$1|1$ PARALLEL TRANSPORT AND CONNECTIONS

FLORIN DUMITRESCU

Abstract. In [4] we defined a notion of parallel transport along superpaths in a supermanifold coming from a vector bundle with connection over the supermanifold. In this note, we show that the converse is also true, at least when the base supermanifold is a manifold.

Contents

1. Introduction and Statement of Result

Let E be a $\mathbf{Z}/2$-graded vector bundle over a compact manifold M. We consider a notion of parallel transport along superpaths in M, generalizing the notion of reparametrization-invariant parallel transport along paths in M, and show that it characterizes even (grading-preserving) connections over M.

Such a problem is motivated by obtaining a characterization of supersymmetric one-dimensional topological field theories (abbreviated TFTs) over a manifold. This would extend the description of one-dimensional TFTs over a space M as vector bundles with connection over M in [5]. The equivalence between connections and usual parallel transport also appears in [3].

For a basic introduction to the theory of supermanifolds the standard reference is Deligne and Morgan [2]. For a brief introduction see also [6] . The notion of $1|1$ parallel transport that we use here appears in [4].

$1|1$ *parallel transport on E over M* is defined by parallel transport along (families of) paths $\mathbf{R} \times S \to M$ in our manifold M (S denotes an arbitrary supermanifold), as lifts of ∂_t, the standard vector field on $\mathbf{R}$, as well as parallel transport along (families of) superpaths $\mathbf{R}^{1|1} \times S \to M$, as lifts of

Date: November 24, 2010.

$D = \partial_\theta + \theta\partial_t$, the standard odd vector field on $\mathbf{R}^{1|1}$, along with a compatibility relation of the parallel transports given by diagrams

$$\begin{tikzcd}
\mathbf{R}^{1|1} \times S \arrow[rr, "\bar{c}"] \arrow[dr, "q\times id"'] & & M, \\
& \mathbf{R} \times S \arrow[ur, "c"'] &
\end{tikzcd}$$

so that a section s along c is ∂_t-parallel if and only if s is D-parallel along $\bar{c}$. Note that a section s of $\bar{c}^*E$ is of the form $s = s_1 + \theta s_2$, with $s_i \in \Gamma(c^*E)$, and s is D-parallel iff $s = s_1 \in \Gamma(c^*E)$, with s_1 a ∂_t-parallel section.

The $1|1$-parallel transport is compatible under gluing of (super)paths, is the identity on constant (super)paths and is invariant under reparametrization. The last condition means (for superpaths) that if $\varphi : \mathbf{R}^{1|1} \times S \to \mathbf{R}^{1|1} \times S$ is a family of diffeomorphisms of $\mathbf{R}^{1|1}$, parametrized by S, that preserve the distribution determined by the standard vector field $D = \partial_\theta + \theta\partial_t$ on $\mathbf{R}^{1|1}$, then a section $s \in \Gamma(\mathbf{R}^{1|1} \times S; c^*E)$ is parallel along the superpath c if and only if $s\varphi$ is parallel along the superpath $c\varphi$. Similarly, reparametrization-invariance for paths means that the parallel transport is invariant under precomposition by (families of) diffeomorphisms of the parametrizing interval $\mathbf{R}$ (see Section 3 of [4] for more details).

Since parallel transport along superpaths is invariant under certain diffeomorphisms of $\mathbf{R}^{1|1}$, and parallel transport along paths is invariant under diffeormorphisms of $\mathbf{R}$, we should have a consistency check read-off in the diagram:

$$\begin{tikzcd}
\mathbf{R}^{1|1} \times S \arrow[rr, "\bar{\varphi}"] \arrow[d, "q"'] \arrow[ddr, bend left, "\bar{c}\bar{\varphi}"] & & \mathbf{R}^{1|1} \times S \arrow[d, "q"] \arrow[ddl, bend right, "\bar{c}"'] \\
\mathbf{R} \times S \arrow[rr, dashed, "\varphi"] \arrow[dr, "c\varphi"'] & & \mathbf{R} \times S. \arrow[dl, "c"] \\
& M &
\end{tikzcd}$$

Here $q : \mathbf{R}^{1|1} \to \mathbf{R}$ denotes the obvious projection map. Note that not every family of diffeomorphisms of $\mathbf{R}^{1|1}$ descends to a family of diffeomorphisms of $\mathbf{R}$. For this to happen, the even part of the family $\bar{\varphi}$ should be independent of the odd variable θ.

The above notion of parallel transport can be word-for-word extended to vector bundles over supermanifolds. In [4] we show that a connection on a vector bundle over a supermanifold gives rise to such a parallel transport. This paper is concerned with showing the equivalence of the two notions when the base space is a manifold. This is enough if we are only interested in describing $1|1$-TFTs over a *manifold*. The case of a base space a supermanifold can probably be easily attained by following the steps in the proof below as all the appearing ingredients make sense in the supercase. Our main result is the following

Theorem 1.1. *There is a 1-1 correspondence*

$$\left\{ \begin{array}{c} 1|1 \textit{ parallel transport} \\ \textit{on } E \textit{ over } M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Even connections} \\ \textit{on } E \textit{ over } M \end{array} \right\}.$$

In other words, if we *define* 1|1-TFTs over M by the above 1|1-parallel transport over M, we can reformulate the theorem as

$$1|1\text{-TFT}(M) \cong \left\{ \begin{array}{c} \mathbf{Z}/2\text{-bundles with even connections} \\ \text{over } M \end{array} \right\},$$

where the left-hand side denotes the space of all 1|1-TFTs over the space M (to avoid set-theoretic issues, we require that the field theories over points are vector spaces in a *fixed* infinite dimensional vector space). For a definition of topological field theories see [1], and for a general definition of field theories see [7].

The proof will be the result of the equivalences expressed in the diagram below.

$$\begin{tikzcd}
\{1|1 \text{ transport on } E\ \} \arrow[r, dashed, leftrightarrow, "\text{Th } 1.1"] \arrow[d, leftrightarrow, "\text{Prop } 2.1"'] & \{\text{even connections on } E\ \} \arrow[d, leftrightarrow, "\text{Prop } 3.1"] \\
\{1|1 \text{ o.t. transport on } \pi^*E\ \} \arrow[r, leftrightarrow, "\text{Prop } 4.1"'] & \{\text{o.t. connections on } \pi^*E\ \}.
\end{tikzcd}$$

The bundle π^*E is the pull-back bundle of te bundle E via the map $\pi : \Pi TM \to M$ from the "odd" tangent bundle of M to M, which on functions is the inclusion of functions on M, as 0-forms, into the space of differential forms on M. The abbreviation o.t. stands for "odd-trivial" (see below).

2. Odd-trivial connections

Proposition 2.1. *Let E be a $\mathbf{Z}/2$-graded vector bundle over M. There is a 1-1 correspondence*

$$\left\{ \begin{array}{c} \textit{Grading-preserving connections} \\ \textit{on } E \textit{ over } M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Odd-trivial connections} \\ \textit{on } \pi^*E \textit{ over } \Pi TM \end{array} \right\}$$

To motivate the definition of odd-trivial connections let us begin by stating the following lemma whose proof is clear.

Lemma 2.2. *Let $\tilde{\nabla}$ denote a connection on the pullback bundle π^*E over ΠTM via the projection map $\pi : \Pi TM \to M$. Then*

$$\tilde{\nabla} = \pi^*\nabla,$$

for some connection ∇ on the bundle E over M if and only if

$$< \tilde{\nabla}(\pi^*s), \iota_X >= 0 \quad \textit{and} \quad < \tilde{\nabla}(\pi^*s), \mathcal{L}_X >\in \pi^*\Gamma(M; E),$$

for any $s \in \Gamma(M;E)$ a section of E and any X a vector field on M.

Here ι_X is the contraction by the vector field X acting on $\Omega^*(M) = \mathcal{C}^\infty(\Pi TM)$, interpreted as an odd derivation (i.e. vector field) on ΠTM. Similarly, $\mathcal{L}_X$ acts as a derivation in the direction of X on differential forms, and interprets as an even vector field on ΠTM. The sharp bracket stands for the pairing between 1-forms and vector fields on ΠTM.

Remark: The zero-equality above is not true for all odd vector fields on ΠTM, for example we have

$$< \tilde{\nabla}(\pi^* s), d >= \pi^*(\nabla s),$$

where d is the standard odd vector field on ΠTM, inducing the exterior derivative d on differential forms. Let us call such connections on pullback bundles $\pi^* E \to \Pi TM$ *odd-trivial connections*.

The proof of Proposition 2.1 is now clear since the lemma is a mere reformulation of the statement. From Lemma 2.2 immediately follows

Lemma 2.3. *If $\tilde{\nabla}$ is an odd-trivial connection, then $\tilde{\nabla}$ is flat in the odd directions, i.e.*

$$[\tilde{\nabla}_X, \tilde{\nabla}_Y] = \tilde{\nabla}_{[X,Y]},$$

for X, Y odd vector fields on ΠTM.

Proof. It is enough to check the relation for odd vector fields of the type ι_X, where X is a vector field on M, since arbitrary odd vector fields on ΠTM can be written as $\Omega^*(M)^{ev}$-combination of these.

□

3. Odd-trivial 1|1-parallel transport

We say that the 1|1-parallel transport on a bundle $\pi^* E$ over ΠTM is *odd-trivial* if the parallel transport along maps

$$\bar{\alpha}_X : \mathbf{R}^{1|1} \times \Pi TM \to \Pi TM,$$

given by the flow of vector fields (see Section 2.6 of [4]) of the form ι_X on ΠTM, where X is a vector field on M, is the identity on sections with initial condition of the form $\pi^* s \in \Gamma(\pi^* E)$, for $s \in \Gamma(E)$. Recall (see [2]) that for such odd vector fields ι_X on ΠTM that square to zero, the flow is actually determined by an $\mathbf{R}^{0|1}$-action on ΠTM, $\alpha_{\iota_X} : \mathbf{R}^{0|1} \times \Pi TM \to \Pi TM$ and the map $\bar{\alpha}_X$ factors as below

$$\begin{tikzcd}
\mathbf{R}^{1|1} \times \Pi TM \arrow[rr, "\bar{\alpha}_X"] \arrow[dr, "p \times id"'] & & \Pi TM, \\
& \mathbf{R}^{0|1} \times \Pi TM \arrow[ur, "\alpha_{\iota_X}"'] &
\end{tikzcd}$$

where $p : \mathbf{R}^{1|1} \to \mathbf{R}^{0|1}$ is the obvious projection map. The identity requirement above makes sense since the pullback of the bundle $\pi^* E$ via the map

$\bar{\alpha}_X$ is the bundle $\mathbf{R}^{1|1} \times \pi^* E$ over $\mathbf{R}^{1|1} \times \Pi TM$, since the bundle is the pullback bundle of the bundle $\alpha_{\iota_X}^* \pi^* E$ via the map $p \times 1$, and the bundle $\alpha_{\iota_X}^* \pi^* E$ is the pullback bundle of E via the map $p_0 \times \pi : \mathbf{R}^{0|1} \times \Pi TM \to M$, i.e. it is the bundle $\mathbf{R}^{0|1} \times \pi^* E$.

We should also require that for parallel transport along paths given by the flows $\alpha_X : \mathbf{R} \times M \to M$ of even vector fields $\mathcal{L}_X$ on ΠTM, coming from vector fields X on M, we have that

$$p^{\Pi TM}(\alpha_X; \pi^* s \in \Gamma(\pi^* E)) \in (1 \times \pi)^* \Gamma(\underline{\alpha}_X^* E),$$

where the map $\underline{\alpha}_X : \mathbf{R} \times M \to M$ is the flow of the vector field X on M. (We use the notation $p^N(c; s_0)$ for parallel sections in the space N along the (super)path c, determined by the initial condition s_0.) Note that there is a compatibility of the flows with the projection map π, as illustrated by the diagram

$$\begin{tikzcd} \mathbf{R} \times \Pi TM \arrow[r, "\alpha_X"] \arrow[d, "1\times\pi"'] & \Pi TM \arrow[d, "\pi"] \\ \mathbf{R} \times M \arrow[r, "\underline{\alpha}_X"'] & M, \end{tikzcd}$$

where the vertical maps are the obvious projections.

Proposition 3.1. *There is a 1-1 correspondence*

$$\left\{ \begin{array}{c} 1|1 \textit{ parallel transport} \\ \textit{on } E \textit{ over } M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} 1|1 \textit{ odd-trivial parallel transport} \\ \textit{on } \pi^* E \textit{ over } \Pi TM \end{array} \right\}$$

Proof. "$\longleftarrow$" Denote by $j : \mathbf{R}^0 \to \mathbf{R}^{1|1}$ the standard inclusion of a point in $\mathbf{R}^{1|1}$, namely mapping to $(0,0) \in \mathbf{R}^{1|1}$. Consider an arbitrary superpath c in M as below

$$\begin{tikzcd} c^* E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \pi^* E \arrow[d] \\ \mathbf{R}^{1|1} \times S \arrow[r, "c"'] & M \arrow[r, "i"'] & \Pi TM. \end{tikzcd}$$

To define a 1|1 parallel transport in M, we need to specify for each such superpath c in M a parallel section $p^M(c; h \otimes s)$ along c, for each initial condition

$$h \otimes s \in \Gamma(S, c_0^* E) \cong \mathcal{C}^\infty(S) \otimes \Gamma(M, E),$$

where $h \in \mathcal{C}^\infty(S)$ and $s \in \Gamma(M, E)$, and $c_0 = c \circ j$. Define

$$p^M(c; h \otimes s) := p^{\Pi TM}(ic; h \otimes s),$$

where $i : M \to \Pi TM$ denotes the standard inclusion. Note that

$$c_0^* E \cong c_0^* i^* \pi^* E,$$

since $\pi i = id$. Let now $\varphi : \mathbf{R}^{1|1} \times S \to \mathbf{R}^{1|1} \times S$ denote a family of diffeomorphisms of $\mathbf{R}^{1|1}$ preserving the conformal structure (the distribution determined by the standard vector field $D = \partial_\theta + \theta\partial_t$ determining the standard metric structure on $\mathbf{R}^{1|1}$) and the point $(0,0)$. Then

$$\begin{aligned} p^M(c\varphi; h\otimes s) &= p^{\Pi TM}(ic\varphi; h\otimes s) \\ &= p^{\Pi TM}(ic; h\otimes s)\circ\varphi \\ &= p^M(c; h\otimes s)\circ\varphi. \end{aligned}$$

The second equality holds since the $1|1$-parallel transport on ΠTM is invariant under reparametrization. This means that the $1|1$-parallel transport on M we constructed is invariant under reparametrization. Compatibility under glueing of superpaths and the identity on constant superpaths are obvious properties of the constructed parallel transport.

Similarly, for a (family of) path(s) c in M as below

$$\begin{tikzcd} c^*E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \pi^*E \arrow[d] \\ \mathbf{R}\times S \arrow[r, "c"'] & M \arrow[r, "i"'] & \Pi TM, \end{tikzcd}$$

we define $p^M(c; h\otimes s) := p^{\Pi TM}(ic; h\otimes s)$, for $h\otimes s\in\Gamma(S, c_0^*E)$ a section along $c_0 : S\to M$. It is clear that the parallel transport along paths is invariant under reparametrization and compatible under glueing of paths.

"$\longrightarrow$" Given a superpath c in ΠTM as below

$$\begin{tikzcd} c^*\pi^*E \arrow[r] \arrow[d] & \pi^*E \arrow[r] \arrow[d] & E \arrow[d] \\ \mathbf{R}^{1|1}\times S \arrow[r, "c"'] & \Pi TM \arrow[r, "\pi"'] & M \end{tikzcd}$$

we need to specify a parallel section $p^{\Pi TM}(c; h\otimes\omega\otimes s)$ along c with initial condition

$$\begin{aligned} h\otimes\omega\otimes s &\in \Gamma(S, c_0^*\pi^*E) \\ &\cong \mathcal{C}^\infty(S)\otimes_{\mathcal{C}^\infty(\Pi TM)}\Gamma(\Pi TM, \pi^*E) \\ &\cong \mathcal{C}^\infty(S)\otimes_{\mathcal{C}^\infty(\Pi TM)}\mathcal{C}^\infty(\Pi TM)\otimes_{\mathcal{C}^\infty(M)}\Gamma(M,E), \end{aligned}$$

where $h\in\mathcal{C}^\infty(S)$, $\omega\in\mathcal{C}^\infty(\Pi TM)\cong\Omega^*(M)$ and $s\in\Gamma(M,E)$. We define such a parallel section by

$$p^{\Pi TM}(c; h\otimes\omega\otimes s) := p^M(\pi c; c_0^*(\omega)h\otimes s),$$

As before, we check that

$$\begin{aligned} p^{\Pi TM}(c\varphi; h\otimes\omega\otimes s) &= p^M(\pi c\varphi; c_0^*(\omega)h\otimes s) \\ &= p^M(\pi c; c_0^*(\omega)h\otimes s)\circ\varphi \\ &= p^{\Pi TM}(c; h\otimes\omega\otimes s)\circ\varphi, \end{aligned}$$

for φ an arbitrary family of diffeomorphisms of $\mathbf{R}^{1|1}$ preserving the conformal structure and the point $(0,0)$. The second equality holds since the 1|1-parallel transport on M is invariant under reparametrization. This means that the 1|1-parallel transport on ΠTM we constructed is invariant under reparametrization. Compatibility under glueing of superpaths and the identity on constant superpaths are as before obvious. Parallel transport along paths in ΠTM is dealt with in a similar manner.

We are left to check the odd-triviality of the 1|1 parallel transport. Let $\bar{\alpha}_X : \mathbf{R}^{1|1} \times \Pi TM \to \Pi TM$ the flow of the odd vector field ι_X on ΠTM, for X a vector field on M. Then

$$\begin{aligned} p^{\Pi TM}(\bar{\alpha}_X; \pi^* s \in \Gamma(\pi^* E)) &= p^M(\pi\bar{\alpha}_X; \pi^* s \in \Gamma(\pi^* E)) \\ &= \bar{\alpha}_X^* \pi^* s, \end{aligned}$$

since the map $\bar{\alpha}_X$ factors through $\alpha_{\iota_X} : \mathbf{R}^{0|1} \times \Pi TM \to \Pi TM$, and the composition $\pi\bar{\alpha}_X$ is the uninteresting projection map.

Now, it is not hard to see that if we apply the construction "$\longrightarrow$"and then the construction "$\longleftarrow$", we obtain the identity. To see that the correspondence in the Proposition is one-to-one, we are left to check that the construction "$\longleftarrow$" is injective. This is a consequence of the following diagram

$$\begin{tikzcd} \{\text{1|1-odd trivial transport in } \Pi TM\} \arrow[r] \arrow[d, leftrightarrow, "\text{Prop 4.1 (to be proven)}"'] & \{\text{1|1-transport in M}\} \\ \{\text{ odd-trivial connections over } \Pi TM\} \arrow[r, leftrightarrow] & \{\text{even connections over } M\} \arrow[u] \end{tikzcd}$$

as well as the diagram

$$\begin{tikzcd} \{\text{even connections over } M\} \arrow[rr] \arrow[dr] & & \{\text{1|1-transport in M}\} \arrow[dl] \\ & \{\text{1-transport in M}\} & \end{tikzcd}$$

being commutative. Now, observe that the lower right arrow map in the last diagram is injective which implies that the right arrow map is injective. This further implies, by looking back at the first diagram, the required injectivity. We conclude that the two constructions are inverses of one another, and so obtain the Proposition.

□

4. An odd-trivial equivalence

Proposition 4.1. *There is a 1-1 correspondence*

$$\left\{ \begin{array}{c} 1|1 \textit{ odd-trivial parallel transport} \\ \textit{on } \pi^* E \textit{ over } \Pi TM \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Odd-trivial connections} \\ \textit{on } \pi^* E \textit{ over } \Pi TM \end{array} \right\}$$

One direction of the proof is clear: connections give rise to 1|1-parallel transport and the odd-trivilaity of a connection implies the odd-triviality of its parallel transport. In the other direction, we start off by lifting the action of vector fields of the type $\mathcal{L}_X$ and ι_X on ΠTM, for X vector fields on M, to actions on the total space of the bundle π^*E, which by differentiation gives us a compatible (under summation and function multiplication of vector fields) family of derivations, i.e. a connection on π^*E. In order to lift such actions we make some preliminary remarks on flows of vector fields- see Subsection 4.1, and then combine the even-odd rules of Subsection 4.2 to obtain the algebraic properties of a connection.

4.1. **Remarks on flows of vector fields.** In this subsection we find a Trotter type formula relating the flow of the sum of two vector fields X and Y, in terms of the flows of X and Y, as well as a relation between the flow of X and the flow of fX, for f a function on the manifold. There is a definite advantage to express geometrically these algebraic operations from a field theoretic perspective.

Proposition 4.2. *Let X and Y be vector fields on M, and let $\alpha, \beta : \mathbf{R} \times M \to M$ denote the flows determined by X, respectively Y. Then the flow γ of the vector field $X+Y$ is given by*

$$\gamma_t(x) = \lim_{n\to\infty} \underbrace{(\alpha_{\frac{t}{n}}\beta_{\frac{t}{n}}) \circ \ldots \circ (\alpha_{\frac{t}{n}}\beta_{\frac{t}{n}})}_{n}(x).$$

Proof. Let us begin with a calculation:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}(\alpha_t \circ \beta_t)(x) &= \frac{d}{dt}\Big|_{t=0}\alpha(t, \beta(t,x)) \\
&= \frac{\partial}{\partial t}\Big|_{t=0}\alpha(t,x) + \sum_{i=1}^{n} \frac{\partial \alpha}{\partial x^i}(0,x)\frac{\partial}{\partial t}\Big|_{t=0}\beta^i(t,x) \\
&= (X+Y)(x).
\end{aligned}$$

By a similar calculation, we have

$$\frac{d}{dt}\Big|_{t=0}(\alpha_{\frac{t}{2}}\beta_{\frac{t}{2}}) \circ (\alpha_{\frac{t}{2}}\beta_{\frac{t}{2}})(x) = (X+Y)(x),$$

and more generally

$$\frac{d}{dt}\Big|_{t=0}\underbrace{(\alpha_{\frac{t}{n}}\beta_{\frac{t}{n}}) \circ \ldots \circ (\alpha_{\frac{t}{n}}\beta_{\frac{t}{n}})}_{n}(x) = (X+Y)(x),$$

for any n. Next, we will show the group property for the family $\{\gamma_t\}$. To simplify notation, denote $\underbrace{f \circ \ldots \circ f}_{n}$ by $f^{(n)}$. We then have

$$\begin{aligned}\gamma_{2t} &= \lim_{n\to\infty}(\alpha_{2t/2n}\beta_{2t/2n})^{(2n)}\\ &= \lim_{n\to\infty}(\alpha_{t/n}\beta_{t/n})^{(n)}(\alpha_{t/n}\beta_{t/n})^{(n)}\\ &= \gamma_t\gamma_t.\end{aligned}$$

By a similar calculation, we obtain $\gamma_{3t} = \gamma_t\gamma_t\gamma_t$, and more generaly

$$\gamma_t = \gamma_{t/n}^{(n)}, \text{ for all } n \geq 1.$$

This implies that

$$\gamma_t\gamma_s = \gamma_{t+s},$$

for all t, s rational numbers, and, by continuity, for all t, s real numbers. Note that the limit in the statement of the proposition exists, as one can check for example by a Taylor expansion in t, for a fixed $x \in M$, and verifying that the Taylor coefficients converge.

□

Remark 4.3. *A word-for-word translation of the proof above shows that the same result holds for X and Y even vector fields on a compact supermanifold M.*

Consider now X a vector field on a (compact) manifold M. This determines an odd vector field ι_X on ΠTM that squares to zero. Its flow is reduced to a map $\alpha : \mathbf{R}^{0|1} \times \Pi TM \to \Pi TM$ given by

$$\alpha^* : \Omega^* M \to \Omega^* M[\theta] : \quad \omega \mapsto \omega + (\iota_X \omega)\theta.$$

Lemma 4.4. *Let X and Y be vector fields on M and ι_X, ι_Y the corresponding odd vector fields on ΠTM with flow maps $\mathbf{R}^{0|1} \times \Pi TM \to \Pi TM$ denoted by α and β. Then the flow γ of $\iota_X + \iota_Y$ is given by*

$$\gamma : \mathbf{R}^{0|1} \times \Pi TM \to \Pi TM, \quad \gamma = \beta \circ (1 \times \alpha) \circ (\Delta \times 1),$$

where $\Delta : \mathbf{R}^{0|1} \to \mathbf{R}^{0|1} \times \mathbf{R}^{0|1}$ is the diagonal map. On S-points, this means

$$\gamma(\theta, x) = \beta(\theta, \alpha(\theta, x)).$$

Proof. We have to check that the following diagram commutes

$$\begin{tikzcd} \mathbf{R}^{0|1} \times \Pi TM \arrow[r, "\gamma"] \arrow[d, "\Delta\times 1"'] & \Pi TM \\ \mathbf{R}^{0|1} \times \mathbf{R}^{0|1} \times \Pi TM \arrow[r, "1\times\alpha"'] & \mathbf{R}^{0|1} \times \Pi TM. \arrow[u, "\beta"'] \end{tikzcd}$$

This, on functions, translates into commutativity of the diagram

$$\begin{tikzcd} \omega + \iota_X\omega\theta + \iota_Y\omega\theta & \omega \arrow[l, mapsto, "\gamma^*"'] \arrow[d, mapsto, "\beta^*"] \\ \omega + \iota_X\omega\theta_2 + (\iota_Y\omega + \iota_X\iota_Y\omega\theta_2)\theta_1 \arrow[u, mapsto, "\theta_1=\theta_2"] & \omega + \iota_Y\omega\theta_1. \arrow[l, mapsto, "\alpha^*"] \end{tikzcd}$$

□

Remark 4.5. *The same proof shows that if X and Y are two odd vector fields on a supermanifold that square to zero and their Lie bracket $[X,Y]$ is also zero, then the sum $X+Y$ is an odd vector field that squares to zero and its flow (an $\mathbf{R}^{0|1}$-action) is the composition of the flows of X and Y.*

Lemma 4.6. *Let $\alpha : \mathbf{R} \times M \to M$ be the flow of a vector field X on the compact manifold M. If f is a positive function on M then the flow of fX is given by*

$$\beta : \mathbf{R} \times M \to M : (t,x) \mapsto \alpha(s(t,x),x),$$

where $s : \mathbf{R} \times M \to \mathbf{R}$ is the solution to

$$\begin{cases} \frac{\partial s}{\partial t}(t,x) = f(\alpha(s(t,x),x)) \\ \\ s(0,x) = 0, \text{ for all } x. \end{cases}$$

The proof is a routine check.

Corollary 4.7. *Let X and Y be vector fields on M. Then X and Y have the same (directed) trajectories if and only if $Y = fX$, for some positive function f on M.*

Corollary 4.8. *If $Y = fX$, for some positive function f on M, and c is an integral curve of X then $c \circ \varphi$ is an integral curve of Y, for some (orientation-preserving) diffeomorphism φ of $\mathbf{R}$.*

When M is a supermanifold, the situation is more involved. We still have as before

Lemma 4.9. *Let $\alpha : \mathbf{R} \times M \to M$ be the flow of an* **even** *vector field X on the compact supermanifold M. If f is a positive* **even** *function on M then the flow of fX is given by*

$$\beta : \mathbf{R} \times M \to M : (t,x) \mapsto \alpha(s(t,x),x),$$

where $s : \mathbf{R} \times M \to \mathbf{R}$ is the solution to

$$\begin{cases} \frac{\partial s}{\partial t}(t,x) = f(\alpha(s(t,x),x)) \\ \\ s(0,x) = 0, \text{ for all } x. \end{cases}$$

Let now f be a positive *even* function and X be an *odd* vector field with flow $\alpha : \mathbf{R}^{1|1} \times M \to M$ on the supermanifold M. Let $\varphi : \mathbf{R}^{1|1} \times M \to \mathbf{R}^{1|1}$ be a family of diffeomorphisms of $\mathbf{R}^{1|1}$ parametrized by M that preserves the 1-dimensional distribution determined by the vector field D on $\mathbf{R}^{1|1}$ so that

$$(D \otimes 1) \circ \varphi^* = M_{f\alpha(\varphi \times 1)(1 \times \Delta)} \circ \varphi^* \circ D.$$

Here $f\alpha(\varphi \times 1)(1 \times \Delta) : \mathbf{R}^{1|1} \times M \to \mathbf{R}^{1|1}$ is an even function on $\mathbf{R}^{1|1} \times M$, and M_g denotes multiplication by the function g. Then we have the following

Lemma 4.10. *The flow of the odd vector field fX is given by the map*

$$\beta : \mathbf{R}^{1|1} \times M \to M, \quad \beta = \alpha(\varphi \times 1)(1 \times \Delta),$$

or, on S-points,

$$\beta(t, \theta, x) = \alpha(\varphi(t, \theta, x), x).$$

Proof. This is just a calculation. We have to check that

$$(D \otimes 1) \circ \beta^* = \beta^* \circ fX.$$

Now

$$\begin{aligned} LHS &= (D \otimes 1) \circ (1 \otimes \Delta^*) \circ (\varphi^* \otimes 1) \circ \alpha^* \\ &= (1 \otimes \Delta^*) \circ (D \otimes 1 \otimes 1) \circ (\varphi^* \otimes 1) \circ \alpha^* \\ &= (1 \otimes \Delta^*) \circ ((D \otimes 1) \circ \varphi^*) \otimes 1 \circ \alpha^* \\ &= (1 \otimes \Delta^*) \circ (M_{f\alpha(\varphi\times 1)(1\times\Delta)} \circ \varphi^* \circ D) \otimes 1 \circ \alpha^* \\ &= M_{f\alpha(\varphi\times 1)(1\times\Delta)} \circ (1 \otimes \Delta^*) \circ ((\varphi^* \circ D) \otimes 1) \circ \alpha^*. \end{aligned}$$

In the fourth equality we used the defining property of the family φ of diffeomorphisms of $\mathbf{R}^{1|1}$. On the other hand,

$$\begin{aligned} RHS &= (1 \otimes \Delta^*) \circ (\varphi^* \otimes 1) \circ \alpha^* \circ fX \\ &= M_{f\alpha(\varphi\times 1)(1\times\Delta)} \circ ((1 \otimes \Delta^*) \circ (\varphi^* \otimes 1) \circ \alpha^* \circ X) \\ &= M_{f\alpha(\varphi\times 1)(1\times\Delta)} \circ ((1 \otimes \Delta^*) \circ (\varphi^* \otimes 1) \circ (D \otimes 1) \circ \alpha^*) \\ &= M_{f\alpha(\varphi\times 1)(1\times\Delta)} \circ ((1 \otimes \Delta^*) \circ ((\varphi^* \circ D) \otimes 1) \circ \alpha^*), \end{aligned}$$

where in the third equality we used the fact that α is the flow of the vector field X. The two expressions coincide, and this verifies the lemma. ∎

4.2. **Even-odd rules.** Consider the following families of even vector fields on ΠTM

$$\bar{e} = \mathbf{R} < \mathcal{L}_X,\ X \text{ vector field on } M >,$$

respectively odd vector fields on ΠTM

$$\bar{o} = \mathbf{R} < \iota_X,\ X \text{ vector field on } M > .$$

The following lemma is easy to check.

Lemma 4.11.

$$e \cdot \bar{o} \oplus o \cdot \bar{e} = \mathcal{X}(\Pi TM)^{odd}$$
$$e \cdot \bar{e} \oplus o \cdot \bar{o} = \mathcal{X}(\Pi TM)^{ev},$$

where e and o denote even, respectively odd functions on ΠTM.

To define a connection on $\pi^* E$ over ΠTM from an odd-trivial parallel transport, we first define $\nabla_{\bar{e}}$ and $\nabla_{\bar{o}}$, using the flows of these vector fields and differentiating the parallel sections along these paths to obtain horizontal lifts, along which we differentiate arbitrary sections. We then define

$$\nabla_{o \cdot \bar{o}} := o \cdot \nabla_{\bar{o}},$$

$$\nabla_{o\cdot\bar{e}} := o \cdot \nabla_{\bar{e}}.$$

It is not hard to check that

$$\nabla_{\bar{e}+\bar{e}} = \nabla_{\bar{e}} + \nabla_{\bar{e}},$$

$$\nabla_{\bar{o}+\bar{o}} = \nabla_{\bar{o}} + \nabla_{\bar{o}}.$$

This is true since in both cases we can express the flow of the sum of two vector fields in terms of the flows of each of the vector fields. If $\mathcal{E}$ and $\mathcal{O}$ denote the even, respectively odd vector fields on ΠTM, we can check that

$$\nabla_{e\cdot\mathcal{O}} = e \cdot \nabla_{\mathcal{O}},$$

$$\nabla_{e\cdot\mathcal{E}} = e \cdot \nabla_{\mathcal{E}},$$

using the Lemmas 4.9 and 4.12. We define

$$\nabla_{\mathcal{O}+\mathcal{O}} := \nabla_{\mathcal{O}} + \nabla_{\mathcal{O}},$$

$$\nabla_{\mathcal{E}+\mathcal{O}} := \nabla_{\mathcal{E}} + \nabla_{\mathcal{O}}.$$

The first relation requires a consistency check. First, if $\sum \omega_j \iota_{X_j} = 0$, then

$$\nabla_{\sum \omega_j \iota_{X_j}} = 0,$$

since $\nabla_{\sum \omega_j \iota_{X_j}}$ acts as the derivation $\sum \omega_j \iota_{X_j}$ on $\Gamma(\Pi TM; \pi^* E) = \Omega^*(M) \otimes \Gamma(M; E)$. Second, if

$$\sum \omega_j \mathcal{L}_{X_j} = 0,$$

then the X_j's are $\mathcal{C}^\infty(M)$-linearly dependent, and the two ways of defining for example $\nabla_{\mathcal{L}_{fX}} = \nabla_{f\mathcal{L}_X}$, for $f \in \mathcal{C}^\infty(M)$, are consistent with each other.

We can summarize the above considerations in the following

Lemma 4.12. *Consider the map*

$$V \in \mathcal{X}(\Pi TM) \longmapsto \nabla_V : \Gamma(\Pi TM; \pi^* E) \to \Gamma(\Pi TM; \pi^* E),$$

so that $\nabla_{\bar{e}}$ and $\nabla_{\bar{o}}$ are $\bar{e}$- respectively $\bar{o}$-derivations. Moreover, we require

$$\nabla_{e\cdot\bar{e}} = e \cdot \nabla_{\bar{e}}, \quad \nabla_{e\cdot\bar{o}} = e \cdot \nabla_{\bar{o}}, \quad \nabla_{\bar{e}+\bar{e}} = \nabla_{\bar{e}} + \nabla_{\bar{e}}, \quad \nabla_{\bar{o}+\bar{o}} = \nabla_{\bar{o}} + \nabla_{\bar{o}}.$$

Then ∇ defines a connection on $\pi^* E$ over ΠTM.

We can now conclude the proof of Proposition 4.1 since an odd-trivial parallel transport defines a map ∇ satisfying the conditions in the Lemma 4.12, so ∇ defines a connection on $\pi^* E$ over ΠTM. This connection is clearly odd-trivial. Let us remark that the Lie bracket of odd vector fields lifts in a compatible way which is consistent with the fact that an odd-trivial connection is flat in the odd directions. This also concludes the proof of our Theorem 1.1, by putting together Propositions 2.1, 3.1 and 4.1.

Acknowledgements. This paper was written while the author was visiting Max Planck Institute for Mathematics in Bonn and the Institute of Mathematics of the Romanian Academy "Simion Stoilow" in Bucharest. We gratefully acknowledge their hospitality and support.

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Institute of Mathematics of the Romanian Academy “Simion Stoilow”
21 Calea Grivitei
Bucharest, Romania.
Email: `florinndo@gmail.com`