MEROMORPHIC EXTENDIBILITY AND RIGIDITY OF INTERPOLATION

MRINAL RAGHUPATHI AND MAXIM L. YATTSELEV

Abstract. Let $\mathbb{T}$ be the unit circle, f be an α-Hölder continuous function on $\mathbb{T}$, $\alpha > 1/2$, and $\mathcal{A}$ be the algebra of continuous function in the closed unit disk $\overline{\mathbb{D}}$ that are holomorphic in $\mathbb{D}$. Then f extends to a meromorphic function in $\mathbb{D}$ with at most m poles if and only if the winding number of $f+h$ on $\mathbb{T}$ is bigger or equal to $-m$ for any $h \in \mathcal{A}$ such that $f+h \neq 0$ on $\mathbb{T}$.

1. Main Results

Let g be a non-vanishing continuous function on a simple Jordan curve T. Denote by $\mathrm{w}_T(g)$ the winding number of $g(T)$ around the origin. That is, $2\pi\mathrm{w}_T(g)$ is equal to the change of the argument of g on T when the curve T is traversed in the positive direction with respect to D, the interior domain of T. Denote by $\mathcal{A}(D)$ the algebra of functions continuous on $\overline{D}$ and holomorphic in D.

Motivated by the work of Alexander and Wermer [2] and Stout [12], Globevnik [4] obtained the following characterization of functions in the disk algebra $\mathcal{A} := \mathcal{A}(\mathbb{D})$, where $\mathbb{D}$ is the unit disk.

Theorem 1 (Globevnik [4]). *A continuous function f on the unit circle $\mathbb{T}$ extends holomorphically through $\mathbb{D}$ if and only if $\mathrm{w}_\mathbb{T}(f+q) \geq 0$ for each polynomial q such that $f+q \neq 0$ on $\mathbb{T}$.*

A shorter proof, based on the notion of badly-approximable functions, was obtained by Khavinson [8].

The polynomials are a dense subalgebra of $\mathcal{A}$. Thus, for any $h \in \mathcal{A}$ such that $f+h \neq 0$ on $\mathbb{T}$, there exists a polynomial q satisfying $|h-q| < |f+h|$ on $\mathbb{T}$. Then

$$\mathrm{w}_\mathbb{T}(f+q) = \mathrm{w}_\mathbb{T}(f+h+q-h) = \mathrm{w}_\mathbb{T}(f+h) + \mathrm{w}_\mathbb{T}\left(1+\frac{q-h}{f+h}\right) = \mathrm{w}_\mathbb{T}(f+h). \tag{1}$$

Hence, the hypothesis of Globevnik's result above could equivalently have been stated as $w_\mathbb{T}(f+q) \geq 0$ for all $q \in \mathcal{A}$.

In later work Globevnik [3] was able to generalize the above result to multiply-connected domains. Namely, let D be a domain whose boundary T consists of finitely many pairwise disjoint Jordan curves. The winding number of a continuous, non-vanishing function g on T, $\mathrm{w}_T(g)$, is defined as the sum of the individual winding numbers on each Jordan curve constituting T oriented positively with respect to D. Then *a continuous function f on T is the trace of a function in $\mathcal{A}(D)$ if and only if $\mathrm{w}_T(f+h) \geq 0$ for any $h \in \mathcal{A}(D)$ such that $f+h \neq 0$ on T.*

2000 *Mathematics Subject Classification.* 30E25.

Key words and phrases. meromorphic extensions, winding number, interpolation.

The research was conducted during M.Y.'s stay at Vanderbilt University as a Visiting Scholar.

The next natural question is to characterize functions that admit *meromorphic* continuation. On this path Globevnik [5] showed the following: *A continuous function f on $\mathbb{T}$ extends meromorphically through $\mathbb{D}$ with at most $m \in \mathbb{Z}_+ := \mathbb{N}\cup\{0\}$ poles*[1] *in $\mathbb{D}$ if and only if $\mathrm{w}_{\mathbb{T}}(pf+q) \geq -m$ for each pair of polynomials p and q such that $pf+q \neq 0$ on $\mathbb{T}$.* Moreover, for sufficiently smooth f it could be assumed that $p \in \mathcal{P}_m$, i.e., p is a polynomial of degree at most m. In fact, this assumption can be made regardless of the smoothness of f. Indeed, it follows from the same method used in [8] for the case $m = 0$. One merely replaces the best holomorphic approximant by the best meromorphic approximant with at most m poles. It is known from AAK-theory [1] that the error of such approximation has constant modulus on $\mathbb{T}$ and winding number at most $-(2m+1)$. Finally, in [6], Globevnik derived a similar result on meromorphic extendibility for multiply connected domains where p and q belong to $\mathcal{A}(D)$.

The work on holomorphic extendibility in [5] and [6] indicates that the above results could be improved. In particular, is it the case that the conclusion of the above results on meromorphic extensibility are still true, under the weaker hypothesis that $\mathrm{w}_T(f+q) \geq -m$ for all $q \in \mathcal{A}(D)$. In this work we give an affirmative answer to this question in the case of the unit circle when f is sufficiently smooth. We now state our main result.

Theorem 2. *Let f be an α-Hölder continuous function on $\mathbb{T}$, $\alpha > 1/2$. Let $m \in \mathbb{Z}_+$. Then f extends to a meromorphic function with at most m poles in $\mathbb{D}$ if and only if*

$$\mathrm{w}_{\mathbb{T}}(f+h) \geq -m \tag{2}$$

for every $h \in \mathcal{A}$ such that $f+h \neq 0$ on $\mathbb{T}$.

As indicated in the argument following (1), this theorem can be stated equivalently with h in the algebra of analytic polynomials.

Notice that the necessity of (2) is trivial. Indeed, if $f = g/q$, $g \in \mathcal{A}$ and $q \in \mathcal{P}_m$, then $\mathrm{w}_{\mathbb{T}}(f+h) = \mathrm{w}_{\mathbb{T}}(g+qh) - \mathrm{w}_{\mathbb{T}}(q) \geq -m$ as the winding number of $g+qh$ is non-negative and $\mathrm{w}_{\mathbb{T}}(q)$ is equal to the number of zeros of q in $\mathbb{D}$. Thus, we need only to show that (2) is sufficient for f to be the trace of a function meromorphic in $\mathbb{D}$ with at most m poles there.

Let $\varphi \in \mathcal{A}$ be bi-Lipschitz in $\overline{\mathbb{D}}$. That is, there exists a finite positive constant c such that

$$(1/c)|z_1 - z_2| \leq |\varphi(z_1) - \varphi(z_2)| \leq c|z_1 - z_2|, \quad z_1, z_2 \in \overline{\mathbb{D}}.$$

Put $D_\varphi := \varphi(\mathbb{D})$ and $L_\varphi := \varphi(\mathbb{T})$. Clearly, it holds that $h \in \mathcal{A}(D_\varphi)$ if and only if $h \circ \varphi \in \mathcal{A}$. Moreover, it is true that $\mathrm{w}_{L_\varphi}(g) = \mathrm{w}_{\mathbb{T}}(g \circ \varphi)$ for any continuous function g on L_φ. It is also true that φ preserves Hölder classes. Thus, the following result is another immediate consequence of Theorem 2.

Corollary 3. *Let $\varphi \in \mathcal{A}$ be bi-Lipschitz in $\overline{\mathbb{D}}$. Then Theorem 2 remains valid when $\mathbb{T}$ and $\mathbb{D}$ are replaced by L_φ and D_φ, respectively.*

In what follows, we suppose that $m \in \mathbb{N}$ since the case $m = 0$ was shown in [3]. Moreover, without loss of generality we may assume that $f \notin \mathcal{A}$. This, in

[1]The counting of poles and zeros is done including multiplicities.

particular, implies that $f_- \not\equiv 0$, where

$$f_-(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(t)dt}{z-t}, \quad z \notin \overline{\mathbb{D}}, \tag{3}$$

is the anti-analytic part of f. It is known that each f_- is a function holomorphic outside of $\overline{\mathbb{D}}$ whose trace on $\mathbb{T}$ is also α-Hölder continuous.

Our approach lies in recasting the condition on the winding number of $f+h$, $h \in \mathcal{A}$, as a certain rigidity property of interpolation of f by polynomials. For a function g holomorphic in $\mathbb{D}$ we denote by $\mathrm{z}(g)$ the number of zeros of g in $\mathbb{D}$ counting multiplicities. The following proposition is central to our approach.

Proposition 4. *Let f be an α-Hölder continuous function on $\mathbb{T}$, $\alpha > 1/2$. Let $m \in \mathbb{N}$. Then (2) is satisfied if and only if*

$$\mathrm{z}(f_n + p) \leq m + n \tag{4}$$

holds for for any $n \in \mathbb{Z}_+$ and any polynomial $p \in \mathcal{P}_n$, where $f_n(z) = z^n f_-(1/z)$, $z \in \mathbb{D}$.

The condition on Hölder continuity of f appearing in the statement of Theorem 2 is needed exactly for the proof of this proposition. It ensures the fact that the image of $\mathbb{T}$ under $f_n + p$ has no interior points.

Motivated by the interpolation rigidity property (4), we define the following classes

$$\mathcal{I}_{n,m} := \{g : \ g \text{ is holomorphic in } \mathbb{D}, \ \mathrm{z}(g+p) \leq m+n \text{ for any } p \in \mathcal{P}_n\},$$

$n \in \mathbb{Z}_+$, $m \in \mathbb{N}$. That is, a function g holomorphic in $\mathbb{D}$ belongs to the class $\mathcal{I}_{n,m}$ if and only if any polynomial of degree at most n interpolates g at no more than $n+m$ points. It follows immediately from the definition that $\mathcal{I}_{n,m} \subset \mathcal{I}_{n,m+1}$. Moreover, $g \in \mathcal{I}_{0,1}$ if and only if g is a univalent function in $\mathbb{D}$ and $g \in \mathcal{I}_{0,m+1} \setminus \mathcal{I}_{0,m}$ if and only if g is an $(m+1)$-valent function.

Observe that a function f, continuous on $\mathbb{T}$, is the trace of a function meromorphic in $\mathbb{D}$ if and only if f_- is a rational function. Thus, the sufficiency part of Theorem 2 is a consequence of Proposition 4 and the following theorem applied with $g(\cdot) = f_-(1/\cdot)$.

Theorem 5. *Let g be a holomorphic function in $\mathbb{D}$ such that $g_n \in \mathcal{I}_{n,m}$ for any $n \in \mathbb{Z}_+$, $g_n(z) = z^n g(z)$. Then g is a rational function of type (m,m) holomorphic in $\mathbb{D}$.*

Clearly if g is a rational function of type (m,m), then the numerator of $g_n + p$, $p \in \mathcal{P}_n$, belongs to $\mathcal{P}_{n+m}$ and therefore $\mathrm{z}(g_n+p) \leq n+m$ for any $n \in \mathbb{Z}_+$.

Theorem 5 was initially proved for the case $m=1$ independently by Ruscheweyh [10] and Kirjackis [9]. In this work, we elaborate on the approach devised in [9].

2. Proofs

Proof of Proposition 4. We prove this proposition in two simple steps. First, we show that (4) is equivalent to

$$\mathrm{z}_{\{|z|>1\}}(f_- + q) \leq \deg(q) + m \tag{5}$$

for any algebraic polynomial q, where $\mathrm{z}_{\{|z|>1\}}(f_- + q)$ stands for the number of zeros of the function $f_- + q$ in $\{|z| > 1\}$. Second, we establish the equivalence of (5) and (2).

Let $q \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$ be given. Set $p(z) := z^n q(1/z)$. Then $p \in \mathcal{P}_n$ and $p(0) \neq 0$. Since

$$z^n(f_- + q)(1/z) = z^n f_-(1/z) + z^n q(1/z) = (f_n + p)(z), \tag{6}$$

it holds that $\mathrm{z}_{\{|z|>1\}}(f_- + q) = \mathrm{z}(f_n + p)$. That is, (4) implies (5). Conversely, let $p \in \mathcal{P}_n$ be given. Denote by k the multiplicity of the zero of p at the origin, $k = 0$ if $p(0) \neq 0$. Set $q(z) := z^n p(1/z)$. Then $q \in \mathcal{P}_{n-k}$ and $\mathrm{z}_{\{|z|>1\}}(f_- + q) = \mathrm{z}(f_n + p) - k$ by (6). Hence, (5) implies (4). Thus, (5) and (4) are equivalent.

Let $h \in \mathcal{A}$ be such that $f + h \neq 0$ on $\mathbb{T}$. Since $f_+ + h \in \mathcal{A}$, $f_+ := f - f_-$, there exists a polynomial q such that $|q - f_+ - h| < |f + h|$ on $\mathbb{T}$. Then we get as in (1) that $\mathrm{w}_{\mathbb{T}}(f + h) = \mathrm{w}_{\mathbb{T}}(f_- + q)$. As $f_- + q$ is a meromorphic function in $\{|z| > 1\}$, $\mathrm{w}_{\mathbb{T}}(f_- + q)$ is simply the difference between the number of poles and the number of zeros of $f_- + q$ there. That is,

$$\mathrm{w}_{\mathbb{T}}(f_- + q) = \deg(q) - \mathrm{z}_{\{|z|>1\}}(f_- + q) \tag{7}$$

since the point at infinity is the only possible pole of $f_- + q$ and the order of this pole is $\deg(q)$. Thus, we get that $\mathrm{w}_{\mathbb{T}}(f + h) = \deg(q) - \mathrm{z}_{\{|z|>1\}}(f_- + q)$. That is, (5) implies (2).

Let q be a polynomial and assume that $f_- + q \neq 0$ on $\mathbb{T}$. Applying (2) with $h = q - f_+$, we get that $\mathrm{w}_{\mathbb{T}}(f_- + q) \geq -m$. The desired conclusion then follows from (7). Assume now that $f_- + q = 0$ for some points on $\mathbb{T}$. As f_- is α-Hölder continuous function on $\mathbb{T}$ with $\alpha > 1/2$, $(f_- + q)(\mathbb{T})$ has no interior points [11]. Thus, for any $\delta > 0$ there exists a complex number ϵ such that $|\epsilon| = \delta$ and $f_- + q + \epsilon \neq 0$ on $\mathbb{T}$. Then (5) follows from the Rouché's theorem. This finishes the proof of the proposition. □

Let g be an m-valent function in $\mathbb{D}$. That is, there exists a constant a such that a interpolates g at m points in $\mathbb{D}$, say $\{z_1, \dots, z_m\}$, in the Hermite sense. Put

$$g^a(z) := \frac{cz^m(g(z) - a)}{(z - z_1)\cdots(z - z_m)}, \tag{8}$$

where the constant c is chosen so $z^{-m} g^a(z) \to 1$ as $z \to 0$. Clearly, g^a is again a holomorphic function in $\mathbb{D}$.

Denote by $\mathcal{S}_m$ the set of m-valent holomorphic functions in $\mathbb{D}$ satisfying $z^{-m} g(z) \to 1$ as $z \to 0$, and set

$$\mathcal{B}_m := \left\{ g \in \mathcal{S}_m : \ g(z) = \frac{z^m}{d(z)}, \ d \in \mathcal{P}_m \right\}.$$

Necessarily, it holds that $d(0) = 1$ and $d(z) \neq 0$ in $\mathbb{D}$. Recall [7, Thm. 5.3] that for any $a \in D_m := \{|z| < 1/4^m\}$, the equation $g(z) = a$ has exactly m roots in $\mathbb{D}$ for any $g \in \mathcal{S}_m$.

For functions in $\mathcal{S}_m$ we adopt the notation

$$g(z) = z^m + (g)_1 z^{m+1} + \cdots + (g)_k z^{m+k} + \cdots.$$

Lemma 6. *There exists a set of polynomials of m variables, say $\{\ell_{m,k}\}$, $k \in \mathbb{N}$, $k > m$, such that $g \in \mathcal{B}_m$ if and only if $(g)_k = \ell_{m,k}((g)_1, \dots, (g)_m)$.*

Proof. Let $g \in \mathcal{B}_m$. That is, $g(z) = z^m/d(z)$, $d(z) = 1 + d_1 z + \cdots + d_m z^m$. Then

$$1 \equiv (1 + d_1 z + \cdots + d_m z^m)(1 + (g)_1 z + \cdots + (g)_m z^m + \cdots).$$

Hence,

$$\left\{\begin{array}{lcl} d_1 & = & -(g)_1, \\ d_k & = & -(g)_k-(g)_{k-1}d_1-\cdots-(g)_1d_{k-1}, \quad k\in\{2,\dots,m\}, \end{array}\right. \tag{9}$$

and

$$(g)_n=-(g)_{n-1}d_1-\cdots-(g)_{n-m}d_m, \quad n\in\mathbb{N}, \quad n>m. \tag{10}$$

Clearly, relations (9) allow us to express d_k polynomially through $(g)_1,\dots,(g)_k$ for each $k\in\{1,\dots,m\}$. Polynomials $\ell_{m,n}$ are constructed then using (10) inductively in n by plugging in the corresponding expressions for d_k.

To prove the "if" part, observe that relations (9) uniquely determine the set of coefficients $d_1,\dots,d_m$ for a given set $\{(g)_1,\dots,(g)_m\}$. Taking into account the way polynomials $\ell_{n,m}$ were constructed, we see that relations (10) take place with these $d_1,\dots,d_m$. That is, $g(z)/z^m=1/d(z)$ for some polynomial $d\in\mathcal{P}_m$. □

Lemma 7. *Let $g\in\mathcal{S}_m$. Then $|(g^a)_k|=|(g)_k|$, $k\in\{1,\dots,m\}$, if and only if $g\in\mathcal{B}_m$.*

Proof. It is a trivial computation to verify that $g^a=g$ for any $g\in\mathcal{B}_m$. Thus, we only need to prove the "only if" part.

Let $g\in\mathcal{S}_m$ and suppose that $|(g^a)_k|=|(g)_k|$, $k\in\{1,\dots,m\}$. Transformation (8) can be equivalently written as

$$g^a(z)=\frac{z^m}{d(z)}\left(1-\frac{g(z)}{a}\right), \quad d(z)=\left(1-\frac{z}{\zeta}\right)\left(1-\frac{z}{\zeta_2}\right)\cdots\left(1-\frac{z}{\zeta_m}\right), \tag{11}$$

where $g(\zeta)=g(\zeta_j)=a$, $j\in\{2,\dots,m\}$. Clearly, $\zeta=\zeta(a)$ and $\zeta_j=\zeta_j(a)$, $j\in\{2,\dots,m\}$, are, in fact, holomorphic functions of $a\in D_m^*:=D_m\setminus[0,1/4^m]$ (or any other domain obtained from D_m by removing a Jordan arc that connects the origin and some point on the boundary of this disk). Write

$$\frac{1}{d(z)}=1+\left(\frac{1}{d}\right)_1 z+\cdots+\left(\frac{1}{d}\right)_m z^m+\cdots.$$

As in (9), it holds that

$$\left(\frac{1}{d}\right)_k=-s_k\left(-\frac{1}{\zeta},-\frac{1}{\zeta_2},\dots,-\frac{1}{\zeta_m}\right)-\sum_{j=1}^{k-1}\left(\frac{1}{d}\right)_{k-j}s_j\left(-\frac{1}{\zeta},-\frac{1}{\zeta_2},\dots,-\frac{1}{\zeta_m}\right), \tag{12}$$

where $s_k(b_1,\dots,b_N)$ is the k-th symmetric function of $b_1,\dots,b_N$, i.e.,

$$\left\{\begin{array}{lcl} s_N(b_1,\dots,b_N) & = & \prod_{j=1}^N b_j \\ s_{N-1}(b_1,\dots,b_N) & = & \sum_{i=1}^N\prod_{j\neq i} b_j \\ & \cdots & \\ s_1(b_1,\dots,b_N) & = & \sum_{j=1}^N b_j. \end{array}\right. \tag{13}$$

It is a simple computation to check using (11) that

$$(g^a)_j=\left(\frac{1}{d}\right)_j, \quad j\in\{1,\dots,m-1\}, \quad \text{and} \quad (g^a)_m=\left(\frac{1}{d}\right)_m-\frac{1}{a}. \tag{14}$$

In particular, this means that each $(g^a)_k$, $k\in\{1,\dots,m\}$, is a holomorphic function of a in D_m^*. Thus, $(g^a)_k\equiv c_k$ for some constants c_k independently of a since by the conditions of the lemma functions $(g^a)_k$ have constant modulus. Hence, (14) and (12) yield that there exist constants c_k^*, $k \in \{1,\dots,m\}$, such that

$$s_k\left(-\frac{1}{\zeta},-\frac{1}{\zeta_2},\dots,-\frac{1}{\zeta_m}\right) \equiv c_k^*, \quad j \in \{1,\dots,m-1\}, \tag{15}$$

and

$$s_m\left(-\frac{1}{\zeta},-\frac{1}{\zeta_2},\dots,-\frac{1}{\zeta_m}\right) \equiv c_m^* - \frac{1}{a} \tag{16}$$

for all $a \in D_m$. It is easy to verify that

$$\begin{cases} s_1\left(-\frac{1}{\zeta},-\frac{1}{\zeta_2},\dots,-\frac{1}{\zeta_m}\right) & = -\frac{1}{\zeta} + s_1\left(-\frac{1}{\zeta_2},\dots,-\frac{1}{\zeta_m}\right) \\ s_k\left(-\frac{1}{\zeta},-\frac{1}{\zeta_2},\dots,-\frac{1}{\zeta_m}\right) & = -\frac{1}{\zeta}s_{k-1}\left(-\frac{1}{\zeta_2},\dots,-\frac{1}{\zeta_m}\right) + s_k\left(-\frac{1}{\zeta_2},\dots,-\frac{1}{\zeta_m}\right) \\ s_m\left(-\frac{1}{\zeta},-\frac{1}{\zeta_2},\dots,-\frac{1}{\zeta_m}\right) & = -\frac{1}{\zeta}s_{m-1}\left(-\frac{1}{\zeta_2},\dots,-\frac{1}{\zeta_m}\right) \end{cases} \tag{17}$$

for $k \in \{2,\dots m-1\}$. Combining (17) with (15), we derive that

$$s_m\left(-\frac{1}{\zeta},-\frac{1}{\zeta_2},\dots,-\frac{1}{\zeta_m}\right) = -\left(\frac{1}{\zeta^m} + \frac{c_1^*}{\zeta^{m-1}} + \dots + \frac{c_{m-1}^*}{\zeta}\right). \tag{18}$$

Finally, since $a = g(\zeta)$, we get from (16) and (18) that

$$g(\zeta) = \frac{\zeta^m}{c_m^*\zeta^m + \dots + c_1^*\zeta + 1}.$$

That is, $g \in \mathcal{B}_m$. □

Lemma 8. *Let ℓ be a polynomial of N variables and $g \in \mathcal{S}_m$. Then $|\ell((g^a)_{k_1},\dots,(g^a)_{k_N})|$ is a subharmonic function of $a \in D_m$.*

Proof. As the modulus of a holomorphic function is subharmonic, we only need to show that $(g^a)_k$ is a holomorphic function of a for any $k \in \mathbb{N}$.

Let $a \in D_m$ and $z_1,\dots,z_m$ be such that $g(z_j) = a$. Then each $z_j(a)$ is a holomorphic function of $a \in D_m^* := D_m \setminus [0,1/4^m]$ and for each j_1 there exists $j_2(\neq j_1)$ such that $z_{j_1}^+ \equiv z_{j_2}^-$ on $[0,1/4^m]$, where z_j^+ and z_j^- are the traces of z_j from above and below on $[0,1/4^m]$. This, in particular, means that $s_j(z_1(a),\dots,z_m(a))$ is a holomorphic function of $a \in D_m$ by the principle of analytic continuation for each $j \in \{1,\dots,m\}$, where s_j is the j-th symmetric function (13). Thus,

$$d(z,a) = (z - z_1(a))\cdots(z - z_m(a))$$

is a holomorphic function of $a \in D_m$ for each z and it vanishes as a function of a only if $g(z) = a$. Hence, $g^a(z)$ is a holomorphic function of $a \in D_m$ for each z. Since,

$$(g^a)_k = \frac{1}{\rho^{2(k+m)}}\int_{|\tau|=\rho} \bar{\tau}^{k+m} g^a(\tau)\frac{d\tau}{2\pi\rho},$$

the conclusion of the lemma follows. □

In the proof of Lemma 7 we pointed out that $\mathcal{B}_m$ is invariant under (8). In fact, there are no larger subset of $\mathcal{S}_m$ with this property.

Lemma 9. *$\mathcal{B}_m$ is the largest subset of $\mathcal{S}_m$ invariant under (8).*

Proof. Denote by $\mathcal{B}$ the largest subset of $\mathcal{S}_m$ invariant under (8). Observe that $\mathcal{B}$ is compact with respect to the locally uniform convergence in $\mathbb{D}$. Indeed, it is known [7, Thm. 5.3] that $\mathcal{S}_m$ is a normal family and therefore $\mathcal{B}$ is bounded. Let $\{g_\alpha\}\subset\mathcal{B}$ be a convergent sequence and g be the limit function. Clearly, g_α^a converge to g^a locally uniformly in $\mathbb{D}$ as well. Then $g,g^a\in\mathcal{S}_m$ by Hurwitz's theorem. By iterating this process, we indeed see that $g\in\mathcal{B}$.

Fix $n\in\mathbb{N}$, $n>m$, and denote by $\mathcal{B}^n$ the subset of $\mathcal{B}$ consisting of functions maximizing the functional

$$\phi_n(g) := |(g)_n - \ell_{m,n}((g)_1,\dots,(g)_m)|,$$

where the polynomial $\ell_{m,n}$ was introduced in Lemma 6. Since $\mathcal{B}$ is compact, $\mathcal{B}^n$ is non-empty and clearly compact. Let $g\in\mathcal{B}^n$. By Lemma 8, $\phi_n(g^a)$ is a subharmonic function of a. As $g^0=g$, the maximum principle for subharmonic functions yields that $g^a\in\mathcal{B}^n$ for all $a\in D_m$.

Among all the functions g in $\mathcal{B}^n$, chose those that have maximal modulus of $(g)_m$. Using the subharmonicity of $|(g)_m|$ and compactness of $\mathcal{B}^n$ as in the previous paragraph, we deduce that this set is compact, non-empty, and invariant under (8). Further, among functions g in the latter set, choose those that maximize $|(g)_{m-1}|$. Once more, it follows that this new set is compact, non-empty, and invariant under (8). Repeat this procedure for $(g)_{m-2},\dots,(g)_1$. After the last step, we reach a non-empty set invariant under (8) with the property $|(g)_k|=|(g^a)_k|$ for all $k\in\{1,\dots,m\}$, $a\in D_m$, and each g in this set. By Lemma 7, this set is contained in $\mathcal{B}_m$.

We have shown that some of the maximizing functions for the functional ϕ_n belong to $\mathcal{B}_m$. However, $\phi_n\equiv 0$ on $\mathcal{B}_m$ by the very definition of $\ell_{n,m}$ and therefore $\phi_n\equiv 0$ on $\mathcal{B}$. Since this is true for any n, Lemma 6 yields that $\mathcal{B}=\mathcal{B}_m$. □

Proof of Theorem 5. Let g be such that $g_n\in\mathcal{I}_{n,m}$ for all $n\in\mathbb{Z}_+$. Without loss of generality we may assume that m is the smallest natural number with this property. Thus, there exists the smallest integer in $\mathbb{Z}_+$, say n_0, such that

$$\mathrm{z}(g_{n_0}+q) = m+n_0 \quad \text{for some} \quad q\in\mathcal{P}_{n_0}.$$

Set $k:=m+n_0$ and v, $\deg(v)=k$, to be the monic polynomial vanishing at the zeros of $g_{n_0}+q$. Define

$$y(z) := cz^k\frac{(g_{n_0}+q)(z)}{v(z)}, \tag{19}$$

where c is a normalizing constant such that $z^{-m}y(z)\to 1$ as $z\to 0$. Then y is a holomorphic function in $\mathbb{D}$ and for any $p\in\mathcal{P}_n$ it holds that

$$\begin{aligned} \mathrm{z}(y_n+p) &= \mathrm{z}\left(\frac{cg_{k+n+n_0}+cz^{k+n}q+pv}{v}\right) = \mathrm{z}\left(g_{k+n+n_0}+z^{k+n}q+\frac{1}{c}pv\right)-k \\ &\leq m+k+n+n_0-k = k+n \end{aligned} \tag{20}$$

as $(cz^{k+n}q+pv)\in\mathcal{P}_{k+n+n_0}$. That is, $y\in\mathcal{S}_k$ and $y_n\in\mathcal{I}_{n,k}$ for all $n\in\mathbb{Z}_+$. Since (19) can be viewed as transformation (8) applied to $g_{n_0}+q$ with $a=0$, estimates analogous to (20) show that $y^a\in\mathcal{S}_k$ and $y_n^a\in\mathcal{I}_{n,k}$, $n\in\mathbb{Z}_+$, for all $a\in D_k$. Applying transformation (8) to y^a, we again get that the newly obtained function belongs to $\mathcal{S}_k$ and its shifts by z^n belong to $\mathcal{I}_{n,k}$. Clearly, this process can be continued indefinitely. That is, y belongs to the subset of $\mathcal{S}_k$ invariant under (8), i.e, $y \in \mathcal{B}_k$ by Lemma 9. Hence, $y(z) = z^k/d(z)$, $d \in \mathcal{P}_k$. Then

$$g(z) = \frac{1}{z^{n_0}}\left(\frac{1}{c}\frac{v(z)}{d(z)} - q(z)\right).$$

Since $\deg(v) = k$ and $\deg(q) = n_0$, g is a rational function of type $(\max\{m,l\}, l)$ with $l := \deg(d) \leq m + n_0$. Thus, it remains to show that $l \leq m$.

Set $\mathbb{D}_\rho := \{z : |z| < \rho\}$ for fixed $\rho < 1$. Clearly, $g \in \mathcal{A}(\mathbb{D}_\rho)$ and it holds that

$$\mathrm{z}_\rho(g_n + p) \leq n + m \tag{21}$$

for any $p \in \mathcal{P}_n$ and $n \in \mathbb{Z}_+$, where $\mathrm{z}_\rho(g_n + p)$ is the number of zeros $g_n + p$ in $\mathbb{D}_\rho$. Since polynomials are contained and dense in $\mathcal{A}(\mathbb{D}_{1/\rho})$, we get as in Proposition 4 that (21) is equivalent to

$$\mathrm{w}_{\mathbb{T}_{1/\rho}}(g(1/\cdot) + h) \geq -m \tag{22}$$

for all $h \in \mathcal{A}(\mathbb{D}_{1/\rho})$ such that $g(1/\cdot) + h \neq 0$ on $\mathbb{T}_{1/\rho} := \{z : |z| = 1/\rho\}$. It is easy to see that $g(1/\cdot) = s/r$, where $s \in \mathcal{P}_l$ and $r \in \mathcal{P}_l \setminus \mathcal{P}_{l-1}$. As $r(z) = z^l d(1/z)$, all the zeros of r, say $\{w_1, \ldots, w_l\}$, belong to $\overline{\mathbb{D}} \subset \mathbb{D}_{1/\rho}$. Fix a determination of each $\log s(w_j)$, $j \in \{1, \ldots, l\}$, and let u be a polynomial interpolating the values $\log s(w_j)$ at the points w_j, respectively. Set $h := (e^u - s)/r$. Then $h \in \mathcal{A}(\mathbb{D}_{1/\rho})$ and

$$\mathrm{w}_{\mathbb{T}_{1/\rho}}(f + h) = \mathrm{w}_{\mathbb{T}_{1/\rho}}\left(\frac{s}{r} + \frac{e^u - s}{r}\right) = \mathrm{w}_{\mathbb{T}_{1/\rho}}\left(\frac{e^u}{r}\right) = -l$$

since $e^u \neq 0$ in $\overline{\mathbb{D}}_{1/\rho}$. Thus, $l \leq m$ by (22). □

References

[1] V.M. Adamyan, D.Z. Arov, and M.G. Krein. Analytic properties of Schmidt pairs for a Hankel operator on the generalized Schur-Takagi problem. *Math. USSR Sb.*, 15:31–73, 1971. English transl. in *Math. USSR Sb.* 15, 1971.

[2] H. Alexander and J. Wermer. Linking numbers and boundaries of varieties. *Ann. Math.*, 151:125–150, 2000.

[3] J. Globevnik. The argument principle and holomorphic extendibility. *J. d'Analyse Math.*, 94:385–395, 2004.

[4] J. Globevnik. Holomorphic extendibility and the argument principle. In M. Agranovsky, L. Karp, and D. Shoikhet, editors, *Complex Analysis and Dynamical Systems II*, volume 382 of *Contemporary Mathematics*, pages 171–175. Amer. Math. Soc., 2005.

[5] J. Globevnik. Meromorphic extendibility and the argument principle. *Publ. Mat.*, 52:171–188, 2008.

[6] J. Globevnik. On meromorphic extendibility. *J. Math. Anal. Appl.*, 351:285–290, 2008.

[7] W.K. Hayman. *Multivalent Functions*, volume 110 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 1994.

[8] D. Khavinson. A note on a theorem of J. Globevnik. In M. Agranovsky, L. Karp, and D. Shoikhet, editors, *Complex Analysis and Dynamical Systems II*, volume 382 of *Contemporary Mathematics*, pages 227–228. Amer. Math. Soc., 2005.

[9] E.G. Kirjackis. A collection of univalent functions. *Lithuanian Math. J.*, 16(2):111–116, 1976.

[10] St. Ruscheweyh. On functions with nonvanishing divided differences. *Math. Nachr.*, 73:143–146, 1976.

[11] R. Salem and A. Zygmund. Lacunary power series and Peano curves. *Duke Math. J.*, 12:569–578, 1945.

[12] E.L. Stout. Boundary values and mapping degree. *Michig. Math. J.*, 47:353–368, 2000.

Department of Mathematics, Vanderbilt University, Nashville, TN, 37240
E-mail address: mrinal.raghupathi@vanderbilt.edu

Corresponding author, Department of Mathematics, University of Oregon, Eugene, OR, 97403
E-mail address: maximy@uoregon.edu