RULED AUSTERE SUBMANIFOLDS OF DIMENSION FOUR

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ABSTRACT. We classify 4-dimensional austere submanifolds in Euclidean space ruled by 2-planes. The algebraic possibilities for second fundamental forms of austere 4-folds M were classified by Bryant, falling into three types which we label A, B, and C. We show that if M is 2-ruled of Type A, then the ruling map from M into the Grassmannian of 2-planes in $\mathbb{R}^n$ is holomorphic, and we give a construction for M starting with a holomorphic curve in an appropriate twistor space. If M is 2-ruled of Type B, then M is either a generalized helicoid in $\mathbb{R}^6$ or the product of two classical helicoids in $\mathbb{R}^3$. If M is 2-ruled of Type C, then M is either a one of the above, or a generalized helicoid in $\mathbb{R}^7$. We also construct examples of 2-ruled austere hypersurfaces in $\mathbb{R}^5$ with degenerate Gauss map.

1. INTRODUCTION

A submanifold in Euclidean space $\mathbb{R}^n$ is *austere* if the eigenvalues of its second fundamental form, in any normal direction, are symmetrically arranged around zero. Thus, austere submanifolds are a special case of minimal submanifolds. Harvey and Lawson [4] showed that austere submanifolds can be used to construct special Lagrangian submanifolds in $T^*\mathbb{R}^n \cong \mathbb{C}^n$ which are ruled by their intersections with the fibres. Austere submanifolds of dimension 3 were completely classified by Bryant [1], but the case of dimension 4 is open. In this paper we are concerned with classifying austere submanifolds of dimension 4 which are themselves ruled by 2-planes. (As we will see below in Corollary 5, austere 4-manifolds ruled by 3-planes are easy to classify, while on the other hand we expect the classification of austere submanifolds ruled by lines to be much more difficult.)

One family of ruled austere submanifolds defined in [1] are simple to describe. A *generalized helicoid* $M \subset \mathbb{R}^n$ is an m-dimensional submanifold swept out by s-planes rotating around a fixed axis, with $n = m + s$ and $s < m$. It can be parametrized by

$$(x_0, \ldots, x_{m-1}) \mapsto$$
$$(\lambda_0 x_0, x_1 \cos(\lambda_1 x_0), x_1 \sin(\lambda_1 x_0), \ldots, x_s \cos(\lambda_s x_0), x_s \sin(\lambda_s x_0), x_{s+1}, \ldots, x_{m-1}), \quad (1)$$

where $\lambda_0, \ldots, \lambda_s$ are constants with $\lambda_1, \ldots, \lambda_s$ nonzero. Of course, this gives the classical helicoid minimal surface when $s = 1$ and $m = 2$. Notice also that if $m > s + 1$ then this splits as a product of a Euclidean factor with an 'irreducible' helicoid swept out by s-planes in $\mathbb{R}^{2s+1}$. Furthermore, when $\lambda_0 = 0$, M is a cone over an austere submanifold in S^{m-1} which is ruled by $(s - 1)$-dimensional totally geodesic spheres.

Helicoids (classical and generalized) will play a significant role in the latter part of this paper. Among austere submanifolds, they have the following characterization:

Theorem (Bryant [1])**.** *If $M \subset \mathbb{R}^n$ is austere, and $|\mathrm{II}|$ is simple (i.e., at each point $p \in M$ the quadratic forms in $|\mathrm{II}_p|$ share a common linear factor) then M is congruent to a generalized helicoid.*

Here, $|\mathrm{II}_p| \subset S^2 T^* M$ is the subspace spanned by the second fundamental form II in various normal directions. In more detail, we recall that the second fundamental form of a submanifold $M \subset \mathbb{R}^n$ is a tensor defined by

$$D_X Y = \nabla_X Y + \mathrm{II}(X, Y),$$

where X, Y are tangent vector fields on M, D is the Euclidean connection on $\mathbb{R}^n$ and the right-hand side is split into the tangential and normal components. Thus, II is a section of $S^2 T^* M \otimes NM$, where NM is the normal bundle, and $|\mathrm{II}_p|$ is the image of II under contraction with a basis for $N_p^* M$. Let δ denote $\dim |\mathrm{II}_p|$, which will be constant on an open dense set of each connected component of M. We will assume M is connected, and refer to the constant δ as the *normal rank* of M.

Using an orthonormal basis to identify $T_p M$ with $\mathbb{R}^4$, we see that $|\mathrm{II}_p|$ must correspond to a subspace of the space $S^2 \mathbb{R}^4$ of 4×4 symmetric matrices which is an *austere subspace*, in which every matrix has eigenvalues symmetrically arranged around zero. The *maximal austere subspaces* $\mathcal{Q} \subset S^2 \mathbb{R}^4$ were determined (up to $O(4)$ conjugation) by Bryant [1], and fall into three types which we have labeled as $\mathcal{Q}_A$, $\mathcal{Q}_B$ and $\mathcal{Q}_C$. We say that an austere submanifold is M is of type A,B,C according to which one of these maximal subspaces $|\mathrm{II}_p|$ lies in, given an appropriate choice of orthonormal basis for $T_p M$. We make the blanket assumption that the type is constant on an open subset of M, with the exception of the case where $\delta = 1$. In that case, the notion of 'type' breaks down, as we can describe all 1-dimensional austere subspaces of $S^2 \mathbb{R}^4$ up to conjugation, using diagonalization.

We now summarize our results. The particular forms for $\mathcal{Q}_A, \mathcal{Q}_B, \mathcal{Q}_C$ will be given later; for now, we recall that austere submanifolds of Type A carry a complex structure, denoted by J, with respect to which the metric on M is Kähler whenever $\delta \geq 2$ (cf. Corollary 7 in [5]). In each of the following theorems, we assume that M is a austere submanifold in $\mathbb{R}^n$, ruled by 2-planes, and $E_p \subset T_p M$ denotes the 2-dimensional subspace tangent to the ruling.

Theorem 1. *If M is Type A, then $E = \mathsf{J}(E)$, $\delta \leq 4$, and the ruling map $\gamma_E : M \to \mathrm{Gr}(2, n)$ is holomorphic. If $\delta = 2$ or $\delta = 4$, then there is a compact complex manifold V which fibers over $\mathrm{Gr}(2, n)$, and a holomorphic mapping $\Gamma : M \to V$, which factors γ_E, and whose image is a complex curve $\mathcal{C}$. Moreover, any generic curve $\mathcal{C}$ is locally the image of an austere submanifold M with these properties.*

It is important to note that, when n is even, the submanifolds M here are not in general holomorphic submanifolds of $\mathbb{C}^{n/2}$; our analysis shows that the space of local solutions is too large for this to be the case.

The particular complex manifolds V are defined in §3 below for various values of δ. We refer to these as 'twistor spaces', with the justification that they are lower-dimensional homogeneous spaces in which we can take solutions of well-known or canonical systems of PDE (e.g., the Cauchy-Riemann equations, in the case of holomorphic curves) and use them to construct austere 4-manifolds with the desired properties.

Theorem 2. *If M is Type B, then $\delta \leq 2$. If $\delta = 2$, then M is either a generalized helicoid in $\mathbb{R}^6$, or the product of two classical helicoids in $\mathbb{R}^3$, or is also of Type A. (In the middle case, the product need not be orthogonal.)*

Theorem 3. *If M is Type C and $\delta = 3$ then M is a generalized helicoid in $\mathbb{R}^7$ (corresponding to $s = 3$ in (1)). If $\delta = 2$ then M is also of Type B.*

We will establish these results using the techniques of moving frames and exterior differential systems. After setting up our basic tools in §2, we discuss types A, B, and C in sections 3,4,5 respectively. Finally in §6 we consider the case of 2-ruled austere submanifolds of normal rank one, showing that they must lie in $\mathbb{R}^5$, and providing an analogous twistor construction for this case.

2. The standard system

Let $\mathcal{F}$ be the semi-orthonormal frame bundle whose fiber at a point p consists of all bases $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ of $T_p\mathbb{R}^n$ such that the vectors $(\mathbf{e}_i)_{i=1\ldots4}$ are orthonormal and orthogonal to the $(\mathbf{e}_a)_{a=5\ldots n}$. For the rest of the paper, we will use the index ranges $1 \leq i, j, k \leq 4$ and $5 \leq a, b, c \leq n$. We choose adapted frames $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ along a submanifold $M^4 \subset \mathbb{R}^n$ such that $\mathbf{e}_1(p), \cdots, \mathbf{e}_4(p)$ are an orthonormal basis of the tangent space T_pM for each point $p \in M$. Note that we will not necessarily choose the vectors $\mathbf{e}_a$ to be orthonormal, as later on we will adapt them accordingly. The frame vectors are regarded as $\mathbb{R}^n$-valued functions on the frame $\mathcal{F}$. We let

$$d\mathbf{p} = \mathbf{e}_i\omega^i + \mathbf{e}_a\omega^a,$$
$$d\mathbf{e}_i = \mathbf{e}_j\omega_i^j + \mathbf{e}_a\omega_i^a \qquad 1 \leq i, j \leq 4, \quad 5 \leq a, b \leq n, \tag{2}$$
$$d\mathbf{e}_a = \mathbf{e}_j\omega_a^j + \mathbf{e}_b\omega_a^b,$$

define the canonical 1-forms ω^i, ω^a and the connection 1-forms $\omega_i^j, \omega_i^a, \omega_a^i$ and ω_b^a on $\mathcal{F}$. This forms span the cotangent space of $\mathcal{F}$ at each point, but they are not linearly independent. Differentiating the equations $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ and $\mathbf{e}_i \cdot \mathbf{e}_a = 0$ yields the relations

$$\omega_i^j = -\omega_j^i, \qquad \omega_a^i = -\omega_i^b g_{ba}, \tag{3}$$

where $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$. Differentiating the first line of (2), we obtain the structure equations

$$d\omega^i = -\omega_j^i \wedge \omega^j + \omega_i^b g_{ba} \wedge \omega^a$$
$$d\omega^a = -\omega_i^a \wedge \omega^i - \omega_b^a \wedge \omega^b \tag{4}$$

Differentiating the last two equation of (2) yields

$$d\omega_j^i = -\omega_k^i \wedge \omega_j^k + \omega_i^b g_{ba} \wedge \omega_j^a,$$
$$d\omega_i^a = -\omega_j^a \wedge \omega_i^j - \omega_b^a \wedge \omega_i^b, \tag{5}$$
$$d\omega_b^a = \omega_i^a \wedge \omega_i^c g_{cb} - \omega_a^c \wedge \omega_b^c$$

along with

$$dg_{ab} = g_{ac}\omega_b^c + g_{bc}\omega_a^c. \tag{6}$$

An adapted frame along a submanifold $M \subset \mathbb{R}^n$ gives a section of $\mathcal{F}|_M$. The following fundamental fact characterizes these sections:

A submanifold $\Sigma^4 \subset \mathcal{F}$ is the image of an adapted frame along the submanifold $M^4 \subset \mathbb{R}^{4+\delta}$ if and only if $\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4|_\Sigma \neq 0$ and $\omega^a|_\Sigma = 0$.

The first of these conditions is a non-degeneracy assumption called the *independence condition*. The second condition implies, by differentiation, that $\omega_i^a|_\Sigma = S_{ij}^a\omega^j$ for some functions S_{ij}^a. These functions give the components of the second fundamental form in this frame, i.e.,

$$\mathrm{II}(\mathbf{e}_i, \mathbf{e}_j) = S_{ij}^a\mathbf{e}_a. \tag{7}$$

In this paper we are classifying the austere 4-folds of type A, B and C which are ruled by 2-planes. We will now describe a class of exterior differential systems, for later use, whose integral submanifolds are adapted frames along 2-ruled austere submanifolds. (Being an integral submanifold of an EDS $\mathcal{I}$ means that the pullback to the submanifold of any 1-forms in $\mathcal{I}$ is zero).

First, suppose we wish to construct an austere submanifold M^4 in $\mathbb{R}^n$ such that at each point p, $|\,\mathrm{II}_p\,|$ is conjugate to an austere subspace $\mathcal{Q}_\lambda$ of dimension δ (i.e, M is of type $\mathcal{Q}$) depending on parameters $\lambda^1, \ldots, \lambda^\ell$ which are allowed to vary along M. Let the symmetric matrices $S^5(\lambda), \ldots, S^{\delta+4}(\lambda)$ be a basis for the subspace $\mathcal{Q}_\lambda$ and suppose the parameters are allowed to range over an open set $L \subset \mathbb{R}^\ell$. Let

$$\theta_i^a := \omega_i^a - S_{ij}^a(\lambda)\omega^j. \tag{8}$$

Then on $\mathcal{F} \times L$ we define the Pfaffian exterior differential system

$$\mathcal{H} = \{\omega^a, \theta_i^a\}$$

whose integral submanifolds correspond to austere manifolds of type $\mathcal{Q}_\lambda$. Namely, given any austere manifold M of this kind, we may construct an adapted frame along M such that

$$\mathrm{II}(\mathbf{e}_i, \mathbf{e}_j) = S_{ij}^a(\lambda)\mathbf{e}_a \tag{9}$$

for functions $\lambda^1, \ldots, \lambda^\ell$ on M. Then the image of the fibered product of the mappings $p \mapsto (p, \mathbf{e}_i(p), \mathbf{e}_a(p))$ and $p \mapsto (\lambda^1(p), \ldots, \lambda^\ell(p))$ will be an integral submanifold of $\mathcal{H}$. Conversely, any integral submanifold of $\mathcal{H}$ satisfying the independence condition gives (by projecting onto the first factor in $\mathcal{F} \times L$) a section of $\mathcal{F}|_M$ which is an adapted frame for an austere manifold M.

Now consider the additional condition that M is ruled by k-dimensional planes. We will let $E \subset TM$ denote a smooth distribution on M with fiber E_p at $p \in M$. The following result characterizes those distributions that are tangent to a ruling of M:

Proposition 4. *Let E be an arbitrary smooth k-plane field on M. Then M is ruled by k-planes in $\mathbb{R}^n$ tangent to E if and only if at each $p \in M$*

$$\mathsf{D}_\mathsf{v}\mathsf{w} \in E \quad \forall\, \mathsf{v} \in E_p, \mathsf{w} \in C^\infty(E), \tag{10}$$

where D denotes the Euclidean connection in $\mathbb{R}^n$.

Projecting both sides of (10) into the normal bundle, we get

$$\mathrm{II}(\mathsf{v}, \mathsf{w}) = 0 \quad \forall\, \mathsf{v}, \mathsf{w} \in E. \tag{11}$$

and we see from (9) that this puts extra conditions on S_{ij}^a. In fact, for $k = 3$ the condition (11) implies the following:

Corollary 5. *Any 4-dimensional austere submanifold that is ruled by 3-planes is a generalized helicoid.*

Proof. If we choose an orthonormal frame in which $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ spans the ruling, then (11) implies that $|\,\mathrm{II}\,|$ lies in the space of matrices with nonzero entries only in the fourth row and column. Thus, $|\,\mathrm{II}\,|$ is simple, and the result follows by Bryant's theorem quoted in §1. $\square$

From now on we will consider submanifolds ruled by 2-planes. We will ensure that the Pfaffian system $\mathcal{H}$ encodes the condition (11) by assuming that E has a certain basis with respect to the orthonormal frame on M, and choosing the space $\mathcal{Q}_\lambda$ so that this condition holds for those basis vectors. However, encoding the *tangential* part of (10) requires additional 1-form generators.

Suppose that $\mathsf{v}_1, \mathsf{v}_2$ are vector fields spanning E, ϕ_1, ϕ_2 are 1-forms that annihilate E, and $\mathsf{w}_1, \mathsf{w}_2$ are vector fields that span the orthogonal complement of E at each point in an open set $U \subset M$. Then the tangential part of (10) is equivalent to

$$\mathsf{w}_i \cdot \nabla \mathsf{v}_j \equiv 0 \mod \phi_1, \phi_2 \tag{12}$$

for all $1 \le i, j \le 2$. (Here, ∇ denotes the Riemannian connection of M, so that $\nabla \mathsf{v}_i$ is a $(1,1)$ tensor on U.) We may encode this condition by defining 1-forms

$$\psi_j^i := \mathsf{w}_i \cdot \nabla \mathsf{v}_j - p_{jk}^i \phi_k, \tag{13}$$

for arbitrary coefficients p_{jk}^i. (We will give the specific forms for the ψ_j^i in later sections.)

Thus, we define the *ruled austere system* $\mathcal{I}$ as the Pfaffian system generated by the forms ω^a and $\theta_i^a = \omega_i^a - S_{ij}^a(\lambda)\omega^j$ from $\mathcal{H}$, plus the extra 1-forms ψ_j^i. The integral submanifolds of this differential ideal

$$\mathcal{I} = \{\omega^a, \theta_i^a, \psi_j^i\} \tag{14}$$

correspond to 2-ruled austere manifolds of type $\mathcal{Q}_\lambda$. The system $\mathcal{I}$ is in general defined on the manifold $\mathcal{F} \times L \times \mathbb{R}^8$, due to the introduction of the p_{jk}^i as new variables. However, for specific types we will restrict to submanifolds of $\mathcal{F} \times L \times \mathbb{R}^8$ on which certain necessary integrability conditions hold.

3. Type A

Recall that an austere 4-fold of Type A carries a complex structure J. We take orthonormal frames on M with respect to which the complex structure is represented by

$$J = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The maximal austere subspace $\mathcal{Q}_A$ consists of symmetric matrices which anticommute with J. This subspace is preserved by the action of $U(2) \subset SO(4)$, the group of orthogonal matrices commuting with J. In Corollary 7 of [5] we showed that any austere 4-fold of Type A with $\delta \ge 2$ is Kähler with respect to J. Consequently, J is parallel along M and the connection forms must satisfy

$$\omega_2^4 = \omega_1^3, \qquad \omega_1^4 = -\omega_2^3. \tag{15}$$

Suppose a type A austere submanifold is 2-ruled. Then $E + \mathsf{J}(E)$ is J-invariant, so is either 4- or 2-dimensional. We treat these cases separately.

Case A.1: $E = \mathsf{J}(E)$. Let M be a 2-ruled austere submanifold of this type. Using the $U(2)$ symmetry, we can choose near each point a semiorthonormal frame with respect to which $|\mathrm{II}|$ lies in $\mathcal{Q}_A$ and E is spanned by $\mathbf{e}_1, \mathbf{e}_2$. Thus, $|\mathrm{II}|$ must lie inside the subspace

$$\mathcal{R} = \left\{ \begin{bmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & b_1 & -a_1 \\ a_1 & b_1 & a_2 & b_2 \\ b_1 & -a_1 & b_2 & -a_2 \end{bmatrix} \right\}, \tag{16}$$

consisting of matrices in $\mathcal{Q}_A$ satisfying the algebraic condition (11). (Thus, $\delta \le 4$.) Moreover, the ruled condition (12), together with (15), implies that

$$\omega_1^3 = \omega_2^4 = u\omega^3 + v\omega^4, \qquad \omega_2^3 = -\omega_1^4 = x\omega^3 + y\omega^4, \tag{17}$$

for some functions u, v, x, y.

Proposition 6. *The map $\gamma_E : M \to \mathrm{Gr}(2, n)$, taking $p \in M$ to the subspace parallel to E_p, is holomorphic.*

Proof. Define a map $\pi : \mathcal{F} \to \mathrm{Gr}(2, n)$ that takes $(p, \mathbf{e}_1, \ldots, \mathbf{e}_n)$ to $\{\mathbf{e}_1, \mathbf{e}_2\}$; then γ_E is locally the composition of π with the lift $f : M \to \mathcal{F}$ provided by the frame. The pullbacks under π of the $(1,0)$-forms on $\mathrm{Gr}(2, n)$ are spanned by $\omega_1^3 - i\omega_2^3, \ldots, \omega_1^n - i\omega_2^n$ (see [7], Chapter XI, Example 10.6). Meanwhile, a basis for the $(1, 0)$-forms on M is given by

$$\tau := \omega^1 + i\omega^2, \qquad \zeta := \omega^3 + i\omega^4. \tag{18}$$

For $a > 4$, the connection forms for the framing satisfy $\omega_i^a = S_{ij}^a \omega^j$ for some matrices S^a taking value in $\mathcal{R}$. We then compute that $\omega_i^a - i\omega_2^a = (S_{13}^a - iS_{14}^a)\zeta$ along the lift of M into $\mathcal{F}$. Thus, $f^*(\omega_1^a - i\omega_2^a)$ is a multiple of ζ for each $a > 4$. We must also show that this is also true for the pullbacks of $\omega_1^3 - i\omega_2^3$ and $\omega_1^4 - i\omega_2^4$.

The subgroup $U(1) \times U(1)$ preserves $\mathcal{R}$. Given any nontrivial subspace of $\mathcal{R}$, we may use this symmetry to arrange that the subspace contains a matrix of the form

$$\begin{bmatrix} 0 & 0 & p & 0 \\ 0 & 0 & 0 & -p \\ p & 0 & 0 & q \\ 0 & -p & q & 0 \end{bmatrix} \tag{19}$$

for p, q not both zero. Thus, along M we can adapt a semiorthonormal frame such that, say, S^5 has the form (19).

The 1-forms θ_i^a, defined by (8), vanish on $\Sigma = f(M)$. Computing their exterior derivatives gives

$$-d\theta_i^a \equiv (dS_{ij}^a - S_{ik}^a \omega_j^k - S_{kj}^a \omega_i^k + S_{ij}^b \omega_b^a) \wedge \omega^j \qquad \mathrm{mod}\ \omega^a, \theta_i^a.$$

In particular, using (17) we have

$$0 = (-d\theta_1^5 + id\theta_2^5) \wedge \zeta = 2p(x - v + i(y + u))\tau \wedge \omega^3 \wedge \omega^4.$$

The 3-form $\tau \wedge \omega^3 \wedge \omega^4$ is nonvanishing since Σ satisfies the independence condition. Thus, wherever $p \neq 0$ we have

$$v = x, \qquad u = -y. \tag{20}$$

Even if p is identically zero on an open set in Σ, we can compute

$$0 = (-d\theta_3^5 + id\theta_4^5) \wedge \zeta = 2q(u + y + i(v - x))\tau \wedge \omega^3 \wedge \omega^4,$$

again showing that (20) must hold along Σ. Substituting (20) into (17) shows that

$$\omega_1^3 - i\omega_2^3 = -i(x - iy)\zeta, \qquad \omega_1^4 - i\omega_2^4 = -(x - iy)\zeta.$$

Thus, γ_E is holomorphic. $\qquad \square$

In the remainder of this subsection, we will analyze the solutions of $\mathcal{I}$ for each possible value of δ. For the 'extra' 1-form generators defined by (13), we may take $\phi_1 = \omega^3$ and $\phi_2 = \omega^4$. In addition, because of the integrability condition (20) holds on all solutions, we will use

$$\begin{aligned} \psi_1^1 &= \omega_1^3 + y\omega^3 - x\omega^4, & \psi_2^1 &= \omega_2^3 - x\omega^3 - y\omega^4, \\ \psi_1^2 &= \omega_1^4 + x\omega^3 + y\omega^4, & \psi_2^2 &= \omega_2^4 + y\omega^3 - x\omega^4. \end{aligned} \tag{21}$$

(a) **$\delta = 2$.** The analysis in section 5 of [5] shows that when M is an austere 4-fold of Type A, with $\delta = 2$, then $|\mathrm{II}|$ has a two-dimensional nullspace E only if either the Gauss map is degenerate, or M lies in $\mathbb{R}^6$ and belongs to the set of ruled submanifolds of "type 2.b" in the

terminology of [5]. In the first case, M belongs to the class of *elliptic* austere submanifolds investigated by Dajczer and Florit [3].

In the second case, we may choose an orthonormal frame $(\mathbf{e}_1, \ldots, \mathbf{e}_6)$ along M such that

$$\mathbf{e}_5 \cdot \mathrm{II} = \begin{bmatrix} 0 & 0 & r & 0 \\ 0 & 0 & 0 & -r \\ r & 0 & 0 & b \\ 0 & -r & b & 0 \end{bmatrix}, \quad \mathbf{e}_6 \cdot \mathrm{II} = \begin{bmatrix} 0 & 0 & 0 & r \\ 0 & 0 & r & 0 \\ 0 & r & a & 0 \\ r & 0 & 0 & -a \end{bmatrix}$$

for some functions a, b, r on M. By regarding the frame vectors as columns in a matrix, we define a mapping $\widehat{\gamma} : M \to SO(6)/U(3)$ which is rank 2 and holomorphic. (See Theorem 15 in [5]; note also that $V = SO(6)/U(3)$ is biholomorphic to $\mathbb{CP}^3$.) Conversely, given a generic holomorphic curve $\mathcal{C}$ in the twistor space V we may, by solving a first-order system of PDE, construct an austere submanifold $M \subset \mathbb{R}^6$ of this type, such that $\widehat{\gamma}(M) = \mathcal{C}$ (see Theorem 16 in [5] for more details).

(b) $\boldsymbol{\delta = 3}$. Using the subgroup $U(1) \times U(1)$ preserving $\mathcal{R}$, we can choose a semiorthonormal frame along M so that (9) holds with

$$S^5 = \begin{bmatrix} 0 & 0 & p & 0 \\ 0 & 0 & 0 & -p \\ p & 0 & q & 0 \\ 0 & -p & 0 & -q \end{bmatrix}, \quad S^6 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad S^7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

for some functions p, q on M, and $S^a = 0$ for $a > 7$. In fact, one can calculate that the prolongation of $|\mathrm{II}|$ has dimension zero; thus, without loss of generality we can assume that δ is the effective codimension, and M lies in $\mathbb{R}^7$ (see Prop. 3 in [5]).

We analyze the structure equations of the standard system $\mathcal{I}$ with 'extra' 1-forms given by (21). In particular, we compute

$$d(\psi_1^1 - \psi_2^2) \equiv 2pg_{56}(\omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3) + (g_{66} - p^2 g_{55})(\omega^2 \wedge \omega^4 - \omega^1 \wedge \omega^3)$$

modulo $\theta^a, \theta_i^a, \psi_j^i$. Thus, integral elements satisfying the independence condition exist only at points where $g_{56} = 0$ and $g_{66} = p^2 g_{55}$ (with p nonzero since the matrix g_{ab} must be positive definite). When we restrict to the submanifold where these integrability conditions hold, the system $\mathcal{I}$ becomes involutive, with last nonzero Cartan character $s_1 = 12$. In what follows, we will show how such submanifolds can be constructed by beginning with a holomorphic curve in a homogeneous complex manifold.

First, note that the integrability conditions imply that we may construct an orthonormal frame $(\mathbf{e}_1, \ldots, \mathbf{e}_4, \tilde{\mathbf{e}}_5, \tilde{\mathbf{e}}_6, \tilde{\mathbf{e}}_7)$ along M such that $\tilde{\mathbf{e}}_5$ and $\tilde{\mathbf{e}}_6$ are multiples of $\mathbf{e}_5$ and $\mathbf{e}_6$, respectively, and with respect to this frame the second fundamental form is represented by

$$\tilde{S}^5 = \begin{bmatrix} 0 & 0 & p & 0 \\ 0 & 0 & 0 & -p \\ p & 0 & q & a \\ 0 & -p & a & -q \end{bmatrix}, \quad \tilde{S}^6 = \begin{bmatrix} 0 & 0 & 0 & p \\ 0 & 0 & p & 0 \\ 0 & p & 0 & b \\ p & 0 & b & 0 \end{bmatrix}, \quad \tilde{S}^7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & c & 0 \end{bmatrix}$$

for some functions a, b, c, p, q with $c, p \neq 0$. (Note that p, q have been scaled up by the length of the original vector $\mathbf{e}_5$.) For convenience, we drop the tildes from now on. Let $\mathcal{I}_{\mathrm{on}}$ denote the Pfaffian system on $\mathcal{F}_{\mathrm{on}} \times L$ for which this orthonormal gives an integral submanifold; that is, $\mathcal{I}_{\mathrm{on}}$ is generated by ω^a for $a = 5, 6, 7$, $\theta_i^a := \omega_i^a - S_{ij}^a \omega^j$ with S^a given by (3), and ψ_j^i. (Here, L is the parameter space with coordinates a, b, c, p, q, x, y.)

At each point $p \in M$, let F_p be the 6-dimensional oriented subspace in $\mathbb{R}^7$ spanned by the oriented basis $\mathbf{e}_1, \ldots, \mathbf{e}_6$, and let $\widehat{\mathbf{J}}_p$ be the complex structure on F_p taking $\mathbf{e}_1$ to $\mathbf{e}_2$, $\mathbf{e}_3$ to $\mathbf{e}_4$ and $\mathbf{e}_5$ to $\mathbf{e}_6$. (This extends the complex structure on $T_p M$.) Define V as the space of triples $(E, F, \widehat{\mathbf{J}})$ where E, F are oriented subspaces of $\mathbb{R}^7$ and $\widehat{\mathbf{J}}$ is an orthogonal complex structure on F preserving E. (By adapting orthonormal frames to each triple $(E, F, \widehat{\mathbf{J}})$, we identify V with the homogeneous space $SO(7)/SO(2) \times U(2)$.) Then we define a smooth mapping $\Gamma : M \to V$ sending p to $(E_p, F_p, \widehat{\mathbf{J}}_p)$.

We will give V a complex structure so that Γ is holomorphic. To this end, define the following complex-valued 1-forms on $SO(7)$:

$$\eta^p := \omega_1^p - \mathrm{i}\omega_2^p, \quad \nu^p := \omega_3^p - \mathrm{i}\omega_4^p \quad \sigma := \omega_5^7 - \mathrm{i}\omega_6^7.$$

Then

$$\eta^3, \ldots, \eta^7, \nu^5 - \mathrm{i}\nu^6, \nu^7, \sigma \tag{22}$$

are semibasic for the quotient map $\pi : SO(7) \to V$, and in fact these span the pullback to $SO(7)$ of the bundle of $(1,0)$-forms for the complex structure on V. To see that Γ is holomorphic, we express the generators of $\mathcal{I}_{\mathrm{on}}$ in terms of the complex 1-forms:

$$
\begin{aligned}
\psi_1^1 - \mathrm{i}\psi_2^1 &= \eta^3 + (y + \mathrm{i}x)\zeta, & \psi_1^2 - \mathrm{i}\psi_2^2 &= \eta^4 + (x - \mathrm{i}y)\zeta, \\
\theta_1^5 - \mathrm{i}\theta_2^5 &= \eta^5 - p\zeta, & \theta_3^5 - \mathrm{i}\theta_4^5 &= \nu^5 - p\tau + (\mathrm{i}a - q)\zeta, \\
\theta_1^6 - \mathrm{i}\theta_2^6 &= \eta^6 + \mathrm{i}p\zeta, & \theta_3^6 - \mathrm{i}\theta_4^6 &= \nu^6 + \mathrm{i}p\tau + \mathrm{i}b\zeta, \\
\theta_1^7 - \mathrm{i}\theta_2^7 &= \eta^7, & \theta_3^7 - \mathrm{i}\theta_4^7 &= \nu^7 + \mathrm{i}c\zeta,
\end{aligned}
\tag{23}
$$

where τ, ζ are as in (18). It follows that the pullbacks of the $(1,0)$-forms on V, when restricted to integral submanifolds of $\mathcal{I}_{\mathrm{on}}$, are multiples of the $(1,0)$-form ζ on M.

The image $\mathcal{C}$ of $\Gamma : M \to V$ is not a generic holomorphic curve; in fact, it is an integral of a well-defined complex rank 3 Pfaffian system on V. To see this, first note that among the semibasic forms listed in (22), the following subsystem vanishes on M:

$$\widehat{\mathcal{K}} = \{\eta^3 - \mathrm{i}\eta^4, \eta^5 - \mathrm{i}\eta^6, \eta^7\}.$$

Then one calculates that

$$
\begin{aligned}
d(\eta^3 - \mathrm{i}\eta^4) &\equiv (\nu^5 - \mathrm{i}\nu^6) \wedge \eta^5, \\
d(\eta^5 - \mathrm{i}\eta^6) &\equiv -(\nu^5 - \mathrm{i}\nu^6) \wedge \eta^3, \\
d\eta^7 &\equiv \eta^3 \wedge \nu^7 + \eta^5 \wedge \sigma
\end{aligned}
$$

modulo the 1-forms of $\widehat{\mathcal{K}}$. The fact that the right-hand sides are pure wedge products of semibasic forms for π indicates that $\widehat{\mathcal{K}}$ is the pullback of a well-defined Pfaffian system $\mathcal{K}$ on V (see [6], Prop. 6.1.19).

Remark 7. The Pfaffian system $\mathcal{K}$ may be characterized in two ways.

First, note that V has the structure of a double fibration over more familiar complex homogeneous spaces, the mappings $\rho_1 : V \to \mathrm{Gr}(2, \mathbb{R}^7)$ and $\rho_2 : V \to SO(7)/U(3)$ being induced by the inclusions $SO(2) \times U(2) \subset SO(2) \times SO(5)$ and $SO(2) \times U(2) \subset U(3)$ respectively. Then $\mathcal{K}$ is spanned by the intersection of the pullbacks of $(1,0)$-forms via ρ_1 and the pullbacks of $(1,0)$-forms via ρ_2.

Next, recall that the tangent space to $\mathrm{Gr}(2, \mathbb{R}^n)$ at E is naturally identified with $E^* \otimes \mathbb{R}^n / E$. Thus, for any tangent vector $\mathbf{v} \in T_p M$, $\gamma_* \mathbf{v}$ is an element of $(E_p)^* \otimes \mathbb{R}^7 / E_p$. However, the

form of the matrices S^a in this case, and the vanishing of the 1-forms ψ^i_j on M, imply that $\Gamma : M \to V$ satisfies the following

> *Contact Condition*: For any $\mathsf{v} \in T_pM$, $\Gamma_*(\mathsf{v})$ takes value in $(E_p)^* \otimes F_p/E_p$ and its value, as a mapping, is complex-linear with respect to $\widehat{\mathbf{J}}_p$.

In fact, any holomorphic mapping into V is an integral of $\mathcal{K}$ if and only if it satisfies this contact condition.

Theorem 8. *Let $\mathcal{C}$ be a holomorphic curve in V which is an integral of $\mathcal{K}$, and has a nonsingular projection onto $\mathrm{Gr}(2, \mathbb{R}^7)$. Given any point $q \in \mathcal{C}$ there is an open neighborhood $U \subset \mathcal{C}$ containing q and an austere $M^4 \subset \mathbb{R}^7$ which is Type A, 2-ruled with $\mathsf{J}(E) = E$, such that $\Gamma(M) = U$.*

Proof. Let $N \subset SO(7)$ be the inverse image of $\mathcal{C}$. Let z be a local holomorphic coordinate defined on open set $U \subset \mathcal{C}$. Then there are functions $f_1, \ldots, f_5$ on $\rho_1^{-1}(U) \subset N$ such that

$$\eta^3 = f_1 dz, \quad \eta^5 = f_2 dz, \quad \nu^5 - \mathrm{i}\nu^6 = f_3 dz, \quad \nu^7 = f_4 dz, \quad \sigma = f_5 dz.$$

(By hypothesis, f_1 and f_2 are never both zero.) Comparing these with (23), we see that to construct an integral manifold of $\mathcal{I}_{\mathrm{on}}$ we will need f_2 to be nonzero and f_4/f_2 purely imaginary. By computing the action of $SO(2) \times U(2)$ on the fiber of ρ_1, we see that there is a smooth, codimension-one submanifold $N' \subset N$ where these conditions hold.

Let $W \subset \mathcal{F}_{\mathrm{on}} \times L$ be the inverse image of N' under the projection to $SO(7)$. The restriction of $\mathcal{I}_{\mathrm{on}}$ to W is spanned by the ω^a and the forms on the right-hand sides in (23). However, these forms are now linearly dependent, because $\eta^4 = \mathrm{i}\eta^3$, $\eta^6 = \mathrm{i}\eta^5$ and $\eta^7 = 0$ on W, and the rest satisfy the following relationships:

$$\eta^3 + (y + \mathrm{i}x)\zeta = f_1 dz + (y + \mathrm{i}x)\zeta$$
$$\eta^5 - p\zeta = f_2 dz - p\zeta$$
$$(\nu^5 - p\tau + (\mathrm{i}a - q)\zeta) - \mathrm{i}(\nu^6 + \mathrm{i}p\tau + \mathrm{i}b\zeta) = f_3 dz + (b - q + \mathrm{i}a)\zeta$$
$$\nu^7 + \mathrm{i}c\zeta = f_4 dz + \mathrm{i}c\zeta,$$

In order to have $\zeta \neq 0$ on solutions, we need the linear combinations on the right to be linearly dependent. Thus, we restrict to the submanifold $W' \subset W$ on which

$$y + \mathrm{i}x = -\frac{f_1 p}{f_2}, \qquad b - q + \mathrm{i}a = -\frac{f_3 p}{f_2}, \qquad c = \mathrm{i}\frac{f_4 p}{f_2}. \tag{24}$$

On W', the remaining linearly independent generators of $\mathcal{I}_{\mathrm{on}}$ are the real and imaginary parts of

$$\beta_1 := p\zeta - f_2 dz, \quad \beta_2 := \nu^5 - p\tau - b\zeta - f_3 dz$$

as well as the ω^a, which lie in the first derived system of $\mathcal{I}_{\mathrm{on}}$. Because (24) determines x, y, a, c, q in terms of b, p and the functions of f_j, the remaining linearly independent 1-forms on W' are $\omega^1, \ldots, \omega^4$, dp, db, and two out of the three forms ω^2_1, ω^4_3 and ω^6_5. These last satisfy a linear relation, due to the condition that f_4/f_2 is purely imaginary, which takes the form

$$\omega^2_1 - \omega^4_3 - \omega^6_5 \equiv 0 \mod \beta_1, \beta_2, \omega^1, \ldots, \omega^4. \tag{25}$$

To test the Pfaffian system spanned by β_1, β_2 for involutivity, we compute on W' that

$$\left.\begin{aligned} d\beta_1 &\equiv \pi_1 \wedge \zeta, \\ d\beta_2 &\equiv -\pi_1 \wedge \tau - \pi_2 \wedge \zeta \end{aligned}\right\} \mod \omega^a, \beta^1, \beta^2$$

where

$$\pi_1 \equiv dp + \mathrm{i}p(\omega_5^6 - \omega_3^4 - \omega_1^2) \qquad\qquad \mathrm{mod}\ \tau, \zeta$$

$$\pi_2 \equiv db - 2\mathrm{i}b\omega_3^4 + \mathrm{i}(b + p(f_3/f_2))\omega_5^6 \qquad\qquad \mathrm{mod}\ \zeta, \overline{\zeta}.$$

In view of the relation (25), the real and imaginary parts of π_1, π_2 will be linearly independent provided that the coefficient $b + p(f_3/f_2)$ is nonzero. Accordingly, we restrict our attention to the open set where this is the case. Then the Pfaffian system is involutive with Cartan character $s_1 = 4$. The exist of the integral manifold of $\mathcal{I}_{\mathrm{on}}$ inside W', which projects to the austere submanifold $M \subset \mathbb{R}^7$, follows by application of the Cartan-Kähler Theorem. $\qquad\square$

The starting ingredient in Theorem 8 is an integral curve of system $\mathcal{K}$ on the homogeneous space V. Because $\mathcal{K}$ is generated by holomorphic 1-forms on V, this system is equivalent to an underdetermined system of three ordinary differential equations in suitable local coordinates. Hence, it is possible that explicit solutions could be written down in terms of 4 arbitrary holomorphic functions.

(c) $\boldsymbol{\delta = 4}$. In this case, $|\,\mathrm{II}\,|$ is all of the space $\mathcal{R}$ defined in (16). In this section, we will outline a twistor-type construction for austere submanifolds $M \subset \mathbb{R}^n$ of this type, similar to that described just above. However, because the prolongation of $\mathcal{R}$ is nonzero, we cannot assume that the codimension of M is equal to δ.

Let $\mathcal{I}$ be the standard system with the following choice of basis for $\mathcal{R}$:

$$S^5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, S^6 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, S^7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, S^8 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

As in the $\delta = 3$ case, by differentiating the ψ_j^i we derive integrability conditions; in this case, these imply that $|\mathbf{e}_5| = |\mathbf{e}_6|$ and $\mathbf{e}_5 \cdot \mathbf{e}_6 = 0$. Once these conditions are added to the system, it is involutive with Cartan character $s_1 = 4n - 16$, indicating that solutions depend on a choice of s_1 functions of one real variable.

To understand the twistor construction, we first observe that the integrability conditions imply that to each point $p \in M$ we can associate a 6-dimensional subspace $F_p \subset \mathbb{R}^n$, spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_6$, and an orthogonal complex structure $\widehat{\mathbf{J}}_p$ on F_p, taking $\mathbf{e}_5$ to $\mathbf{e}_6$, which extends the complex structure on T_pM. We may thus define a mapping $\Gamma : M \to V$ sending p to the triple $(E_p, F_p, \widehat{\mathbf{J}}_p)$, a point in the homogeneous space $V = SO(n)/SO(2) \times U(2) \times SO(n-6)$.

Then, exactly as before, the form of II and the vanishing of the 1-forms ψ_j^i imply that Γ satisfies the contact condition from Remark 7. Furthermore, this means that the image of Γ is an integral of a certain complex contact system $\mathcal{K}$ on V.

We define the complex structure and contact system on V as follows. Define the projections $\rho_1 : V \to SO(n)/SO(2) \times SO(n-2)$, the Grassmannian $\mathrm{Gr}(2, n)$, and $\rho_2 : V \to SO(n)/U(3) \times SO(n-6)$. On $SO(n)$, define the complex-valued 1-forms

$$\eta^p := \omega_1^p - \mathrm{i}\omega_2^p, \qquad \nu^p := \omega_3^p - \mathrm{i}\omega_4^p, \qquad \sigma^p = \omega_5^p - \mathrm{i}\omega_6^p.$$

Then the $(1,0)$-forms for the usual complex structure on $\mathrm{Gr}(2, \mathbb{R}^n)$ are precisely those that pull back to $SO(n)$ to be in the span of the η^p for $p > 2$. Likewise, we define a homogeneous complex structure on the flag manifold $SO(n)/U(3) \times SO(n-6)$ by specifying that the $(1,0)$-forms pull back to $SO(n)$ to give the span

$$\{\eta^3 - \mathrm{i}\eta^4, \eta^5 - \mathrm{i}\eta^6, \nu^5 - \mathrm{i}\nu^6, \eta^7, \nu^7, \sigma^7, \ldots, \eta^n, \nu^n, \sigma^n\}.$$

The complex structure on V is uniquely specified by requiring ρ_1, ρ_2 to be holomorphic, and the Pfaffian system $\mathcal{K}$ is defined as the intersection of the pullback of the $(1,0)$-forms under ρ_1 and the pullback of the $(1,0)$-forms under ρ_2. (Thus, $\mathcal{K}$ has complex rank $n-4$.) Moreover, holomorphic integrals of $\mathcal{K}$ are exactly those mappings into V that satisfy the contact condition.

With this machinery in place, we can state the analogue of Theorem 8 for this case.

Theorem 9. *Let $\mathcal{C}$ be a holomorphic curve in V which is an integral of $\mathcal{K}$, and has a nonsingular projection onto $\mathrm{Gr}(2, \mathbb{R}^n)$. Given any point $p \in \mathcal{C}$, there is an open neighborhood $U \subset \mathcal{C}$ containing p and an austere $M^4 \subset \mathbb{R}^n$ which is Type A, 2-ruled satisfying $\mathsf{J}(E) = E$, such that $\Gamma(M) = U$.*

The proof is completely analogous to that of the earlier theorem; at the last stage, the Cartan-Kähler Theorem is required to construct an integral for an involutive system with character $s_1 = 4$.

Case A.2: $E \cap \mathsf{J}(E) = 0$. In this case we can use the $U(2)$ symmetry to choose orthonormal frames so that E is spanned by $\mathbf{e}_1$ and $\mathbf{e}_2 + k\mathbf{e}_3 + m\mathbf{e}_4$ for some functions k, m. Then the subspace $\mathcal{R} \subset \mathcal{Q}_A$ of matrices satisfying (11) is 3-dimensional. One can check that $\mathcal{R}^{(1)} = 0$, so that $\delta = \dim |\,\mathrm{II}\,|$ is the effective codimension of M.

Theorem 15 in [5] implies that any 2-ruled submanifolds of Type A with $\delta = 2$ are those with $\mathsf{J}(E) = E$, discussed in Case A.1(a) above. In the case where $\delta = 3$, an extensive analysis of the exterior differential system $\mathcal{I}$ yields integrability conditions which imply that no such submanifolds exist.

4. Type B Ruled Austere Submanifolds

In this section we discuss the 2-ruled austere 4-folds of Type B. Recall from [5] that the maximal austere subspace of Type B is given by

$$\mathcal{Q}_B = \left\{ \begin{bmatrix} m & 0 & b_1 & b_2 \\ 0 & m & b_3 & b_4 \\ b_1 & b_3 & -m & 0 \\ b_2 & b_4 & 0 & -m \end{bmatrix} \right\}, \tag{26}$$

and may be characterized as the span of a matrix representing a reflection R that fixes a 2-plane in $\mathbb{R}^4$, together with the symmetric matrices that commute with that reflection. Depending on the position of the ruling plane E relative to the eigenspaces of R, the intersection $\mathsf{R}(E) \cap E$ is a vector space of dimension equal to 0, 1 or 2. We examine these cases separately.

Case B.1: $\dim \mathsf{R}(E) \cap E = 2$. In this case, E can be an eigenspace of R or can be a sum of the $+1$ and -1 eigenspaces of R.

(a) E **is a eigenspace of** R. The symmetry group of $\mathcal{Q}_B$ is generated by conjugation by $O(2) \times O(2) \subset O(4)$ and the permutation $\mathbf{e}_1 \leftrightarrow \mathbf{e}_3, \mathbf{e}_2 \leftrightarrow \mathbf{e}_4$. Using the permutation we can assume $E = \{\mathbf{e}_1, \mathbf{e}_2\}$. Since the second fundamental form vanishes on the ruling E, this implies that $m = 0$. Therefore the subspace of $\mathcal{Q}_B$ satisfying the algebraic condition (11) is

$$\mathcal{R} = \left\{ \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \,\middle|\, B \text{ is a } 2 \times 2 \text{ matrix} \right\}.$$

Since $\mathcal{R}^{(1)}$ is zero, the effective codimension of M is equal to δ We break into subcases according to the value of δ. If $\delta = 4$ or $\delta = 3$, an analysis of the EDS $\mathcal{I}$ shows that there are

no integral manifolds satisfying the independence condition. In the case $\delta = 2$ the analysis is more involved, and will be described in what follows.

In this case, $|\,\mathrm{II}\,|$ is a 2-dimensional subspace of $\mathcal{R}$, and we let $\mathcal{B}$ denote the 2-dimensional of the matrices B in the upper right corner. The conjugate action of $O(2) \times O(2)$ on 2×2 matrices is given by $(S, T) \cdot B = SBT^{-1}$ and the determinant is invariant up to sign under this action. We distinguish several subcases depending on the type of the quadratic form det given by the determinant restricted to $\mathcal{B}$:

(a.i) $\mathsf{det}|_{\mathcal{B}}$ **has rank 2**. In this case we normalize the space of second fundamental forms so that $\mathcal{B}$ is spanned by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ q & 0 \end{smallmatrix}\right)$ where $pq < 0$ if det is definite and $pq > 0$ if det is indefinite.

With respect to the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ of the tangent space to M, $|\,\mathrm{II}\,|$ is spanned by the matrices

$$
S^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \end{pmatrix}, \qquad
S^6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & q & 0 \\ 0 & q & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
$$

Let $\mathcal{I}$ be the Pfaffian system, described in §2, whose integral submanifolds correspond to semiorthonormal frames such that

$$
\mathrm{II}(\mathbf{e}_i, \mathbf{e}_j) = S^a_{ij}\mathbf{e}_a, \qquad 1 \le i, j \le 4, \ 5 \le a \le 6 \tag{27}
$$

The forms $\phi_1 = \omega^3, \phi_2 = \omega^4$ represent a basis of the annihilator of E. The ruling condition (12) amounts to requiring that $\omega^3_1, \omega^3_2, \omega^4_1, \omega^4_2 \equiv 0 \mod \{\omega^3, \omega^4\}$. Therefore, the 'extra' 1-forms of $\mathcal{I}$ are

$$
\begin{aligned}
\psi^1_1 &= \omega^3_1 - u_1\omega^3 - u_2\omega^4 \\
\psi^1_2 &= \omega^3_2 - v_1\omega^3 - v_2\omega^4 \\
\psi^2_1 &= \omega^4_1 - x_1\omega^3 - x_2\omega^4 \\
\psi^2_2 &= \omega^4_2 - y_1\omega^3 - y_2\omega^4
\end{aligned}
\tag{28}
$$

where $u_1, u_2, v_1, v_2, x_1, x_2, y_1, y_2$ are arbitrary functions on M.

Proposition 10. *The only austere 4-manifolds of this kind are those where pq is identically equal to 1.*

Proof. We analyze the structure equations of the standard system $\mathcal{I}$ with the 1-forms ψ^i_j given above. We compute:

$$
\left.\begin{aligned}
d\theta^5_1 &= \pi_1 \wedge \omega^3 + \pi_2 \wedge \omega^4 \\
d\theta^5_2 &= \pi_3 \wedge \omega^3 + \pi_4 \wedge \omega^4 \\
d\theta^5_3 &= \pi_1 \wedge \omega^1 + \pi_3 \wedge \omega^2 + (v_1 p + x_1 - 2u_2)\omega^3 \wedge \omega^4 \\
d\theta^5_4 &= \pi_2 \wedge \omega^1 + \pi_4 \wedge \omega^2 + (2y_1 p - x_2 - v_2 p)\omega^3 \wedge \omega^4 \\
d\theta^6_1 &= \pi_5 \wedge \omega^3 + \pi_6 \wedge \omega^3 \\
d\theta^6_2 &= \pi_7 \wedge \omega^3 + \pi_8 \wedge \omega^4 \\
d\theta^6_3 &= \pi_5 \wedge \omega^1 + \pi_7 \wedge \omega^2 + (u_1 + qy_1 - 2qv_2)\omega^3 \wedge \omega^4 \\
d\theta^6_4 &= \pi_6 \wedge \omega^1 + \pi_8 \wedge \omega^2 + (2x_1 - u_2 - qy_2)\omega^3 \wedge \omega^4
\end{aligned}\right\}
\quad \mod \omega^a, \theta^a_i, \psi^i_j \tag{29}
$$

for certain forms $\pi_1, \ldots, \pi_8$ which are linearly independent combinations of the connection forms $\omega^i_j, \omega^a_b, dp, dq$ and ω^i. The 2-forms on the right hand side must vanish on an integral

element. Wedging the third, fourth, seventh and eighth 2-forms with $\omega^1 \wedge \omega^2$ gives the following integrability conditions:

$$v_1 p + x_1 - 2u_2 = 0, \ 2y_1 p - x_2 - v_2 p = 0, \ u_1 + qy_1 - 2qv_2 = 0, \ 2x_1 - u_2 - qy_2 = 0. \quad (30)$$

Moreover, equations (29) show that all the forms $\pi_1, \ldots, \pi_8$ must vanish on any integral 4-plane also, so they have to be added to the ideal. For example, the vanishing of the 2-form $d\theta_1^5$ implies that, on any integral element satisfying the independence condition, π_1 must be a linear combination of ω^3 and ω^4, while the vanishing of $d\theta_3^5$ implies that π_1 must be a linear combination of ω^1 and ω^2. Therefore, $\pi_1 = 0$ on any such integral element.

Let $\mathcal{J}$ be the differential ideal obtained from adding the forms $\pi_1, \ldots, \pi_8$ to the original ideal $\mathcal{I}$, and restricting to the submanifold where the integrability conditions (30) hold (by solving for u_1, u_2, x_1, x_2). We analyze this new ideal. While $d\theta_i^a \equiv 0$ modulo the one-forms in $\mathcal{J}$, we also compute that:

$$
\begin{aligned}
d\psi_1^1 &= (2q\pi_{10} - q\pi_{11}) \wedge \omega^3 + \left(\tfrac{1}{3}q\pi_{12} + \tfrac{2}{3}p\pi_9\right) \wedge \omega^4 + F_1\omega^{13} + F_2\omega^{23} + F_3\omega^{14} + F_4\omega^{24} \\
d\psi_2^1 &= \pi_9 \wedge \omega^3 + \pi_{10} \wedge \omega^4 \\
d\psi_1^2 &= \left(\tfrac{1}{3}p\pi_9 + \tfrac{2}{3}q\pi_{12}\right) \wedge \omega^3 + (2p\pi_{11} - p\pi_{10}) \wedge \omega^4 + F_5\omega^{13} + F_6\omega^{23} + F_7\omega^{14} + F_8\omega^{24} \\
d\psi_2^2 &= \pi_{11} \wedge \omega^3 + \pi_{12} \wedge \omega^4
\end{aligned}
\tag{31}
$$

where $\pi_9, \ldots, \pi_{12}$ are linearly independent combinations of dv_1, dv_2, dy_1, dy_2 and the ω^i, and $F_1, \ldots F_8$ are certain polynomial functions in $p, q, v_1, v_2, y_1, y_2, g_{55}, g_{56}$ and g_{66}. (We also abbreviate $\omega^i \wedge \omega^j$ by ω^{ij} in the above.)

In the equations (31), the forms $\pi_9, \pi_{10}, \pi_{11}$ and π_{12} are not unique, but can be adjusted only by multiples of the forms ω^3 and ω^4; therefore all the expressions $F_1, \ldots, F_8$ must vanish at points where admissible integral elements of the ideal $\mathcal{J}$ exist. These give a system of 8 linear equations in g_{55}, g_{56}, g_{66} with coefficients depending on p, q, v_1, v_2, y_1, y_2. Eliminating the g_{ab} gives 5 homogeneous quadratic equations in v_1, v_2, y_1, y_2 with coefficients p and q. Of these, three equations constitute a homogeneous linear system in $v_1v_2, v_1y_1, y_1y_2, v_2y_2$. The coefficient matrix of this system has rank 3, unless $pq = 1$ or $pq = -1$ or $p + q = 0$. The latter cases will be considered separately later on; for now, suppose that $p + q \neq 0$ and $pq \neq \pm 1$. Then the vector $(v_1v_2, v_1y_1, y_1y_2, v_2y_2)$ must be a multiple of (q, q, p, p), which spans the kernel of the matrix.

We distinguish two possible subcases. First, suppose that v_1 and y_2 are identically zero on an integral submanifold. Then $F_2 = 0$ implies that $g_{56} = 0$, and the 2-forms (31) determine the values of dv_2 and dy_1 uniquely on an integral element. Setting $d^2v_2 = 0$ and $d^2y_1 = 0$ implies that $g_{55} = g_{66} = 0$, which is impossible. Second, if v_1 or y_2 is nonzero, then $v_1 = (q/p)y_1$ and $v_2 = y_1$. Then $F_1 = \ldots = F_8 = 0$ implies that $g_{55} = 0$, which again is impossible.

In the case that $p + q = 0$, a computation of integrability conditions similar to (29) gives that $v_1 = y_2$ and $v_2 = -y_1$; then, restricting to the submanifold where these relations hold and computing $d\psi_j^i$ yields contradictory integrability conditions. We eliminate the case $pq = -1$ using similar computations. $\square$

In what follows, we deal with the last case, when $pq = 1$. By making a convenient change of frame we can assume that $|\,\mathrm{II}\,|$ is spanned by matrices

$$
S^5 = \begin{pmatrix} 0 & 0 & 1 & p \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}, \qquad
S^6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -p \\ 0 & 1 & 0 & 0 \\ 0 & -p & 0 & 0 \end{pmatrix},
$$

where p is some positive function on M. The new basis for the normal space is distinguished by the fact that the components of II in the direction of each of $\mathbf{e}_5, \mathbf{e}_6$ has rank 2, with a 3-dimensional nullspace. Within E, the lines spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ are distinguished as intersections with these nullspaces. In fact, with respect to the orthonormal coframe $(\omega^1, \omega^2, \omega^3, \omega^4)$ on the tangent space, we may write

$$
\mathrm{II} = \mathbf{e}_5 \otimes \omega^1 \circ \kappa_1 + \mathbf{e}_6 \otimes \omega^2 \circ \kappa_2,
$$

where, for later convenience, we define

$$
\kappa_1 = \omega^3 + p\,\omega^4, \qquad \kappa_2 = \omega^3 - p\,\omega^4.
$$

Using that $d\theta_i^a \equiv 0$ mod the one-forms in the ideal $\mathcal{I}$ gives the integrability conditions:

$$
v_1 = -\frac{y_1}{p},\, y_2 = -py_1,\, u_1 = \frac{x_1}{p},\, x_2 = px_1,\, u_2 = x_1,\, v_2 = y_1
$$

Let us denote $x_1 = x$ and $y_1 = y$. Therefore, the standard system $\mathcal{I}$ is defined on $\mathcal{F} \times \mathbb{R}^3$, with coordinates x, y and $p > 0$ on the last factor, and the extra 1-forms are given by:

$$
\psi_1^1 = \omega_1^3 - \frac{x}{p}\kappa_1, \qquad\qquad \psi_2^1 = \omega_2^3 + \frac{y}{p}\kappa_2,
$$

$$
\psi_1^2 = \omega_1^4 - x\kappa_1, \qquad\qquad \psi_2^2 = \omega_2^4 - y\kappa_2.
$$

By computing

$$
(d\psi_1^3 \wedge \omega^3 + d\psi_1^4 \wedge \omega^4) \wedge \omega^1
$$

modulo $\mathcal{I}$, we conclude that integral elements occur only at points where the following extra integrability condition holds:

$$
g_{56} = xy(p^{-2} - 1). \tag{32}
$$

Restricting to the codimension-one submanifold $N \subset \mathcal{F} \times \mathbb{R}^3$ where this condition holds yields an EDS $\mathcal{I}'$ with a unique integral element at each point. Adjoining to $\mathcal{I}'$ the 1-forms that vanish on these yields a Frobenius system $\mathcal{J}$ on N, whose solutions depend on 22 constants.

We now wish to interpret the solutions.

Proposition 11. *Let M be a connected, 2-ruled austere submanifold of type B, substantial in $\mathbb{R}^6$, such that $E = \mathsf{R}(E)$ and the rank of $\det|_{\mathcal{B}}$ is 2. Then M is congruent to an open subset of a Cartesian product $H_1 \times H_2 \subset \mathbb{R}^3 \oplus \mathbb{R}^3$ where $H_\alpha \subset \mathbb{R}^3$ is a helicoid, in which the rulings of the two helicoids are mutually orthogonal, but the axes of the helicoids are not necessarily orthogonal.*

Proof. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6)$ be a semiorthonormal framing on M, which induces an integral Σ of system $\mathcal{J}$. If we define vectors

$$
\mathbf{f}_5 := \mathbf{e}_5 + x(\mathbf{e}_4 + p^{-1}\mathbf{e}_3), \quad \mathbf{f}_6 := \mathbf{e}_6 + y(\mathbf{e}_4 - p^{-1}\mathbf{e}_3)
$$

then

$$
\left.\begin{array}{ll} d\mathbf{e}_1 \equiv \mathbf{f}_5\kappa_1, & d\mathbf{e}_2 \equiv \mathbf{f}_6\kappa_2 \\[2mm] d\left(\dfrac{\mathbf{f}_5}{L_1}\right) \equiv -L_1\mathbf{e}_1\kappa_1, & d\left(\dfrac{\mathbf{f}_6}{L_2}\right) \equiv -L_2\mathbf{e}_2\kappa_2 \end{array}\right\} \quad \bmod \mathcal{J},
$$

where $L_1 = |\mathbf{f}_5|$ and $L_2 = |\mathbf{f}_6|$. (Note that (32) implies that $\mathbf{f}_5$ and $\mathbf{f}_6$ are orthogonal.) This shows that the orthogonal planes $\mathsf{P}_1, \mathsf{P}_2$ through the origin in $\mathbb{R}^6$, spanned by $\{\mathbf{e}_1, \mathbf{f}_5\}$ and $\{\mathbf{e}_2, \mathbf{f}_6\}$ respectively, are fixed, independent of the choice of point $p \in M$. Within the rulings, the lines spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ are parallel to P_1 and P_2, respectively.

In addition, we compute that $\omega^1, \omega^2, L_1\kappa_1, L_2\kappa_2$ are all closed modulo $\mathcal{J}$, so we may introduce coordinates s, t, u, v along M such that

$$ds = \omega^1, \quad dt = \omega^2, \quad du = L_1\kappa_1, \quad dv = L_2\kappa_2.$$

We also find that

$$d\left(\frac{x}{pL_1^2}\right) \equiv \omega^1, \qquad d\left(\frac{y}{pL_2^2}\right) \equiv -\omega^2 \qquad \mathrm{mod}\ \mathcal{J},$$

so that we may choose the arclength coordinates s, t along the $\mathbf{e}_1$- and $\mathbf{e}_2$-lines satisfying

$$x = pL_1^2 s, \qquad y = -pL_2^2 t.$$

In terms of these coordinates, the derivative of the basepoint map $\mathbf{p} : \mathcal{F} \to \mathbb{R}^6$ is

$$d\mathbf{p} \equiv \mathbf{e}_1 ds + \mathbf{e}_2 dt + \mathbf{f}_3 du + \mathbf{f}_4 dv \quad \mathrm{mod}\ \mathcal{J}, \tag{33}$$

where

$$\mathbf{f}_3 = \frac{1}{2L_1}(\mathbf{e}_3 + p^{-1}\mathbf{e}_4), \qquad \mathbf{f}_4 = \frac{1}{2L_2}(\mathbf{e}_3 - p^{-1}\mathbf{e}_4).$$

Computing the projections of these vectors onto the orthogonal complements of of $\mathbf{f}_5$ and $\mathbf{f}_6$ respectively, we find that

$$d\left(\mathbf{f}_3 - \frac{s}{L_1}\mathbf{f}_5\right) \equiv 0, \quad d\left(\mathbf{f}_4 - \frac{t}{L_2}\mathbf{f}_6\right) \equiv 0 \qquad \mathrm{mod}\ \mathcal{J}.$$

In other words, these vectors are constant along M.

Now consider the splitting

$$\mathbb{R}^6 = \{\mathbf{e}_1, \mathbf{f}_3, \mathbf{f}_5\} \oplus \{\mathbf{e}_2, \mathbf{f}_4, \mathbf{f}_6\}, \tag{34}$$

and let π_1, π_2 be the projections onto the fixed $\mathbb{R}^3$ summands on the right. Then (33) shows that $\pi_1|_M$ and $\pi_2|_M$ are rank two, with coordinates s, u and t, v respectively on the images. In fact, the images are open subsets of classical helicoids. For example, on the surface $\pi_1(M)$ the vectors $\mathbf{e}_1$ and $\mathbf{f}_3$ span the tangent space of the surface, and are tangent to the s- and u-coordinate curves respectively. The s-coordinate curves are straight lines. Since p, L_1, L_2 are functions of s, t only along M, $|\mathbf{f}_3|$ is constant along the u-coordinate curves. We compute

$$\frac{\partial}{\partial u}\mathbf{f}_3 = -s\mathbf{e}_1, \quad \frac{\partial}{\partial u}\mathbf{e}_1 = \frac{1}{L_1}\mathbf{f}_5,$$

which shows that the u-coordinate curves are helices with curvature $s/|\mathbf{f}_3|$ and torsion $1/|\mathbf{f}_3|$.

We note that the splitting (34) is not orthogonal, unless p is identically equal to 1. In fact, the inner product of the fixed vectors pointing along the axes of the helicoids is given by

$$\left(\mathbf{f}_3 - \frac{s}{L_1}\mathbf{f}_5\right) \cdot \left(\mathbf{f}_4 - \frac{t}{L_2}\mathbf{f}_6\right) = \frac{1 - p^{-2}}{4L_1 L_2}.$$

$\square$

(a.ii) $\det|_{\mathcal{B}}$ **has rank 1.** In this case we can normalize so that $\mathcal{B}$ is spanned by $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ for $p \neq 0$. Thus, with respect to an orthonormal basis for the tangent space, $|\,\mathrm{II}\,|$ is spanned by the matrices

$$S^5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S^6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \end{pmatrix}.$$

Let $\mathcal{I}$ be the Pfaffian system for ruled austere submanifolds, with semiorthonormal frames such that (27) holds for the above matrices S^5, S^6. The 1-forms encoding the tangential ruled condition are as in (28).

Computing the 2-forms of this system yields immediate integrability conditions

$$u_2 = 0, \quad x_1 = -pv_1, \quad x_2 = 3pv_2, \quad y_1 = 2v_2, \quad y_2 = 0.$$

We restrict to the submanifold where these conditions hold and derive further integrability conditions on the remaining variables, which imply that $v_2 = 0$ and either $u_1 = 0$ or $v_1 = 0$. In both cases, new integrability conditions arise that imply that $\mathbf{e}_6 \cdot \mathbf{e}_6 = 0$ which is impossible. Therefore, there are no integral submanifolds in this case.

(a.iii) $\det|_{\mathcal{B}} \equiv 0$. In this case we can normalize so that $\mathcal{B} = \{\left(\begin{smallmatrix} x & 0 \\ y & 0 \end{smallmatrix}\right)\}$ or $\{\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right)\}$. In either case, the quadratic forms in $|\,\mathrm{II}\,|$ have a common linear factor, hence $|\,\mathrm{II}\,|$ is *simple*, and M is a generalized helicoid. We will determine what are the particular values for the constants in (1).

We consider the first case, namely when $\mathcal{B}$ is of a form $\{\left(\begin{smallmatrix} x & 0 \\ y & 0 \end{smallmatrix}\right)\}$. Integrability conditions imply that the extra 1-forms (28) of the ideal $\mathcal{I}$ must have $u_2, v_2, x_1, x_2, y_1, y_2$ all zero. The first prolongation $\mathcal{I}^{(1)}$ of the ideal is involutive—in fact, Frobenius, with solutions depending on 21 arbitrary constants. Moreover, we compute that

$$d\mathbf{e}_4 \equiv 0 \qquad \mod \mathcal{I}^{(1)},$$

indicating that $\mathbf{e}_4$ is parallel to a fixed line. Thus, M is congruent to the product of a line with a generalized helicoid in $\mathbb{R}^5$ (i.e., $s = 2$ in (1)).

The second case is when $\mathcal{B}$ is of the form $\{\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right)\}$. In this case, integrability conditions imply that all coefficients in (28) vanish except for y_2. Again, the first prolongation is Frobenius, with solutions depending on 17 arbitrary constants. However, in this case M is a cone, with the vector $\mathbf{e}_2$ tangent to the generators, and is ruled by 3-planes spanned by $\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. This corresponds to $s = 3$, λ_0 and $\lambda_1 = \lambda_2 = \lambda_3$ in (1). Thus, M is a cone over a generalized Clifford torus, the embedding of the product of the unit spheres $S^1 \subset \mathbb{R}^2$ and $S^2 \subset \mathbb{R}^3$ into $S^5 \subset \mathbb{R}^6$.

We summarize this case as follows:

Proposition 12. *Let M be a connected, 2-ruled austere submanifold of type B, substantial in $\mathbb{R}^6$, such that $E = \mathsf{R}(E)$ and $\delta = 2$. Then M is congruent to an open subset of one of the following: a product of helicoids in $\mathbb{R}^3$, a generalized helicoid in $\mathbb{R}^6$ with $s = 2$ and $\lambda_0 \neq 0$, or a cone over $S^1 \times S^2 \subset S^5$.*

(b) E **is a sum of** $+1$ **and** -1 **eigenspaces of** R. In this case we can assume that $E = \{\mathbf{e}_1, \mathbf{e}_3\}$. This implies that $m = b_1 = 0$ and therefore $\delta \leq 3$. Analyzing the standard system $\mathcal{I}$ in either the case $\delta = 3$ and $\delta = 2$ leads to impossible integrability conditions. Thus, no submanifolds of this type exist.

Case B.2: dim $R(E) \cap E = 1$. Here we can arrange that $E = \{\mathbf{e}_1, \mathbf{e}_3 + a\mathbf{e}_2\}$ for $a \neq 0$. In this case the standard system is involutive with $s_2 = 2$, so that solutions depending on 2 functions of 2 variables. However, the algebraic condition (11) implies that $|\mathrm{II}|$ is simple, and thus M is a generalized helicoid. In fact, it is a cone, ruled by 3-planes spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. (The 2 functions come in when we choose a 2-plane field within the ruling.) This corresponds to $s = 2$ and $\lambda_0 = 0$ in (1), but in this instance the constants $\lambda_1, \lambda_2, \lambda_3$ may be distinct.

Case B.3: $R(E) \cap E = \{0\}$. In this case we can normalize so that $E = \{\mathbf{e}_1 + a\mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_4\}$ for a, b both nonzero. It follows from (11) that $\delta \leq 2$. Assuming that $\delta = 2$, analyzing the standard system shows that solutions only exist when $a = b$. In this case, $|\mathrm{II}|$ is also of type A, with $\mathsf{J}(E) = E$, so these submanifolds are described in Case A.1(a) above.

5. TYPE C RULED AUSTERE SUBMANIFOLDS

In this section we discuss the ruled austere 4-folds of Type C. We recall from [1] that a maximal austere subspace of Type C is defined by

$$\mathcal{Q}_C = \left\{ \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & \lambda_3 x_3 & \lambda_2 x_2 \\ x_2 & \lambda_3 x_3 & 0 & \lambda_1 x_1 \\ x_3 & \lambda_2 x_2 & \lambda_1 x_1 & 0 \end{bmatrix} \right\}. \tag{35}$$

where x_1, x_2, x_3 are linear coordinates on the subspace, and $\lambda_1, \lambda_2, \lambda_3$ are real parameters satisfying

$$\lambda_1 \lambda_2 \lambda_3 + \lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{36}$$

An austere 4-manifold M is of Type C if near any point there is an orthonormal frame field $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ with respect to which $|\mathrm{II}| \subset \mathcal{Q}_C$, for parameters λ_i which may vary smoothly along M. Notice that this condition is invariant under simultaneously permuting the frame vectors $(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ and the parameters λ_i.

We distinguish two cases, depending on whether or not $\mathbf{e}_1$ is orthogonal to the ruling plane E.

Case C.1: $\mathbf{e}_1 \cdot E = 0$. In this case, the algebraic condition (11) says that E, lying in the span of $\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$, is a 2-dimensional nullspace for every matrix in $|\mathrm{II}|$. Thus, matrices in $|\mathrm{II}|$ of the form in (35) must satisfy

$$0 = \det \begin{bmatrix} 0 & \lambda_3 x_3 & \lambda_2 x_2 \\ \lambda_3 x_3 & 0 & \lambda_1 x_1 \\ \lambda_2 x_2 & \lambda_1 x_1 & 0 \end{bmatrix} = 2\lambda_1 \lambda_2 \lambda_3 x_1 x_2 x_3.$$

implying that either one of x's is zero, or one of the λ's, and one of these must hold on an open set in M.

In the first case, we can assume that $|\mathrm{II}| \subset \mathcal{Q}_C$ is a 2-dimensional subspace defined by $x_1 = 0$ at each point. Then the ruling plane is spanned by $\{\mathbf{e}_3, \mathbf{e}_4\}$, and $|\mathrm{II}|$ is also a subspace of $\mathcal{Q}_B$. Swapping $\mathbf{e}_3, \mathbf{e}_4$ with $\mathbf{e}_1, \mathbf{e}_2$, we see that M falls into case B.1 above.

In the second case, we can assume that λ_1 is identically zero along M. Assuming first that $\delta = 3$, an analysis of the standard system $\mathcal{I}$ shows that the remaining parameters must be constant along M. Then Prop. 14 in [5] implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and M is a generalized helicoid, swept out by 3-planes in $\mathbb{R}^7$. On the other hand, if $\delta = 2$ then $E = \{\mathbf{e}_3, \mathbf{e}_4\}$ and the coordinates x_1, x_2, x_3 are linearly related at each point. An analysis

of the system $\mathcal{I}$ in this case, similar to that in the proof of Prop. 10, shows that no such manifolds exist.

Case C.2: $\mathbf{e}_1 \cdot E \neq 0$. In this case, by permuting the frame vectors, we can assume that E is spanned by $\mathbf{e}_1 + a_3 \mathbf{e}_3 + a_4 \mathbf{e}_4$ and $\mathbf{e}_2 + b_3 \mathbf{e}_3 + b_4 \mathbf{e}_4$ for some functions a_3, a_4, b_3, b_4. The algebraic condition (11) implies that $\delta \leq 2$. We assume that $\delta = 2$. Requiring that a 2-dimensional subspace of $\mathcal{Q}_C$ satisfy the algebraic condition leads to the following subcases:
(i) $a_3 = a_4 = 0$, $b_3 \neq 0$, $b_4 \neq 0$. In this case, (36) implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Then a change of basis shows that $|\,\mathrm{II}\,|$ is of Type B and simple, falling into case B.1(a.iii) above; in particular M is a generalized helicoid cone in $\mathbb{R}^6$.
(ii) $a_3 = a_4 = b_4 = 0$, $b_3 \neq 0$, $\lambda_3 = 0$. An analysis of $\mathcal{I}$ shows that the only possibility is that λ_1, λ_2 vanish identically, in which case M is again a generalized helicoid cone.
(iii) $a_3 = a_4 = b_3 = b_4 = 0$. In this case, $|\,\mathrm{II}\,|$ is of Type B, falling into case B.1 above.
(iv) $a_4 = b_3 = 0$, $a_3 \neq 0, b_4 \neq 0$, $\lambda_1 = -1/(a_3 b_4)$, $\lambda_3 = -b_4/a_3$. A lengthy analysis of the integrability conditions for this case shows that the only possibility is that $a_3 = \pm 1$ and $b_4 = \pm 1$. Then an orthogonal change of basis shows that $|\,\mathrm{II}\,|$ is precisely of the type described in Prop. 10, and thus M is a product of helicoids as described by Prop. 11.

We summarize our results for Type C in Theorem 3 stated in the Introduction.

6. Normal Rank One

In this section we investigate austere submanifolds M^4 where $|\,\mathrm{II}\,|$ is one-dimensional at each point, and the Gauss map has rank two. As we will see, this implies that M is a hypersurface in $\mathbb{R}^5$ ruled by 2-planes; however, it is not known if a 2-ruled austere hypersurface of this dimension must have a degenerate Gauss map.

For a submanifold in Euclidean space, the kernel of the differential of the Gauss map is known as the *relative nullity distribution* (see, e.g., [2]), and consists at $p \in M$ of those vectors X such that $\mathrm{II}(X, Y) = 0$ for all $Y \in T_p M$. Thus, if M^4 is austere with a rank two Gauss map, then we may choose an orthonormal frame so that $\mathbf{e}_1, \mathbf{e}_2$ span the relative nullity distribution, and $|\,\mathrm{II}\,|$ is spanned by matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & q & -p \end{bmatrix}. \tag{37}$$

Accordingly, let $\mathcal{H}_{\mathrm{on}}$ be the standard system for austere submanifolds (described in §2) defined on $\mathcal{F}_{\mathrm{on}} \times \mathbb{R}_*^2$ with p, q not both zero as coordinates on the second factor, and where matrix S^5 is given by (37) and $S^6, \ldots, S^n$ are zero. Any austere $M^4 \subset \mathbb{R}^n$, equipped with an orthonormal frame such that $\mathbf{e}_1, \mathbf{e}_2$ span the relative nullity distribution, gives an integral of $\mathcal{H}_{\mathrm{on}}$, and conversely.

Theorem 13. *M is ruled by planes tangent to the relative nullity distribution, and is contained in a totally geodesic $\mathbb{R}^5$.*

Proof. The 'normal part' of the ruled condition (11) holds automatically for the relative nullity distribution. The remaining requirement (12) for $\mathbf{e}_1, \mathbf{e}_2$ to span a ruling is that $\omega_1^3, \omega_2^3, \omega_1^4, \omega_2^4$ be multiples of ω^3, ω^4 along any integral manifold.

We will use complex-valued 1-forms to compute the structure equations of $\mathcal{H}_{\mathrm{on}}$ and other related systems. For example, when $n = 5$ the 1-forms generating $\mathcal{H}_{\mathrm{on}}$ are ω^5, $\theta_1^5 = \omega_1^5$,

$\theta_2^5 = \omega_2^5$ and

$$\theta_3^5 - \mathrm{i}\theta_4^5 = (\omega_3^5 - \mathrm{i}\omega_4^5) - (p - \mathrm{i}q)(\omega^3 + \mathrm{i}\omega^4) \tag{38}$$

Defining complex-valued 1-forms

$$\zeta = \omega^3 + \mathrm{i}\omega^4, \quad \eta_1 = \omega_1^3 + \mathrm{i}\omega_1^4, \quad \eta_2 = \omega_2^3 + \mathrm{i}\omega_2^4,$$

we compute the nontrivial 2-forms of $\mathcal{H}_{\mathrm{on}}$ as

$$\begin{aligned}
d\theta_1^5 &\equiv \mathrm{Re}[(p - \mathrm{i}q)\eta_1 \wedge \zeta], \\
d\theta_2^5 &\equiv \mathrm{Re}[(p - \mathrm{i}q)\eta_2 \wedge \zeta], \\
d(\theta_3^5 - \mathrm{i}\theta_4^5) &\equiv (-d(p - \mathrm{i}q) + 2\mathrm{i}(p - \mathrm{i}q)\omega_3^4) \wedge \zeta + (p - \mathrm{i}q)(\eta_1 \wedge \omega^1 + \eta_2 \wedge \omega^2)
\end{aligned} \tag{39}$$

modulo the 1-forms of $\mathcal{H}_{\mathrm{on}}$. Thus, on any integral element

$$\eta_1 = T_1\zeta, \quad \eta_2 = T_2\zeta \tag{40}$$

for some complex-valued functions T_1, T_2. In particular, M is ruled.

When $n > 5$, the additional 1-form generators are ω^b and ω_i^b for $b > 5$. It is easy to check that $d\omega^b, d\omega_1^b, d\omega_2^b$ are congruent to zero modulo the 1-forms of $\mathcal{H}_{\mathrm{on}}$, while

$$d(\omega_3^b - \mathrm{i}\omega_4^b) \equiv -(p - \mathrm{i}q)\omega_5^b \wedge \zeta.$$

We deduce that the real-valued 1-forms ω_5^b must vanish on all integral elements. From the vanishing of $\omega_1^b, \ldots, \omega_5^b$ it follows, using (2), that the subspace spanned by $\mathbf{e}_1, \ldots \mathbf{e}_5$ is fixed, and thus M contained in a 5-dimensional plane in $\mathbb{R}^n$. $\square$

Without loss of generality, we will restrict our attention to austere hypersurfaces in $\mathbb{R}^5$ for the rest of this section. Let $\mathcal{I}_{\mathrm{on}}$ be the Pfaffian system obtained by adding to $\mathcal{H}_{\mathrm{on}}$ the 1-forms encoding the tangential part of the ruling condition; these are obtained from the real and imaginary parts of the equations in (40). Altogether, the 1-form generators of $\mathcal{I}_{\mathrm{on}}$ are

$$\omega^5, \ \omega_1^5, \ \omega_2^5, \ \omega_3^5 - p\omega^3 - q\omega^4, \ \omega_4^5 - q\omega^3 + p\omega^4,$$
$$\omega_1^3 - s_1\omega^3 + t_1\omega^4, \ \omega_1^4 - t_1\omega^3 - s_1\omega^4, \ \omega_2^3 - s_2\omega^3 + t_2\omega^4, \ \omega_2^4 - t_2\omega^3 - s_2\omega^4,$$

where s_i, t_i are the real and imaginary parts of T_i, $i = 1, 2$. Then $\mathcal{I}_{\mathrm{on}}$ is defined on $\mathcal{F}_{\mathrm{on}} \times \mathbb{R}_*^2 \times \mathbb{R}^4$, with the s_i, t_i added as new variables. It is easy to check that $\mathcal{I}_{\mathrm{on}}$ is involutive with Cartan character $s_1 = 6$. Below, we will use the involutivity to show that such hypersurfaces exist, passing through any generic curve in $\mathbb{R}^5$.

Remark 14. An analogue of the twistor spaces V, used to construct various Type A.1 ruled submanifolds in §3, exists for this hypersurface case. Namely, let V be the space of flags $E \subset F$ in $\mathbb{R}^5$, where the subspaces have dimensions 2 and 4 respectively. (This manifold is a homogeneous 8-dimensional quotient of $SO(5)$.) Given an austere hypersurface M with rank 2 Gauss map, we define a rank 2 mapping $\Gamma : M \to V$ taking $p \in M$ to (E_p, T_pM), where E_p is parallel to the ruling through p. Then $\Gamma(M)$ is an integral surface of a certain exterior differential system $\mathcal{J}$ on V, and any such surface is the locally the image $\Gamma(M)$ for some austere hypersurface M of this type. (Details are left to the interested reader.) The main difference between this case and the Type A twistor spaces is that there is no homogeneous complex structure on V with respect to which $\Gamma(M)$ is a holomorphic curve.

In closing, we consider a Cauchy problem for austere hypersurfaces. Let $\gamma(s)$ be a regular real-analytic curve, parametrized by arclength, which is substantial in $\mathbb{R}^5$ (i.e., it lies in

no lower-dimensional totally geodesic submanifold). It follows that there exist real-analytic orthonormal vectors T, N_1, N_2 and real-analytic functions k_1, k_2 such that

$$\frac{d\gamma}{ds} = T, \quad \frac{dT}{ds} = k_1 N_1, \quad \frac{dN_1}{ds} = -k_1 T + k_2 N_2.$$

We lift γ to obtain a real-analytic integral curve of $\mathcal{I}_{\text{on}}$ in $\mathcal{F}_{\text{on}} \times \mathbb{R}^6$ as follows. First, letting $\mathbf{e}_3 = T$, $\mathbf{e}_5 = N_1$, $\mathbf{e}_4 = N_2$, and letting $\mathbf{e}_1, \mathbf{e}_2$ be any analytic orthonormal vectors along γ that are perpendicular at each point to the plane spanned by $\{T, N_1, N_2\}$, gives a lift into $\mathcal{F}_{\text{on}}$ which is an integral of ω^5, ω_1^5 and ω_2^5. (As well, $\omega^1, \omega^2, \omega^4$ pull back to be zero along γ, while ω^3 pulls back to equal ds.) Next, we choose values for the functions p, q, s_1, t_1, s_2, t_2 along γ so as to give a lift into $\mathcal{F}_{\text{on}} \times \mathbb{R}^6$ which is an integral curve of the remaining generators of $\mathcal{I}_{\text{on}}$. (For example, since $\omega_3^5 = \mathbf{e}_5 \cdot (d\mathbf{e}_3/ds)$ and $\omega_4^5 = \mathbf{e}_5 \cdot (d\mathbf{e}_4/ds)$, we set $p = k_1$ and $q = -k_2$.)

It follows by the Cartan-Kähler Theorem (see, e.g., [6]), that there exists an integral 4-manifold of $\mathcal{I}_{\text{on}}$, satisfying the independence condition, containing the lift of γ. Our hypersurface M is the projection of this 4-manifold into $\mathbb{R}^5$. Note that the fact that N_1 is orthogonal to the hypersurface means that γ is a geodesic in M.

Proposition 15. *Let γ be a real-analytic curve, substantial in $\mathbb{R}^5$. Then there exists an austere 4-fold $M^4 \subset \mathbb{R}^5$ with rank 2 Gauss map, such that γ is a geodesic in M and along γ the ruling plane is orthogonal to the span of $\gamma', \gamma'', \gamma'''$. M is unique in the sense that any two such hypersurfaces must coincide in a neighborhood of γ.*

The 'function count' $s_1 = 6$ provided by the Cartan's Test may be interpreted as follows. Choosing a curve in $\mathbb{R}^5$ (up to rigid motions) depends on four functions of one variable, namely the Frenet curvatures. Then we use up the remaining two function's worth of freedom by choosing a lift into $\mathcal{F}_{\text{on}}$ along which $\omega^1, \omega^2, \omega^4, \omega^5, \omega_1^5, \omega_2^5$ are zero and $\mathbf{e}_5$ lies in the osculating plane of γ.

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