THE ORBIFOLD COHOMOLOGY OF MODULI OF HYPERELLIPTIC CURVES

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Abstract. We study the inertia stack of $[\mathcal{M}_{0,n}/S_n]$, the quotient stack of the moduli stack of smooth genus 0 curves with n marked points via the action of the symmetric group S_n. Then we see how from this description we can obtain a description of the inertia stack of $\mathcal{H}_g$, the moduli stack of hyperelliptic curves of genus g. From this, we can compute additively the Chen–Ruan (or orbifold) cohomology of $\mathcal{H}_g$.

1. Introduction

A hyperelliptic curve of genus g is a smooth algebraic curve that admits a $2:1$ map to $\mathbb{P}^1$, and thus has $2g+2$ branch points. From its very definition, it is clear that the moduli stack of genus g hyperelliptic curves $\mathcal{H}_g$ admits a map onto the moduli stack $[\mathcal{M}_{0,2g+2}/S_{2g+2}]$, which is an isomorphism at the level of coarse moduli spaces. The foundations for moduli of hyperelliptic curves, as well as the precise definition of the previous map, can be found in [LK79] (in particular Theorem 5.5).

The last decade has seen tremendous improvements in our understanding of the moduli space of hyperelliptic curves $\mathcal{H}_g$. We mention here some of the recent achievements that are relevant to the present work. In the paper [AV04], $\mathcal{H}_g$ is described as a moduli stack of cyclic covers of the projective line. As a consequence of this description, the authors are able to determine its Picard group. Along these lines, the Picard group of the Deligne-Mumford compactification $\overline{\mathcal{H}}_g$ was computed in [C07], and very recently the whole integral Chow ring of $\mathcal{H}_g$ was computed in [FV10] (see also [EF09], [GV08]). In the last years, much effort was also made in studying the automorphism groups of hyperelliptic curves [GSS], [GD05], [Sh03], [MSSV].

In this paper we deal with rational cohomology and Chow group with rational coefficient. From both these points of view, the moduli stacks $\mathcal{H}_g$ are trivial. The triviality of $H^*(\mathcal{H}_g, \mathbb{Q})$ follows from [KL02, Theorem 2.13], while the triviality of $A^*_{\mathbb{Q}}(\mathcal{H}_g)$ follows from its description as finite quotient of the affine variety $\mathcal{M}_{0,n}$. Still some non-triviality can be measured with rational coefficients, but one has to consider instead the *orbifold cohomology* or the *stringy Chow group*. The orbifold cohomology as a vector space (or Chen–Ruan cohomology) of an orbifold $\mathcal{X}$, is obtained by adding to the usual cycles of $\mathcal{X}$ the cycles of all

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the *twisted sectors* of $\mathcal{X}$. The twisted sectors are orbifolds that parametrize couples (x, g) where x is a point of $\mathcal{X}$ and $g \in \mathrm{Aut}(x)$. The new cycles are then given an unconventional degree, which is the sum of their original degree as cycles inside their twisted sector Y, plus a rational number (called *age* or *degree shifting number*) that depends on the normal bundle $N_Y \mathcal{X}$.

The orbifold cohomology of moduli spaces of curves is studied in [Pa08], [Pa10], [Sp06] (see also the PhD thesis [Pa09], [Sp04]). The present work has some nontrivial intersection with [Pa10] and [Sp06], since in these two papers in particular the orbifold cohomology and stringy Chow group of $\mathcal{M}_2 = \mathcal{H}_2$ are described.

The main result of this paper is Theorem 5.1, where we give a closed formula for the *orbifold Poincaré polynomial* of $\mathcal{H}_g$, *i.e.* a "polynomial"[1], whose coefficient of q^i corresponds to the dimension of the group H^i.

To achieve this result, we first describe the cohomology of the twisted sectors of $[\mathcal{M}_{0,n}/S_n]$, in Section 3 as quotients of certain $\mathcal{M}_{0,k}$ modulo a subgroup of S_k.

Then, in Section 4, we study the twisted sectors of $\mathcal{H}_g$. If g is odd, we see that the twisted sectors of $\mathcal{H}_g$ are simply the twisted sectors of $[\mathcal{M}_{0,2g+2}/S_{2g+2}]$ repeated twice. If g is even, the same happens for the twisted sectors of $[\mathcal{M}_{0,2g+2}/S_{2g+2}]$ whose distinguished automorphism is not an involution. The remaining few twisted sectors are still described as quotients of moduli of genus 0, pointed curves modulo the action of a certain subgroup of the symmetric group on markings.

Finally, in Section 5 we compute all the degree shifting numbers, and we write the explicit results by recollecting the results of the previous sections.

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1.b. **Notation.** We work over $\mathbb{C}$, cohomologies and Chow groups are taken with rational coefficients. Orbifold for us means smooth Deligne–Mumford stack, and we always work within the category of Deligne–Mumford stacks. If a finite group G acts on a scheme (stack) X, $[X/G]$ is the stack quotient and X/G is the quotient as a scheme. We call $\mu_N := \mathbb{Z}_N^\vee$ the group of characters of $\mathbb{Z}_N$, and μ_N^* the subgroup whose elements are the injective characters. We make an implicit use of the relative language of schemes. For instance, when no confusion can arise, we speak of a genus g smooth curve, meaning a family of genus g smooth curve over a certain base S.

[1]We call it polynomial in analogy with the ordinary Poincaré polynomial, although the exponents of the variable q are not natural but rational.

2. Definition of Orbifold Cohomology

In this section we define orbifold cohomology. For a more detailed study of this topic, we address the reader to [AGV08, Section 3] for the various inertia stacks, and to [AGV08, Section 7.1] for the degree shifting number (the original reference is [CR04]). What we call orbifold cohomology is the graded vector space underlying the Chen–Ruan cohomology ring (or algebra): the latter is a more refined object that we will not introduce in this work.

We introduce the following natural stack associated to a Deligne–Mumford stack X, which points to where X fails to be an algebraic space.

Definition 2.1. ([AGV02, 4.4], [AGV08, Definition 3.1.1]) Let X be an algebraic stack. The *inertia stack* $I(X)$ of X is defined as:

$$I(X) := \coprod_{N \in \mathbb{N}} I_N(X)$$

where $I_N(X)(S)$ is the following groupoid:

1. The objects are pairs (ξ, α), where ξ is an object of X over S, and $\alpha : \mu_N \to \mathrm{Aut}(\xi)$ is an injective homomorphism,
2. The morphisms are the morphisms $g : \xi \to \xi'$ of the groupoid $X(S)$, such that $g \cdot \alpha(1) = \alpha(1) \cdot g$.

We also define $I_{TW}(X) := \coprod_{N>1} I_N(X)$, in such a way that:

$$I(X) = I_1(X) \coprod I_{TW}(X)$$

The connected components of $I_{TW}(X)$ are called *twisted sectors* of the inertia stack of X, or also twisted sectors of X. The inertia stack comes with a natural forgetful map $f : I(X) \to X$.

We observe that, by our very definition, $I_N(X)$ is an open and closed substack of $I(X)$, but it rarely happens that it is connected. One special case is when N equals to 1: in this case the map f restricted to $I_1(X)$ induces an isomorphism of the latter with X. The connected component $I_1(X)$ will be referred to as the *untwisted sector*.

We also observe that given a generator of μ_N, we obtain an isomorphism of $I(X)$ with $I'(X)$, where the latter is defined as the (2-)fiber product $X \times_{X \times X} X$ where both morphisms $X \to X \times X$ are the diagonals.

Remark 2.2. There is an involution $\iota : I_N(X) \to I_N(X)$, which is induced by the map $\iota' : \mu_N \to \mu_N$, that is $\iota'(\zeta) := \zeta^{-1}$.

Proposition 2.3. [AGV08, Corollary 3.1.4] *Let X be a smooth algebraic stack. Then the stacks $I_N(X)$ (and therefore $I(X)$ itself) are smooth.*

We now study the behaviour of the inertia stack under arbitrary morphisms of stacks.

Definition 2.4. Let $f : X \to Y$ be a morphism of stacks. We define $f^*(I(Y))$ as the stack that makes the following diagram 2−cartesian:

$$\begin{array}{ccc} f^*(I(Y)) & \xrightarrow{I(f)} & I(Y) \\ \downarrow & \square & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

and $I(f)$ as the map that lifts f in the diagram. Obviously, there is an induced map that we call $I'(f)$, which maps $I(X) \to f^*(I(Y))$.

We now define the degree shifting number for the twisted sectors of the inertia stack of a smooth stack X. With $R\mu_N$, we denote the representation ring of μ_N.

Definition 2.5. [AGV08, Section 7.1] Let $\rho : \mu_N \to \mathbb{C}^*$ be a group homomorphism. It is determined by an integer $0 \leq k \leq N-1$ as $\rho(\zeta_N) = \zeta_N^k$. We define a function *age*:

$$\operatorname{age}(\rho) = k/N$$

This function extends to a unique group homomorphism:

$$\operatorname{age} : R\mu_N \to \mathbb{Q}$$

We now define the age of a twisted sector Y.

Definition 2.6. ([CR04, Section 3.2], [AGV08, Definition 7.1.1]) Let Y be a twisted sector and $g : \operatorname{Spec}\mathbb{C} \to Y$ a point. Then the pull-back via g of the tangent sheaf, $g^*(T_X)$, is a representation of μ_N on a finite dimensional vector space. We define:

$$a(Y) := \operatorname{age}(g^*(T_X))$$

We can then define the orbifold, or Chen–Ruan, degree.

Definition 2.7. ([CR04, Definition 3.2.2]) We define the $d-th$ degree orbifold cohomology group as follows:

$$H^d_{CR}(X, \mathbb{Q}) := \bigoplus_i H^{d-2a(X_i,g_i)}(X_i, \mathbb{Q})$$

where the sum is over all twisted sectors. The *orbifold Poincaré polynomial* of X is:

$$P_X^{CR}(q) := \sum_{i \in \mathbb{Q}^+} \dim \left(H^i_{CR}(X) \right) q^i$$

Remark 2.8. One can also define the stringy Chow group and its unconventional grading in complete analogy with the above definition. See [AGV08, Section 7.3] for this construction.

3. The inertia stack of the configuration of unordered points on the Riemann sphere

In this section, we study the cohomology of the inertia stack of $[\mathcal{M}_{0,n}/S_n]$ (also known in the literature as $\widetilde{\mathcal{M}_{0,n}}$). For this, it is enough to give a description of the coarse moduli spaces of the twisted sectors of the inertia stack of $[\mathcal{M}_{0,n}/S_n]$. We thus describe the coarse moduli spaces of the twisted sectors of the latter stack as quotients of the kind $\mathcal{M}_{0,k}/S$, for S a certain subgroup of S_k. The cohomology of these quotients is well-known. The cohomology of $\mathcal{M}_{0,n}$ was first computed as a representation of the symmetric group S_n by Getzler [Ge94, 5.6] (see also [KL02]).

In particular, we shall use the following result:

Proposition 3.1. *The Poincaré polynomial of $\mathcal{M}_{0,n+2}/S_n$ is:*

$$P^0_{n+2;n,1,1}(q) = \sum_{i=0}^{n-1} q^i$$

The Poincaré polynomial of $\mathcal{M}_{0,n+2}/S_n \times S_2$ is:

$$P^0_{n+2;n,2}(q) = \begin{cases} 1 & n = 1 \\ \sum_{i=0}^{\lfloor \frac{n-2}{4} \rfloor} q^i + q^{i+1} & n > 1 \end{cases}$$

Proof. It follows from [KL02, Theorem 2.9]. □

It will be convenient to have a definition for the set where each injective character of $\mathbb{Z}_N$ is identified with its inverse:

Definition 3.2. We define $\widetilde{\mu_N}^*$ as the quotient set $\mu_N^*/\mathbb{Z}_2$ where $\overline{1}(\zeta_N) := \zeta_N^{-1}$. If N is even, we define $\overline{\mu}_N^*$ to be the quotient set $\mu_N^*/\mathbb{Z}_2$ where the action of $\overline{1}$ is defined to be: $\overline{1}(\zeta_N) := -\zeta_N^{-1}$.

The following proposition describes the inertia stack of $[\mathcal{M}_{0,n}/S_n]$.

Proposition 3.3. *We describe the coarse moduli space of the inertia stack of $[\mathcal{M}_{0,n}/S_n]$.*

(1) if $N > 2$, let $n = kN + a$ where $a \in \{0, 1, 2\}$. Then:

$$I_N([\mathcal{M}_{0,n}/S_n]) = \begin{cases} \coprod_{\chi \in \widetilde{\mu_N}^*} (\mathcal{M}_{0,k+2}/S_k \times S_2, \chi) & a = 0, 2 \\ \coprod_{\chi \in \mu_N^*} (\mathcal{M}_{0,k+2}/S_k, \chi) & a = 1 \end{cases}$$

(2) if n is even, $n =: 2g + 2$:

$$I_2([\mathcal{M}_{0,n}/S_n]) = (\mathcal{M}_{0,g+2}/S_g \times S_2, -1) \coprod (\mathcal{M}_{0,g+3}/S_{g+1} \times S_2, -1)$$

(3) if n is odd, $n =: 2g + 1$:

$$I_2([\mathcal{M}_{0,n}/S_n]) = (\mathcal{M}_{0,g+2}/S_g, -1)$$

Proof. Let C be a smooth genus 0 curve and α an automorphism of finite order N of it. From the Riemann–Hurwiz formula, α has at least two fixed points, and if it had more it would be the identity. We can choose coordinates on C in such a way that the two fixed points are 0 and ∞, and α is the multiplication by a primitive $N-$th root of unity. Now let $C' = C/\langle \alpha \rangle$ be the quotient curve. Upon a suitable choice of coordinates on C and C', the quotient map $C \to C'$ becomes the map $z \to z^N$.

Let us first deal with the case when $N > 2$. In this case there is exactly one choice of $k \in \mathbb{N}^+$ and $a \in \{0, 1, 2\}$ such that $n = kN + a$. The set of $n = kN + a$ marked points corresponds to a set of $k + a$ marked points on C', where the a points are a subset of the branch divisor.

If a is equal to 1, there is a choice of a point p in C that is the only point that is both in the set of n points, and a fixed point for α. Then α determines a $\chi \in \mu_N^*$: the character of the representation of α on T_pC.

If a equals 0 or 2, there is no such a choice. Then α acts on the set of two fixed points, thus determining two inverse characters in μ_N. In the same way as before, α then gives an equivalence class in the set $\widetilde{\mu_N}^*$ (see Definition 3.2).

If N is equal to 2, the argument is the same, the only difference being that $\widetilde{\mu_2}^* = \mu_2^*$. □

Remark 3.4. In [AGV08], the authors introduce two notions related to the inertia stack: the *stack of cyclotomic gerbes* ([AGV08, Definition 3.3.6]) and the *rigidified inertia stack* ([AGV08, 3.4]), showing in [AGV08, 3.4.1] that they are equivalent. By substituting $\mathcal{M}_{0,k+2}/S_k \times S_2$ with the stack quotient $[\mathcal{M}_{0,k+2}/S_k \times S_2]$ (respectively, $\mathcal{M}_{0,k+2}/S_k$ with $[\mathcal{M}_{0,k+2}/S_k]$) one obtains a stacky description of the rigidified inertia stack of $[\mathcal{M}_{0,n}/S_n]$. We have stated the above proposition in this simplified way because this is enough for our purposes, and in this way we could avoid having to introduce the whole theory of inertia stack and its variants (see [AGV08, Section 3]).

4. The inertia stack of moduli of smooth hyperelliptic curves

In this section we study the inertia stack of $\mathcal{H}_g$. We will implicitly use the fact that any family of hyperelliptic curves has a globally defined hyperelliptic involution, a result that follows from [LK79, Theorem 5.5]. Let:

$$f : \mathcal{H}_g \to [\mathcal{M}_{0,2g+2}/S_{2g+2}] = \widetilde{\mathcal{M}_{0,2g+2}}$$

be the map that associates to every hyperelliptic genus g curve, the corresponding genus 0 curve, together with the degree $2g + 2$ étale Cartier divisor D obtained by considering the branch locus of the hyperelliptic involution. This map is well defined on families as a consequence of [LK79, Theorem 5.5].

Let $C \to C' = C/\langle \tau \rangle$ be a hyperelliptic curve, and α an automorphism of it. Then α induces an automorphism α^{red} of C'. If D is the degree $2g + 2$ branch divisor of $C \to C'$, then α^{red} induces a bijection on the set of reduced points of D.

Definition 4.1. Let $I_N^{red}(\mathcal{H}_g)$ be the open and closed substack of $I(\mathcal{H}_g)$ whose objects correspond to couples (C, α), where C is an object of $\mathcal{H}_g$ and $\alpha^{red} : \mu_N \to \mathrm{Aut}(C/\tau)$ is an injective homomorphism.

For our purposes, it is more convenient to work with $I_N^{red}(\mathcal{H}_g)$ than with the usual $I_N(\mathcal{H}_g)$. Note that of course we have in the end that:

$$I(\mathcal{H}_g) = \coprod_{N \in \mathbb{N}} I_N(\mathcal{H}_g) = \coprod_{N \in \mathbb{N}} I_N^{red}(\mathcal{H}_g)$$

but with the latter decomposition, we have that the natural map of 2.4, $I'(f) : I(\mathcal{H}_g) \to f^* \left(I([\mathcal{M}_{0,2g+2}/S_{2g+2}]) \right)$ induces maps:

$$I'(f)_N : I_N^{red}(\mathcal{H}_g) \to f^* \left(I_N([\mathcal{M}_{0,2g+2}/S_{2g+2}]) \right)$$

This is not the case for the standard decomposition of the inertia stacks, since an automorphism of order N of the genus 0 curve can lift to an automorphism of order N or to an automorphism of order $2N$ on the corresponding hyperelliptic curve.

Let $n = 2g+2$ be the number of Weierstrass points, and $N = \mathrm{ord}(\alpha^{red})$. It is convenient to write $n = kN + a$ for $k \in \mathbb{N}^+, a \in \{0, 1, 2\}$ (following the results of Section 3). If $N > 2$ such a decomposition of n is unique. The number a is the number of Weierstrass points whose image in the quotient via the hyperelliptic involution is a branch point for α^{red}.

We label each twisted sector by a character. If a is equal to zero, then there are four points in C whose image in C/τ consists of the two points fixed by α^{red}. In this case the automorphism α can:

(1) fix the four points;
(2) exchange them two-by-two;
(3) fix two of them and exchange the other two.

In the first two cases, we label the twisted sectors by a couple $(\chi, 1)$ or $(\chi, -1)$ respectively.

Theorem 4.2. *We describe the coarse moduli space of the inertia stack of the moduli stack of smooth hyperelliptic curves of genus g: $\mathcal{H}_g$.*

(1) if $N > 2$, let $n = kN + a$ where $a \in \{0, 1, 2\}$. Then:

$$I_N^{red}(\mathcal{H}_g) = \begin{cases} \coprod_{\chi \in \widetilde{\mu_N}^*, \lambda \in \pm 1} (\mathcal{M}_{0,k+2}/S_k \times S_2, (\chi, \lambda)) & a = 0, \ k \ \textit{even} \\ \coprod_{\chi \in \mu_N{}^*} (\mathcal{M}_{0,k+2}/S_k \times S_2, \chi) & a = 0, \ k \ \textit{odd} \\ \coprod_{\chi \in \mu_N{}^* \sqcup \mu_{2N}^*} (\mathcal{M}_{0,k+2}/S_k, \chi) & a = 1 \\ \coprod_{\chi \in \widetilde{\mu_{2N}}^*} (\mathcal{M}_{0,k+2}/S_k \times S_2, \chi) & a = 2, \ k \ \textit{even}, \ N \textit{even} \\ \coprod_{\chi \in \widetilde{\mu_N}^* \sqcup \widetilde{\mu_{2N}}^*} (\mathcal{M}_{0,k+2}/S_k \times S_2, \chi) & a = 2, \ k \ \textit{even}, \ N \textit{odd} \\ \coprod_{\chi \in \overline{\mu}_{2N}^*} (\mathcal{M}_{0,k+2}/S_k \times S_2, \chi) & a = 2, \ k \ \textit{odd} \end{cases}$$

(2) if g is odd:

$$I_2^{red}(\mathcal{H}_g) = (\mathcal{M}_{0,g+2}/S_g \times S_2, \zeta_4) \coprod \left(\mathcal{M}_{0,g+2}/S_g \times S_2, \zeta_4^3\right) \coprod$$

$$\coprod (\mathcal{M}_{0,g+3}/S_{g+1} \times S_2, (-1,1)) \coprod (\mathcal{M}_{0,g+3}/S_{g+1} \times S_2, (-1,-1))$$

where $\{\zeta_4, \zeta_4^3\} = \overline{\mu}_4^* = \mu_4^*$.

(3) if g is even:

$$I_2^{red}(\mathcal{H}_g) = (\mathcal{M}_{0,g+2}/S_g, -1) \coprod (\mathcal{M}_{0,g+3}/S_{g+1}, -1)$$

Proof. First we observe that the morphism of 2.4:

$$I(f)_N : f^* \left(I_N([\mathcal{M}_{0,2g+2}/S_{2g+2}])\right) \to I_N([\mathcal{M}_{0,2g+2}/S_{2g+2}])$$

is a μ_2-gerbe, and as such it induces an isomorphism at the level of coarse moduli spaces.

Let us consider then:

$$I'(f)_N : I_N^{red}(\mathcal{H}_g) \to f^* \left(I_N([\mathcal{M}_{0,2g+2}/S_{2g+2}])\right)$$

This map is a $2:1$ étale cover, because every automorphism of a genus 0 curve with an invariant divisor of degree $2g+2$ can be lifted exactly to two automorphisms of the corresponding hyperelliptic curve. To prove the two points (1) and (2), we prove that this is the trivial cover, and then apply the result of Proposition 3.3. To prove point (3), we show that in the particular case when $N = \text{ord}(\alpha^{red}) = 2$ and g is even, a lifting of α^{red} corresponds to a choice of a distinguished point p in D, the branch divisor of $C \to C'$.

Let C be a hyperelliptic curve, α an automorphism of it, and τ the hyperelliptic involution. We have the two projections on the quotient:

$$C \xrightarrow{\pi} C/\langle\tau\rangle \xrightarrow{p_N} C/\langle\tau, \alpha^{red}\rangle$$

After choosing suitable coordinates on $C/\langle\tau\rangle \cong \mathbb{P}^1$ and $C/\langle\tau, \alpha^{red}\rangle \cong \mathbb{P}^1$, the map p_N is simply the map $z \to z^N$. Let R be the set of ramification points of p_N. The number of points in R that are branch points for π is then a, by its very definition.

Now we study separately the three cases $a = 0, 1, 2$.

If a is equal to 0, then a hyperelliptic curve C that admits an automorphism α of reduced order N can be written as:

$$y^2 = (x^N - \alpha_1)(x^N - \alpha_2)\dots(x^N - \alpha_k)$$

with the automorphism α:

$$\begin{cases} x \to \zeta_N^i x \\ y \to \pm y \end{cases}$$

Exchanging the coordinates $0, \infty$, the action of α becomes:

$$\begin{cases} x \to \zeta_N^{-i} x \\ y \to \pm(\zeta_N)^{i(g+1)} y \end{cases}$$

If k is odd, then α fixes two of the points in $\pi^{-1}(R)$ and exchanges the other two. The action of α on the two fixed fibers determines the same character in μ_N^*. If k is even, then α can either fix the four points of $\pi^{-1}(R)$ or exchange them two-by-two. In both the cases, the action of α or $\alpha\tau$ on the four fixed points determines an element of $\widetilde{\mu_N}^*$.

If a is equal to 1, then a hyperelliptic curve C that admits an automorphism α of reduced order N can be written as:

$$y^2 = x(x^N - \alpha_1)(x^N - \alpha_2) \ldots (x^N - \alpha_k)$$

with the automorphism α:

$$\begin{cases} x \to \zeta_N^i x \\ y \to \pm\zeta_{2N}^i y \end{cases}$$

In this case, if we call p the point in R that is also a branch point of π, then the action of α on $T_{\pi^{-1}(p)}$ determines a well-defined element of μ_N^* or μ_{2N}^*.

If a is equal to 2, then a hyperelliptic curve C that admits an automorphism α of reduced order N can be written as:

$$y^2 = x(x^N - \alpha_1)(x^N - \alpha_2) \ldots (x^N - \alpha_k)$$

with the automorphism α:

$$\begin{cases} x \to \zeta_N^i x \\ y \to \pm\zeta_{2N}^i y \end{cases}$$

Exchanging the coordinates $0, \infty$, the action of α becomes:

$$\begin{cases} x \to \zeta_N^{-i} x \\ y \to \pm\zeta_{2N}^{-i}(\zeta_N)^{ig} y \end{cases}$$

In this case, the action of α fixes the two points in $\pi^{-1}(R)$. Then α induces a well-defined element of $\widetilde{\mu_{2N}}^*$ when k is even and N is even, of $\widetilde{\mu_N}^* \sqcup \widetilde{\mu_{2N}}^*$ when k is even and N is odd, and of $\overline{\mu}_{2N}^*$ when k is odd (and therefore N is even).

Now for the point (2), it is enough to check that our separate study in the different cases $a = 0, 1, 2$ carries on also when $N = \mathrm{ord}(\alpha^{red}) = 2$, if g is odd.

The two remaining cases are when g is even, $N = 2$; therefore a is equal to zero (then $k = g + 1$ is odd), or a is equal to two (then $k = g$ is also even). We have that $\mu_2^* = \widetilde{\mu_4}^* = \widetilde{\mu_2}^*$. In these cases, the action of α on $\pi^{-1}(R)$ distinguishes the two points of R. For example, if $a = 2$, k even, then α acts on the two points of $\pi^{-1}(R)$, on one of them with character ζ_4 and on the other with character ζ_4^3. In these cases therefore, the two $2:1$ étale covers, at the level of coarse moduli spaces, are the two quotient maps:

$$\mathcal{M}_{0,g+2}/S_g \to \mathcal{M}_{0,g+2}/S_g \times S_2 \quad \text{and} \quad \mathcal{M}_{0,g+3}/S_{g+1} \to \mathcal{M}_{0,g+3}/S_{g+1} \times S_2$$

□

Remark 4.3. The above theorem could be restated as a stack-theoretic description of the rigidified inertia stack of $\mathcal{H}_g$ (cfr. Remark 3.4), by substituting each occurrence of a quotient $\mathcal{M}_{0,k+2}/S$ (S a subgroup of the symmetric group S_n), with the stack quotient $[\mathcal{M}_{0,k+2}/S]$.

Remark 4.4. As a consequence of [AV04, Section 4], one can see that the μ_2-gerbe:

$$\mathcal{H}_g \to [\mathcal{M}_{0,2g+2}/S_{2g+2}]$$

is trivial when g is odd. Therefore, when g is odd, the map:

$$I'(f)_N : I_N^{red}(\mathcal{H}_g) \to f^* \left(I_N([\mathcal{M}_{0,2g+2}/S_{2g+2}]) \right)$$

is the trivial μ_2-torsor, so that Theorem 4.2 follows from Proposition 3.3 in the odd genus case. Our explicit study permits however a unified description of the inertia stack of $\mathcal{H}_g$ in terms of moduli of genus 0 pointed curves regardless of the parity of g.

5. The Orbifold cohomology of smooth hyperelliptic curves

Here we compute the orbifold Poincaré polynomial for moduli of smooth hyperelliptic curves (see Definition 2.7).

Let us fix a hyperelliptic curve C of genus g. A basis for the cotangent space $(T_C\mathcal{H}_g)^\vee$ is given by:

$$\left(\frac{dX}{Y}\right)^2 \quad X\left(\frac{dX}{Y}\right)^2 \quad \dots \quad X^{2g-2}\left(\frac{dX}{Y}\right)^2$$

If α is an automorphism of C, it is straightforward to compute its action on each element of such a basis. What we have done so far gives us the possibility of writing a closed formula for $P_{\mathcal{H}_g}^{CR}$ for fixed $g \geq 2$.

Theorem 5.1. *The orbifold Poincaré polynomial of moduli of smooth hyperelliptic curves is given by the formula:*

$$P_{\mathcal{H}_g}^{CR}(q) = \sum_{(k,N,i)\in A_{2g+2}} q^{a_g(i,N)} P^0_{k+2;k,2}(q) + \sum_{(k,N,i)\in A_{2g+1}} 2q^{b_g(i,N)} P^0_{k+2;k,1,1}(q) +$$

$$+ \sum_{(k,N,i)\in A_{2g}} q^{b_g(i,N)} P^0_{k+2;k,2}(q) + 2 + \begin{cases} q^{\frac{g-1}{2}} P^0_{g+3;g+1,1,1}(q) + q^{\frac{g}{2}} P^0_{g+2;g,1,1}(q) & \text{if } g \text{ is even} \\ 2q^{\frac{g-1}{2}} P^0_{g+3;g+1,2}(q) + 2q^{\frac{g}{2}} P^0_{g+2;g,2}(q) & \text{if } g \text{ is odd} \end{cases}$$

Where the sets of indices are defined as:

$$A_n := \left\{ (k,N,i) \in \mathbb{N}^2 \times \mathbb{Z}_N^* |\ N > 2,\ kN = n \right\}$$

and the exponents are:

$$a_g(i,N) := 2\left(2g-1-\sum_{j=1}^{2g-1}\left\{\frac{i(j+1)}{N}\right\}\right)$$

$$b_g(i,N) := 2\left(2g-1-\sum_{j=1}^{2g-1}\left\{\frac{ij}{N}\right\}\right)$$

Remark 5.2. By substituting all the Poincaré polynomials on the right hand side of Theorem 5.1 with 1, one gets the "polynomial" whose coefficients in degree $i \in \mathbb{Q}^{\geq 0}$ are the dimensions of the stringy Chow group of degree i (cfr. Remark 2.8). This is so because all the twisted sectors of $\mathcal{H}_g$ have trivial Chow group, since their coarse moduli spaces are quotients of affine sets.

In particular, we can write closed formulas for the total dimensions of the orbifold cohomology of $\mathcal{H}_g$. Let us define:

$$h^h_{CR}(g) := \dim H^*_{CR}(\mathcal{H}_g)$$

We denote with ϕ the Euler totient function. Then we can give a corollary of Theorem 4.2:

Corollary 5.3. *The following explicit formula for the function just introduced hold:*

(1) If g is even, $n = 2g+2$:

$$h^g_{CR}(n) = 3 + 2g + 2\sum_{N>2|\ n=kN+1} k\phi(N) + 2\sum_{N>2|\ n=kN, or\ n=kN+2} \lfloor\frac{k-2}{4}\rfloor\phi(N)$$

(2) If g is odd, $n = 2g+2$:

$$h^g_{CR}(n) = 2+4\left(\lfloor\frac{n-2}{4}\rfloor + \lfloor\frac{n-1}{4}\rfloor\right)+2\sum_{N>2|\ n=kN+1} k\phi(N)+2\sum_{N>2|\ n=kN, or\ n=kN+2} \lfloor\frac{k-2}{4}\rfloor\phi(N)$$

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