

Exact solution for eigenfunction statistics at the center-of-band anomaly in the Anderson localization model

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An exact solution is found for the problem of the center-of-band ($E = 0$) anomaly in the one-dimensional (1D) Anderson model of localization. By deriving and solving an equation for the generating function $\Phi(u, \phi)$ we obtained an exact expression in quadratures for statistical moments $I_q = \langle |\psi_E(\mathbf{r})|^{2q} \rangle$ of normalized wavefunctions $\psi_E(\mathbf{r})$ which show violation of one-parameter scaling and emergence of an additional length scale at $E \approx 0$.

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–Introduction. Anderson localization (AL) enjoys an unusual fate of being a subject of advanced research during a half of century. The seminal paper by P.W.Anderson¹ opened up a direction of research on the interplay of quantum mechanics and disorder which is of fundamental interest up to now². The one-dimensional (1D) tight-binding model with diagonal disorder –the Anderson model (AM)– which is the simplest and the most studied model of this type, became a paradigm of AL:

$$\hat{H} = \sum_i \varepsilon_i c_i^\dagger c_i - \sum_i t_i \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right). \quad (1)$$

In this model the hopping integral is deterministic $t_i = t = 1$ and the on-site energy ε_i is a random Gaussian variable uncorrelated at different sites and characterized by the variance $\langle (\delta\varepsilon_i)^2 \rangle = w$.

The best studied is the continuous limit of this model in which the lattice constant $a \rightarrow 0$ at ta^2 remaining finite^{3,4}. There was also a great deal of activity⁵ aimed at a rigorous mathematical description of 1D AL. However, despite considerable efforts invested, some subtle issues concerning 1D AM still remain unsolved. One of them is the effects of commensurability between the de-Broglie wavelength λ_E (which depends on the energy E) and the lattice constant a .

It was known for quite a while^{6,7} that at weak disorder $w \ll 1$ the Lyapunov exponent takes anomalous values at the ratio $f = \frac{2a}{\lambda_E}$ equal to $\frac{1}{2}$ and $\frac{1}{3}$ (compared to those at f beyond the window of the size w around $f = \frac{1}{2}$ and $f = \frac{1}{3}$). The Lyapunov exponent sharply *decreases* at $f = \frac{1}{2}$ (which is usually associated with *increasing* the localization length) and may both increase or decrease at $f = \frac{1}{3}$ depending on the third moment of the on-site energy distribution⁷. More recently^{8,9} it was found that the statistics of conductance in 1D AM is anomalous at the center of the band that corresponds to $f = \frac{1}{2}$. We want to stress that all these anomalies were observed for the AM Eq.(1) in which the on-site energy ε_i is random. This Hamiltonian does not possess the *chiral symmetry*^{2,10} which is behind the statistical anomalies at the center of the band $E = 0$ in the

Lifshitz model described by Eq.(1) with the deterministic $\varepsilon_i = 0$ and a random hopping integral t_i . Thus the statistical anomaly at $f = \frac{1}{2}, \frac{1}{3}$ raises a question about a *hidden symmetry* that does not merely reduce to the two-sublattice division^{2,9,10}.

A similar phenomenon may occur in dynamical systems. An elegant analogy between the 1D localization and the classical system of kicked oscillator was studied in Ref.¹¹. According to this analogy the energy-dependent de-Broglie wavelength λ_E is encoded in the frequency of the oscillator and the lattice constant a determines the period of the δ -function "kicks" of the external force, their amplitude being proportional to disorder.

The interest to one-dimensional Anderson localization is greatly increased recently after several groups reported about successful experiments on localization of cold atoms^{12,13}, where even tiny details of localized wavefunctions were observed. Kicked rotors and kicked oscillator can also be realized in systems of cold atoms¹⁴.

There are numerous questions concerning physics behind the anomalies. One of puzzles is the sign of the variation of the Lyapunov exponent which corresponds to *weaker* localization at $f = \frac{1}{2}$. Such a tendency can be considered as a remnant of the chiral symmetry spoiled by fluctuating on-site energy. There is, however, a completely different view on the problem which predicts the *stronger* localization at the band center. It involves the notion of *Bragg mirrors*¹⁵ created by disorder realizations with alternating on-site energies which double the period, at least locally. A possible resolution of this conflict between different mechanisms of the center-of-band anomaly could be a typical wavefunction sketched in Fig.1. It contains *two* length scales: one of them ℓ_{1oc} which is somewhat larger than the localization length ℓ away from the anomaly, is due to remnants of the chiral symmetry, while the other, much smaller one $d \ll \ell_{1oc}$ (but $d \gg a$), is due to the formation of the Bragg mirror fluctuation. If the weight of the narrow peak $p \ll 1$ is small, the statistical moments $I_q = \langle |\psi(\mathbf{r})|^{2q} \rangle$ of the normalized wavefunctions $\psi(\mathbf{r})$ with relatively small q will follow the standard¹⁷ behavior (L is the length of the

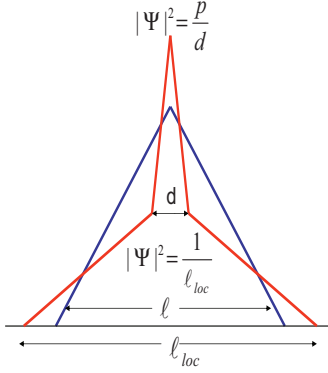


FIG. 1: The cartoon of the standard (blue) and the $E = 0$ typical state (red). At the anomaly the second scale d emerges.

chain):

$$I_q^{(\text{st})} = (q-1)! \ell_{\text{loc}}^{1-q} L^{-1}. \quad (2)$$

while at large q , the higher and more narrow peak will

dominate in I_q , leading to the $d^{-(q-1)}$ instead of $\ell_{\text{loc}}^{-(q-1)}$ behavior of moments. The simplest crossover between the two regimes is described by

$$I_q = I_q^{(\text{st})} \left[1 + p \left(\frac{p \ell_{\text{loc}}}{d} \right)^{q-1} \right]. \quad (3)$$

Eq.(3) can be obtained from the following qualitative arguments. The average moment is the sum of two contributions. The first one leading to the standard moment Eq.(2), is equal to the q -th power of the typical amplitude $|\psi(\mathbf{r})|^2 \sim 1/\ell_{\text{loc}}$ inside the localization radius but outside of the narrow peak, multiplied by the probability $\sim \ell_{\text{loc}}/L$ that the observation point \mathbf{r} falls inside this region. The second contribution $(p/d)^q d/L$ arises when with small probability d/L the observation point falls inside the narrow peak where the amplitude $|\psi(\mathbf{r})|^2 \sim p/d$. It is this contribution which corresponds to the d -dependent term in Eq.(3).

In general, the information about a typical shape of localized wave functions is encoded in statistical moments I_q . In this Letter we solve exactly the problem of statistical moments I_q at $E = 0$ for the 1D Anderson disordered chain Eq.(1) of the length $L \rightarrow \infty$ and show that the behavior Eq.(3) indeed emerges.

– *Generating function (GF), moments of $|\psi|^2$ and the probability distribution function (PDF) of phase.* Moments I_q of normalized eigenfunctions with integer $q > 1$

$$I_q(\mathbf{r}) = \langle |\psi_E(\mathbf{r})|^{2q} \rangle = \frac{2\pi}{L(q-2)!} \int_0^\pi d\phi \cos^{2q}(\phi) \int_0^\infty dz z^{q-2} \Phi_{r-1}(z, \phi - k) \Phi_{N-r}(z, -\phi - k), \quad (E = 2 \cos k). \quad (4)$$

can be expressed¹⁶ in terms of a *generating function* $\Phi_j(u, \phi)$ on the lattice site j . The starting point of our analysis is the recursive equation for GF which can be de-

rived elementary¹⁸ starting from Eq.(1) as well as using the super-symmetry method¹⁹:

$$\Phi_{j+1}(z, \phi) = \frac{\sin k e^{-z \cos^2 \phi}}{\sqrt{2\pi w \cos^2 \phi}} \int_0^\pi d\phi' \exp \left[-\frac{\sin^2 k}{2w} (\tan \phi - \tan \phi')^2 \right] \Phi_j \left(z \frac{\cos^2 \phi}{\cos^2 \phi'}, \phi' - k \right), \quad \Phi_{j=0}(z, \phi) = \delta(\phi - \pi/2). \quad (5)$$

This equation is *exact* and holds both for weak and strong disorder controlled by the parameter w for any energy $E = E(k)$ parametrized by $E(k) = 2 \cos k$.

The way the variables z and ϕ enter Eq.(4) suggests their physical meaning⁶: they determine the values of a wave function $\psi(i+1)$ and $\psi(i)$ on a link $\{i, i+1\}$:

$$\psi(i) = \sqrt{z_i} \cos(\phi_i), \quad \psi(i+1) = \sqrt{z_i} \cos(\phi_i - k). \quad (6)$$

It is remarkable that both the "elementary"¹⁸ and the "super-symmetric"¹⁹ derivations of eigenfunction statistics involve naturally the *two* link variables Eq.(6). In contrast to the moments I_q , the Lyapunov exponent

$$\gamma = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \ln \left(\frac{|\psi(i+1)|}{|\psi(i)|} \right) \quad (7)$$

depends on only one of the two sets of variables, ϕ ($\psi(i+1)/\psi(i) = \cos(\phi_i - k)/\cos(\phi_i)$), which determines completely its statistics^{6,7}. That is why the problem of moments is more complicated and more general than that of the Lyapunov exponent.

The integrand in Eq.(4) is *bi-linear* in Φ . This effectively takes into account the boundary conditions at the *two* ends of the chain¹⁷ which is necessary to describe the *normalized eigenfunctions*. In contrast to that in the problem of Lyapunov exponent⁶ one considers essentially a *semi-infinite* chain and does not require of the solution to Eq.(1) to be an eigenfunction.

Despite the fact that $\Phi_j(z, \phi)$ is *not* the joint PDF of z and ϕ , its descender $\Phi_j(z = 0, \phi)$ is the PDF of phase:

$$\Phi_j(z = 0, \phi) = P_j(\phi), \quad \int_0^\pi P_j(\phi) d\phi = 1. \quad (8)$$

This statement can be formally proven¹⁸ but the key properties of PDF, the positivity of $P_j(\phi)$ and the conservation of normalization, are easily seen directly from Eq.(5) and the boundary condition $\Phi_0(\phi) = \delta(\phi - \pi/2)$.

-*Evolution equation for weak disorder.* Eq.(5) is valid for an arbitrary strength of disorder. However, the anomaly we are going to study is sharp only at weak disorder and is rounded off as disorder increases. For weak disorder when the localization length $\ell = \frac{2a \sin^2 k}{w}$ is large compared to the the lattice constant a one can reduce Eq.(5) to a partial differential equation (DE) of the Fokker-Planck type, where the coordinate $x = j a/\ell$ along 1D chain plays a role of time and the *two-dimensional* space of variables $u = \ell z$ and ϕ stands for the coordinate space.

In the first order in $a/\ell \ll 1$ one obtains by a proper expansion in Eq.(5):

$$\Phi_{j+1}(u, \phi) = \left(1 + \frac{a}{\ell} \left[\hat{\mathcal{L}}(\phi) - c_1(\phi) u \right]\right) \Phi_j(u, \phi - k), \quad (9)$$

where the evolution operator $\left(1 + \frac{a}{\ell} \left[\hat{\mathcal{L}}(\phi) - c_1(\phi) u \right]\right)$ contains the differential part

$$\begin{aligned} \hat{\mathcal{L}}(\phi) = & c_2(\phi) u^2 \partial_u^2 + c_3(\phi) (u \partial_u - 1) \\ & + c_4(\phi) u \partial_u \partial_\phi + c_5(\phi) \partial_\phi + c_6(\phi) \partial_\phi^2 \end{aligned} \quad (10)$$

and $c_i(\phi)$ are certain linear combinations of 1, $\sin(2\phi)$, $\cos(2\phi)$, and $\sin(4\phi)$, $\cos(4\phi)$.

The formal reason for the center-of-band anomaly at $k = \frac{\pi}{2}$ (as well as of the weaker anomaly at any $k = \pi p/q$, where p, q are positive integers) is the shift by k of the ϕ argument in r.h.s. of Eq.(9). Because of this shift and the periodicity $\Phi_j(u, \phi) = \Phi_j(u, \phi + \pi)$, one has to apply the evolution operator q times in order to get a closed recursive equation which expresses $\Phi_{j+q}(u, \phi)$ in terms of $\Phi_j(u, \phi)$ and its derivatives. For weak disorder and not very large $q \ll \ell/a$, one can expand $\Phi_{j+q} - \Phi_j \approx (aq/\ell) \partial \Phi(u, \phi; x) / \partial x$, where we introduce a function $\Phi(u, \phi; x) = \Phi_{\ell x/a}(u, \phi)$ of a continuous dimensionless coordinate $x = j a/\ell$. Thus in the lowest order

in a/ℓ we obtain for $k = \pi p/q$:

$$\partial_x \Phi = \left[\sum_{s=0}^{q-1} \hat{\mathcal{L}}(\phi - \frac{s\pi p}{q}) - u \sum_{s=0}^{q-1} c_1(\phi - \frac{s\pi p}{q}) \right] \Phi. \quad (11)$$

The sum over s arises because the small corrections to the evolution operator proportional to a/ℓ add up in the product of q evolution operators, each time entering with a shift $c_i(\phi) \rightarrow c_i(\phi - k)$ according to Eq.(9). The crucial point for emergence of anomaly at $k = \frac{\pi}{2}$ ($q = 2, p = 1$) is the identity:

$$\sum_{s=0}^{q-1} e^{2i\phi - 2is\pi p/q} = 0, \quad \sum_{s=0}^{q-1} e^{4i\phi - 4is\pi p/q} = \begin{cases} 0, & q > 2 \\ q e^{4i\phi}, & q = 2 \end{cases} \quad (12)$$

One observes that at $k = \pi p/q$ with all q but $q = 2$ the ϕ -dependent terms disappear from the r.h.s. of Eq.(11). At $k = \frac{\pi}{2}$, however, one obtains the anomalous, ϕ -dependent, evolution equation. It appears to have a nice $SL(2)$ group structure:

$$\begin{aligned} \partial_x \Phi(u, \phi; x) = & \{ \hat{L}_1^2 + \hat{L}_3^2 - u \} \Phi(u, \phi; x), \\ \hat{L}_1 = & \cos \theta \partial_\theta + \sin \theta u \partial_u, \quad \hat{L}_3 = \partial_\theta, \quad (\theta = 2\phi), \end{aligned} \quad (13)$$

where \hat{L}_1 and \hat{L}_3 and $\hat{L}_2 = [\hat{L}_3, \hat{L}_1] = -\sin \theta \partial_\theta + \cos \theta u \partial_u$ form a closed $sl(2)$ algebra.

Note that Eq.(13) contains all the known particular results. For instance, omitting all the ϕ -dependent terms one obtains the standard equation for GF away from the anomaly¹⁷ which allows for the ϕ - and x -independent (*zero-mode*) solution^{6,7}:

$$\Phi^{(st)}(u) = \frac{2}{\pi} \sqrt{u} K_1(2\sqrt{u}). \quad (14)$$

Alternatively, in agreement with Eq.(8), by setting $u = 0$ in Eq.(13) one arrives at the second order ordinary DE for the non-trivial phase-distribution function $P_0(\phi)$ at the $k = \frac{\pi}{2}$ anomaly with the zero-mode solution:

$$P_0(\phi) = \Phi(0, \phi) = \frac{C}{\sqrt{3 + \cos(4\phi)}}, \quad C = \frac{4\sqrt{\pi}}{\Gamma^2(\frac{1}{4})}, \quad (15)$$

resulting in the anomaly of the Lyapunov exponent^{6,7}:

$$\frac{\gamma(E = 0)}{\gamma(E \neq 0)} = \int_0^\pi (1 + \cos(4\phi)) P_0(\phi) = \frac{8\Gamma^2(\frac{3}{4})}{\Gamma^2(\frac{1}{4})} \approx 0.9139. \quad (16)$$

Derivation of Eq.(13), and its *exact* solution is the main result of this Letter.

-*Separation of variables and the zero-mode solution.* The variables u and ϕ are entangled in Eq.(13). However, there is a hidden symmetry which allows to separate variables in this equation, provided that the term $\partial_x \Phi(u, \phi; x) = 0$. This *zero mode* solution is sufficient to describe anomalous eigenfunction statistics in a very long chain $L \gg \ell$ far from its ends.

"Correct variables" ξ and η are suggested by Eq.(6):

$$\xi = u \cos^2 \phi, \quad \eta = u \sin^2 \phi. \quad (17)$$

Defining also the "correct function":

$$\tilde{\Phi}(\xi, \eta) = \frac{(\xi\eta)^{\frac{1}{4}}}{(\xi + \eta)} \Phi(u(\xi, \eta), \phi(\xi, \eta)), \quad (18)$$

one casts the zero-mode variant of Eq.(13) in the form of the Schroedinger equation:

$$\left[\hat{H}(\xi) + \hat{H}(\eta) \right] \tilde{\Phi} = 0, \quad \hat{H}(\xi) = -\partial_\xi^2 - \frac{3}{16\xi^2} + \frac{1}{4\xi}. \quad (19)$$

Note that the singular operator $\hat{H}(\xi)$ is not Hermitian for generic wave function. Its spectrum is *continuous* and, in general, complex. The zero-mode solution corresponds to a zero sum of the two eigenvalues ($\pm\Lambda$) of the 1D Hamiltonians $\hat{H}(\xi)$ and $\hat{H}(\eta)$. Thus the solution to Eq.(19) emerges as an integral over a continuous variable $\lambda \propto 1/\sqrt{\Lambda}$ which can be taken real without loss of generality¹⁸. The integrand involves the product of two eigenfunctions $\Psi_\Lambda(\xi)$ and $\Psi_{-\Lambda}(\eta)$, and an arbitrary function $C(\lambda)$. Yet, one can find this function $C(\lambda)$ uniquely¹⁸ using the conditions of (i) smoothness of $\Phi(u, \phi)$ at $\phi = 0$ and $\phi = \frac{\pi}{2}$ and (ii) normalization of the phase distribution function $P_0(\phi) = \Phi(u = 0, \phi)$:

$$\Phi_{\text{an}}(u, \phi) = \frac{u^{\frac{1}{2}}}{\Gamma^4\left(\frac{1}{4}\right) |\cos \phi \sin \phi|^{\frac{1}{2}}} \int_0^\infty d\lambda \frac{\Gamma\left(\frac{1}{4} + \epsilon\lambda\right) \Gamma\left(\frac{1}{4} + \bar{\epsilon}\lambda\right)}{\lambda^{\frac{3}{2}}} \text{Re} \left[W_{-\epsilon\lambda, \frac{1}{4}}\left(\frac{\bar{\epsilon}\xi}{4\lambda}\right) W_{-\bar{\epsilon}\lambda, \frac{1}{4}}\left(\frac{\epsilon\eta}{4\lambda}\right) \right], \quad (20)$$

where $W_{\alpha, \frac{1}{4}}(x)$ is the Whittaker function, $\Gamma(x)$ is the Euler Gamma-function, and $\epsilon = e^{i\pi/4}$, $\bar{\epsilon} = e^{-i\pi/4}$.

Eq.(20) is the main analytic result of the Letter.

-*Moments of normalized eigenfunctions.* A convenient way to present the results is to plot the *reduced moments* $R_q = I_q(E = 0)/I_q(E \neq 0)$,

$$R_q = C_q \int_0^\infty du \int_0^{\pi/2} d\phi \cos^{2q}(\phi) u^{q-2} \Phi_{\text{an}}^2(u, \phi). \quad (21)$$

Here $I_q(E \neq 0) = L^{-1} (q-1)! \ell^{1-q}$ are the moments away from the anomaly, where $\Phi(u, \phi) = \Phi^{(\text{st})}(u)$ is given by Eq.(14), and $C_q = \frac{\pi 4^q}{(q-1)!(q-2)!}$. Using the solution Eq.(20) we evaluated the reduced moments R_q numerically up to $q = 10$. The results are given in Fig.2. One can see that at $E = 0$ the moments $R_q \approx (\ell/\ell_{\text{loc}})^{q-1}$ with small q follow Eq.(2), albeit with a localization length ℓ_{loc} larger than that away from the anomaly. The best exponential fit of moments with $q < 6$ gives $\ell_{\text{loc}}/\ell \approx 1.252$. This reflects the same tendency as Eq.(16). However, larger moments are significantly greater than the prediction of one-parameter scaling Eq.(2). The *excess factor* $S_q = I_q(E = 0)/I_q^{(\text{st})}$ which should be compared with that in the square brackets of Eq.(3), is plotted in the insert of Fig.2. A comparison with Eq.(3) shows a very satisfactory (for a crude qualitative interpretation in terms of two scales sketched in Fig.1) agreement for moments up to $q = 10$ which can be interpreted as an emergence of a very narrow (but still much wider than the lattice constant) peak in an "average" eigenfunction at the anomaly.

In conclusion, we solved exactly the problem of statistical moments I_q of the amplitude $|\psi_E(\mathbf{r})|^2$ of random wave functions in the 1D Anderson model at energies $E \approx 0$. It is shown that the statistics of such wave-

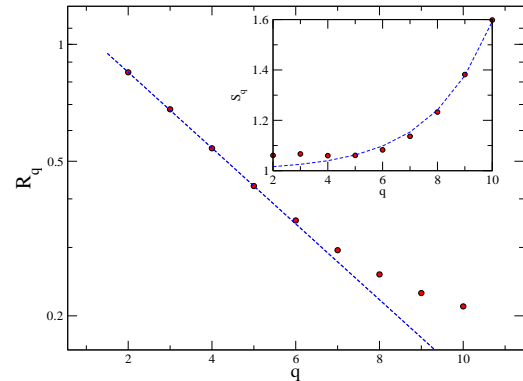


FIG. 2: (color line) Reduced moments R_q (red points) in the log-linear scale. The dashed line is the exponential fit $R_q = (\ell/\ell_{\text{loc}})^{q-1}$ with $\ell_{\text{loc}}(E = 0)/\ell = 1.252$. In the insert: The excess factor (red points). The dashed line is a fit by Eq.(3) with $p \approx 0.01$ and $\ell_{\text{loc}}(E = 0)/d \approx 160$.

functions is anomalous which anomaly does not merely reduce to the variation of the localization length or the Lyapunov exponent. The enhancement of the localization length $\ell_{\text{loc}}/\ell \approx 1.252$ derived from $I_q \propto \ell_{\text{loc}}^{1-q}$ with $q < 6$ is different from that obtained from the inverse Lyapunov exponent $\gamma(E \neq 0)/\gamma(E = 0) \approx 1.094$. This fact together with the anomalous enhancement of moments with large $q > 6$ implies a significant change of the form of the typical eigenfunction at $E \approx 0$ which requires more than one characteristic length for its description.

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- ¹ P.W.Anderson, Phys.Rev. **109**, 1492 (1958).
² F.Evers and A.D.Mirlin, Rev.Mod.Phys. **80**, 1355 (2008).
³ V.L.Berezinskii, Zh.Exp.Teor.Fiz. **65**, 1251 (1973)[Sov.Phys.JETP **38**, 620 (1974)].
⁴ V.I.Melnikov, JETP Lett. **32**, 225 (1980).
⁵ I.M.Lifshitz, S.A.Gredeskul, and L.A.Pastur, *Introduction to the theory of disordered systems* (Wiley, N.Y., 1988).
⁶ M.Kappus and F.Wegner, Z.Phys. B **45**, 15 (1981).
⁷ B.Derrida and E.Gardner, J.Phys. (Paris) **45**, 1283 (1984).
⁸ H.Schomerus and M.Titov, Phys.Rev. B **67**, 100201(R) (2003).
⁹ L.I.Deych, et al., Phys.Rev.Lett. **91**, 096601 (2003).
¹⁰ F.J.Dyson, Phys.Rev. **92**, 1331 (1958).
¹¹ L.Tessieri and F.M.Izrailev, Phys.Rev. E **62**, 3090 (2000).
¹² J.Billy, V.Josse,Z.Zuo, A.Bernard, B.Hambrecht, P.Lugan, D.Clement, L.Sanches-Palencia, P.Bouyer, and A.Aspect, Nature (London) **453**, 891 (2008).
¹³ G.Roati, C.D'Errico, L.Fallani, M.Fattori, C.Fort, M.Zaccanti, G.Modugno, and M.Inguscio, Nature (London) **453**, 895 (2008).
¹⁴ F.L.Moore, J.C.Robinson, C.F.Bharucha, S.Sundaram, and M.G.Raizen, Phys.Rev.Lett. **75**, 4598 (1995).
¹⁵ B.A.Muzykantskii and D.E.Khmelnitskii, Phys.Rev.B **51**, 5480 (1995); I. E. Smolyarenko and B. L. Altshuler, Phys. Rev. B **55**, 10451 (1997); V.M.Apalkov, M.E.Raikh, and B.Shapiro, Phys.Rev.Lett. **92**, 066601 (2004).
¹⁶ The analogous expression away from anomaly is given in Ref.¹⁷, Sec.3.1.
¹⁷ A.D.Mirlin, Phys.Rep. **326**, 259 (2000).
¹⁸ V.E.Kravtsov and V.I.Yudson, arXiv:1011.1480 (unpublished)
¹⁹ A.Ossipov and V.E.Kravtsov, Phys.Rev. B **73**, 033105 (2006).