$. Let M_0 be the quadratic module in $M_\nu(\mathbb{R}[x])$ generated by p_0. Since M_0 contains $(1-\|x\|^2)I_\nu$, it is archimedean by Lemma 11. By Theorem 10 applied to $U = F^\nu$, every element $p \in \mathbb{R}[x] \otimes A$ which is strictly positive definite on K_0 belongs to M_0'. From the definition of M_0', we get that $p = \sum_{j\in J} \left(s_j^* s_j + q_j^* p_0 q_j\right)$ for a finite J, $s_j \in \mathbb{R}[x] \otimes A$ and $q_j \in M_{\nu\times 1}(\mathbb{R}[x]\otimes A)$, hence

$$p \;=\; \sum_{j\in J} \left(s_j^* s_j + w_j^*(1-\|x\|^2)w_j + z_j^*(\|x\|^2-1)z_j + \sum_{k=1}^m s_{jk}^* p_k s_{jk}\right)$$

for a finite J, $s_j, w_j, z_j \in \mathbb{R}[x]\otimes A$ and $s_{jk} \in M_{\nu_k\times 1}(\mathbb{R}[x]\otimes A)$. Replacing x by $\frac{x}{\|x\|}$ and multiplying with a large power of $\|x\|^2$ we get that

$$\|x\|^{2\theta} p(x) = \sum_{j\in J} \Big((u_j(x)+\|x\|v_j(x))^*(u_j(x)+\|x\|v_j(x)) + \\ + \sum_{k=1}^m (u_{jk}(x)+\|x\|v_{jk}(x))^* p_k(x)(u_{jk}(x)+\|x\|v_{jk}(x))\Big)$$

where $u_j, v_j \in\in \mathbb{R}[x]\otimes A$ and $u_{jk}, v_{jk} \in M_{\nu_k\times 1}(\mathbb{R}[x]\otimes A)$ for every $j \in J$. Finally, we can get rid of the terms containing $\|x\|$ by replacing $\|x\|$ with $-\|x\|$ and adding the old and the new equation. $\square$

5. Theorems E and F

Our next result, Theorem 12, is a special case of Theorem 10 for $U = F$. It extends Theorem E from $\mathbb{R}[x]$ to R and from C^*-algebras to algebraically bounded $*$-algebras. For $R = \mathbb{R}[\cos\phi_1, \sin\phi_1, \ldots, \cos\phi_n, \sin\phi_n]$

we get a generalization of Theorem F to algebraically bounded $*$-algebras.

Theorem 12. *Let R be a commutative real algebra with trivial involution, A an algebraically bounded $*$-algebra over $F \in \{\mathbb{R}, \mathbb{C}\}$ and M an archimedean quadratic module in R. Write $M' = M \cdot \Sigma^2(R \otimes A)$ for the quadratic module in $R \otimes A$ which consists of all finite sums of elements of the form mq^*q with $m \in M$ and $q \in R \otimes A$. For every element $p \in R \otimes A_h$, the following are equivalent:*

1. *$p \in \epsilon + M'$ for some real $\epsilon > 0$.*
2. *For every multiplicative state ϕ on R such that $\phi(M) \geq 0$, we have that $(\phi \otimes \mathrm{id}_A)(p) > 0$.*

If we combine Theorem 12 with a suitable version of the Riesz representation theorem (namely, Theorem 1 in [9]) we get the following existence result for operator-valued moment problems which extends Theorem 3 and Lemma 5 from [1].

Theorem 13. *Let A be an algebraically bounded $*$-algebra, R a commutative real algebra and M an archimedean quadratic module on R. For every linear functional $L\colon R \otimes A \to \mathbb{R}$ such that $L(mq^*q) \geq 0$ for every $m \in M$ and $q \in R \otimes A$, there exists an A'-valued nonnegative measure m on $M^\vee$ such that $L(p) = \int_{M^\vee}(dm, p)$ for every $p \in R \otimes A$. (Note that p defines a function $\phi \mapsto (\phi \otimes \mathrm{id}_A)(p)$ from $M^\vee$ to A.)*

Proof. Recall that the set $M^\vee$ is compact in the weak*-topology. We assume that A is equipped with its natural C^*-seminorm induced by the archimedean quadratic module $\Sigma^2(A)$, see [3, Section 3], hence it is a locally convex $*$-algebra. We will write $\mathcal{C}^+(M^\vee, A) := \mathcal{C}(M^\vee, \overline{\Sigma^2(A)})$ for the positive cone of $\mathcal{C}(M^\vee, A)$. Let i be the mapping from $R \otimes A$ to $\mathcal{C}(M^\vee, A)$ defined by $i(p)(\phi) = (\phi \otimes \mathrm{id}_A)(p)$ for every $p \in R \otimes A$ and $\phi \in M^\vee$. By Theorem 12, we have that $\mathcal{C}^+(M^\vee, A) \cap i(R \otimes A) = i(\overline{M'})$ where $M' = M \cdot \Sigma^2(R \otimes A)$. Note that L is a $\overline{M'}$-positive functional on $R \otimes A$ and that is defines in the natural way an $i(\overline{M'})$-positive functional L' on $i(R \otimes A)$. By the Riesz extension theorem for positive functionals, L' extends to a $\mathcal{C}^+(M^\vee, A)$-positive functional on $\mathcal{C}(M^\vee, A)$ which has the required integral representation by Theorem 1 in [9]. Hence L also has the required integral representation. □

Finally, we would like to prove a nichtnegativsemidefinitheitsstellensatz that corresponds to Theorem 12.

Theorem 14. *Let H be a separable infinite-dimensional complex Hilbert space and R a commutative real algebra with trivial involution. Let M*

be an archimedean quadratic module in R and $M' = M \cdot \Sigma^2(R \otimes B(H))$. For every $p \in R \otimes B(H)_h$, the following are equivalent:

(1) *There are finitely many $c_i \in R \otimes B(H)$ such that $\sum_i c_i^* p c_i \in 1 + M'$.*
(2) *For every multiplicative state ϕ on R such that $\phi(M) \geq 0$, the operator $(\phi \otimes \mathrm{id}_{B(H)})(p)$ is not the sum of a negative semidefinite and a compact operator.*

Note that for finite-dimensional H, (1) is equivalent to the following: For every multiplicative state ϕ on R such that $\phi(M) \geq 0$, the operator $(\phi \otimes \mathrm{id}_{B(H)})(p)$ is not negative semidefinite; cf. Theorem 9.

Proof. The equivalence (1") $\Leftrightarrow$ (2") of Theorem 10 (with $U = \mathbb{C}$ and $A = B(H)$) says that our assertion (1) is equivalent to the following:

(3) *For every multiplicative state ϕ on R such that $\phi(M) \geq 0$, there exist finitely many operators $D_i \in B(H)$ such that $\sum_i D_i^*(\phi \otimes \mathrm{id}_A)(p) D_i \in 1 + P(H)$.*

Therefore it suffices to prove the following claim:

Claim. For every operator $C \in B(H)_h$, the following are equivalent:

(i) C is not the sum of a negative semidefinite and a compact operator,
(ii) the positive part of C is not compact,
(iii) there exists an operator D such that $D^* C D = 1$,
(iv) there exist finitely many $D_i \in B(H)$ such that $\sum_i D_i^* C D_i \in 1 + P(H)$.

The implications (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) are clear. To prove that (ii) implies (iii) we first note that $C_+ := E_0 C = E_0^* C E_0$ where E_0 is the spectral projection belonging to the interval $[0, \infty)$. Since C_+ is not compact, there exists by the spectral theorem a real number $\gamma > 0$ such that the spectral projection E_γ belonging to the interval $[\gamma, \infty)$ has infinite-dimensional range. The operator $C_\gamma := E_\gamma C_+ = E_\gamma^* C E_\gamma$ has decomposition $C_\gamma = \tilde{C}_\gamma \oplus 0$ with respect to $H = E_\gamma H \oplus (1 - E_\gamma) H$ where $\tilde{C}_\gamma \geq \gamma$. Write $F = (\tilde{C}_\gamma)^{-1/2} \oplus 0$ and note that $(E_\gamma F)^* C (E_\gamma F) = 1 \oplus 0$. Since $E_\gamma H$ is infinite-dimensional, it is isometric to H. If G is an isometry from H onto $E_\gamma H$ then $D := E_\gamma F G$ satisfies (iii). $\square$

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