

Lowest weight representations of super Schrödinger algebras in low dimensional spacetime

N Aizawa

Department of Mathematics and Information Sciences, Graduate School of Science,
Osaka Prefecture University, Nakamozu Campus, Sakai, Osaka 599-8531, Japan

E-mail: aizawa@mi.s.osakafu-u.ac.jp

Abstract. We investigate the lowest weight representations of the super Schrödinger algebras introduced by Duval and Horváthy. This is done by the same procedure as the semisimple Lie algebras. Namely, all singular vectors within the Verma modules are constructed explicitly then irreducibility of the associated quotient modules is studied again by the use of singular vectors. We present the classification of irreducible Verma modules for the super Schrödinger algebras in $(1+1)$ and $(2+1)$ dimensional spacetime with $\mathcal{N} = 1, 2$ extensions.

1. Introduction

Throughout the last four decades nonrelativistic conformal symmetries have been observed in wide area of physics ranging from condensed matter to high energy (see for example [1–18] and references therein). In many cases the symmetries are described by the groups extending the Galilei symmetry group of nonrelativistic systems [19, 20]. Those extended Galilei groups are non-semisimple as the original Galilei group. This may be an obstacle to develop systematic study of representations for such groups.

The simplest conformal extension of the Galilei group may be the Schrödinger groups [1, 2] introduced in the studies of symmetry for free Schrödinger equation. Geometrical interpretation of the Schrödinger group revealed that the AdS/CFT correspondence has its nonrelativistic analogue [14, 15] (see also [8]). Supersymmetric extensions of the Schrödinger group and its Lie algebra have also been discussed in connection with various physical systems such as fermionic oscillator [21, 22], spinning particles [23], nonrelativistic Chern-Simons matter [24–26], Dirac monopole and magnetic vortex [25] and so on. Some super Schrödinger algebras were constructed from the viewpoint of infinite dimensional Lie super algebra [27] or by embedding them to conformal superalgebras [28, 29].

Despite the numerous publications on the Schrödinger algebra and its supersymmetric extensions, the representation theories of them are not studied well. Especially there are a few works based on representation theoretic viewpoint. We mention the followings: Projective representations of the Schrödinger group in 3 spatial dimension were constructed in [30]. Irreducible representations of the Schrödinger algebras up to 3 spatial dimension were investigated in [31–33]. The author of [34] studied the highest weight representations of the $\mathcal{N} = 2$ super Schrödinger algebra in 2 spatial dimension (“exotic” algebra in the terminology of [25]). To apply the (super) Schrödinger algebras for physical problems detailed study of irreducible representations of them will be helpful.

In the present work, we provide a classification of the Verma modules (a lowest weight representations) over the super Schrödinger algebras for simple cases, namely, one and two spatial dimensions with $\mathcal{N} = 1, 2$ supersymmetric extensions. The Schrödinger algebra discussed in this paper is centrally extended one. The central term corresponds to the mass of a system. In general, the Schrödinger algebra of fixed spacetime dimension has some distinct supersymmetric extensions even for fixed value of \mathcal{N} . A systematic method to extend the Schrödinger algebra in $(n + 1)$ dimensional spacetime to arbitrary \mathcal{N} has developed in [25]. We restrict our study of the representations to the super Schrödinger algebras introduced in [25]. We denote the centrally extended Schrödinger algebra in $(n + 1)$ dimensional spacetime by $\mathfrak{s}(n)$. The $\mathcal{N} = 1, 2$ extension of $\mathfrak{s}(1)$ are denoted by $\mathfrak{s}(1/1)$ and $\mathfrak{s}(1/2)$, respectively. The $\mathcal{N} = 1$ extension of $\mathfrak{s}(2)$ is also denoted by SS21. On the other hand there exist two $\mathcal{N} = 2$ extensions for $\mathfrak{s}(2)$. We denote one of them so-called standard by SS22 and the another one so-called exotic by $\hat{\mathfrak{s}}(2/2)$. The superalgebra $\mathfrak{s}(n/2)$ corresponds to the $N_+ = 1, N_- = 0$ extended algebra in the notation of [25].

The plan of this paper is as follows. We present the definition and structure of the super Schrödinger algebras in the next section. After giving the Verma modules in section 3 we give all the singular vectors and list of irreducible lowest weight modules without detailed calculation. The detailed proof is found in [35, 36]. The employed procedure in the references is the same one as [31, 32]. This shows that the standard method of representation theory for semisimple Lie algebras is able to apply the super Schrödinger algebras, too.

2. Definition and structure of super Schrödinger algebras

The bosonic Schrödinger algebra in $(n + 1)$ dimensional spacetime $\mathfrak{s}(n)$ is generated by the following transformations: time translation (H), space translations (P_a), Galilei boosts (G_a), spatial rotations (J_{ab}), dilatation (D), conformal transformation (K) and the central element (M). Contrast to the relativistic conformal algebra, there exists only one conformal generator. In this section we give the definition of the super Schrödinger algebras studied in this paper and their triangular decomposition which will be the key to study representations.

2.1. Algebras in $(1 + 1)$ dimensional spacetime

The bosonic sector has six generators. Their nonvanishing commutators are given by

$$\begin{aligned} [H, D] &= 2H, & [H, K] &= D, & [D, K] &= 2K, & [P, G] &= M, \\ [H, G] &= P, & [D, G] &= G, & [P, D] &= P, & [P, K] &= G. \end{aligned} \quad (1)$$

The $\mathcal{N} = 1$ extension $\mathfrak{s}(1/1)$ has three fermionic generators: $\mathcal{Q}, \mathcal{S}, \mathcal{X}$. They satisfy the anti-commutation relations

$$\begin{aligned} \{\mathcal{Q}, \mathcal{Q}\} &= -2H, & \{\mathcal{S}, \mathcal{S}\} &= -2K, & \{\mathcal{X}, \mathcal{X}\} &= -M, \\ \{\mathcal{Q}, \mathcal{X}\} &= -P, & \{\mathcal{S}, \mathcal{X}\} &= -G, & \{\mathcal{Q}, \mathcal{S}\} &= -D. \end{aligned} \quad (2)$$

As seen from the relations none of them are nilpotent. Nonvanishing bosonic-fermionic relations are listed below:

$$\begin{aligned} [\mathcal{Q}, D] &= \mathcal{Q}, & [\mathcal{Q}, K] &= \mathcal{S}, & [D, \mathcal{S}] &= \mathcal{S}, & [H, \mathcal{S}] &= \mathcal{Q}, \\ [\mathcal{Q}, G] &= \mathcal{X}, & [P, \mathcal{S}] &= \mathcal{X}. \end{aligned} \quad (3)$$

The $\mathcal{N} = 2$ extension $\mathfrak{s}(1/2)$ has six fermionic generators $\mathcal{Q}_a, \mathcal{S}_a, \mathcal{X}_a$ ($a = \pm$) and one additional bosonic generator R corresponding to the R -parity. The generator R commutes with

all bosonic elements of $\mathfrak{s}(1/2)$. The nilpotent fermionic generators satisfy the anti-commutation relations

$$\begin{aligned} \{\mathcal{Q}_\pm, \mathcal{Q}_\mp\} &= -2H, & \{\mathcal{S}_\pm, \mathcal{S}_\mp\} &= -2K, & \{\mathcal{X}_\pm, \mathcal{X}_\mp\} &= -M, \\ \{\mathcal{Q}_\pm, \mathcal{X}_\mp\} &= -P, & \{\mathcal{S}_\pm, \mathcal{X}_\mp\} &= -G, & \{\mathcal{Q}_\pm, \mathcal{S}_\mp\} &= -D \mp R. \end{aligned} \quad (4)$$

Bosonic-fermionic commutators are given by

$$\begin{aligned} [\mathcal{Q}_\pm, D] &= \mathcal{Q}_\pm, & [\mathcal{Q}_\pm, K] &= \mathcal{S}_\pm, & [D, \mathcal{S}_\pm] &= \mathcal{S}_\pm, & [H, \mathcal{S}_\pm] &= \mathcal{Q}_\pm \\ [\mathcal{Q}_\pm, G] &= \mathcal{X}_\pm, & [P, \mathcal{S}_\pm] &= \mathcal{X}_\pm, & [R, \mathcal{A}_\pm] &= \pm \mathcal{A}_\pm, \end{aligned} \quad (5)$$

where $\mathcal{A}_a = \mathcal{Q}_a, \mathcal{S}_a, \mathcal{X}_a$.

2.2. Algebras in (2 + 1) dimensional spacetime

In this spacetime there exists one spatial rotation J . The bosonic algebra $\mathfrak{s}(2)$ has nine generators. Their nonvanishing commutators are given by

$$\begin{aligned} [H, D] &= 2H, & [H, K] &= D, & [D, K] &= 2K, & [P_\pm, G_\mp] &= 2M, \\ [H, G_\pm] &= P_\pm, & [D, G_\pm] &= G_\pm, & [P_\pm, D] &= P_\pm, & [P_\pm, K] &= G_\pm, \\ [J, G_\pm] &= \pm G_\pm, & [J, P_\pm] &= \pm P_\pm. \end{aligned} \quad (6)$$

The $\mathcal{N} = 1$ extension SS21 has four fermionic elements $\mathcal{Q}, \mathcal{S}, \mathcal{X}_\pm$ subject to the relations:

$$\begin{aligned} \{\mathcal{Q}, \mathcal{Q}\} &= -2H, & \{\mathcal{S}, \mathcal{S}\} &= -2K, & \{\mathcal{X}_\pm, \mathcal{X}_\mp\} &= -2M, \\ \{\mathcal{Q}, \mathcal{X}_\pm\} &= -P_\pm, & \{\mathcal{S}, \mathcal{X}_\pm\} &= -G_\pm, & \{\mathcal{Q}, \mathcal{S}\} &= -D. \end{aligned} \quad (7)$$

The generators \mathcal{X}_\pm are nilpotent, however, \mathcal{Q}, \mathcal{S} are not. Nonvanishing bosonic-fermionic commutators are given by

$$\begin{aligned} [\mathcal{Q}, D] &= \mathcal{Q}, & [\mathcal{Q}, K] &= \mathcal{S}, & [D, \mathcal{S}] &= \mathcal{S}, & [H, \mathcal{S}] &= \mathcal{Q}, \\ [\mathcal{Q}, G_\pm] &= \mathcal{X}_\pm, & [P_\pm, \mathcal{S}] &= \mathcal{X}_\pm, & [J, \mathcal{X}_\pm] &= \pm \mathcal{X}_\pm. \end{aligned} \quad (8)$$

For $\mathfrak{s}(2)$ there exist two distinct $\mathcal{N} = 2$ extensions, called *standard* and *exotic*. The standard extension SS22 has eight fermionic generators $\mathcal{Q}_\pm, \mathcal{S}_\pm, \mathcal{X}_{\pm\pm}, \mathcal{X}_{\pm\mp}$ and additional bosonic R -parity operator. The fermionic sector is defined by the relations:

$$\begin{aligned} \{\mathcal{Q}_\pm, \mathcal{Q}_\mp\} &= -2H, & \{\mathcal{S}_\pm, \mathcal{S}_\mp\} &= -2K, & \{\mathcal{X}_{++}, \mathcal{X}_{--}\} &= -2M, \\ \{\mathcal{X}_{+-}, \mathcal{X}_{-+}\} &= -2M, & \{\mathcal{Q}_\pm, \mathcal{X}_{\sigma\mp}\} &= -P_\sigma, & \{\mathcal{S}_\pm, \mathcal{X}_{\sigma\mp}\} &= -G_\sigma, \\ \{\mathcal{Q}_\pm, \mathcal{S}_\mp\} &= -D \mp R, \end{aligned} \quad (9)$$

where $\sigma = +, -$. Nonvanishing bosonic-fermionic relations are listed below:

$$\begin{aligned} [\mathcal{Q}_\pm, D] &= \mathcal{Q}_\pm, & [\mathcal{Q}_\pm, K] &= \mathcal{S}_\pm, & [D, \mathcal{S}_\pm] &= \mathcal{S}_\pm, & [H, \mathcal{S}_\pm] &= \mathcal{Q}_\pm \\ [G_\pm, \mathcal{Q}_\pm] &= -\mathcal{X}_{\pm\pm}, & [G_\pm, \mathcal{Q}_\mp] &= -\mathcal{X}_{\pm\mp}, & [P_\pm, \mathcal{S}_\pm] &= \mathcal{X}_{\pm\pm}, & [P_\pm, \mathcal{S}_\mp] &= \mathcal{X}_{\pm\mp}, \\ [R, \mathcal{A}_\pm] &= \pm \mathcal{A}_\pm, & [R, \mathcal{X}_{\sigma\pm}] &= \pm \mathcal{X}_{\sigma\pm}, & [J, \mathcal{X}_{\sigma\pm}] &= \sigma \mathcal{X}_{\sigma\pm}, \end{aligned} \quad (10)$$

where $\mathcal{A} = \mathcal{Q}, \mathcal{S}$.

The exotic extension $\hat{\mathfrak{s}}(2/2)$ has six fermionic generators $\mathcal{Q}_\pm, \mathcal{S}_\pm, \mathcal{X}_\pm$ and additional bosonic R -parity operator. The fermionic generators are subject to the relations:

$$\begin{aligned} \{\mathcal{Q}_\pm, \mathcal{Q}_\mp\} &= -2H, & \{\mathcal{S}_\pm, \mathcal{S}_\mp\} &= -2K, & \{\mathcal{X}_\pm, \mathcal{X}_\mp\} &= -M, \\ \{\mathcal{Q}_\pm, \mathcal{X}_\mp\} &= -P_\mp, & \{\mathcal{S}_\pm, \mathcal{X}_\mp\} &= -G_\mp, & \{\mathcal{Q}_\pm, \mathcal{S}_\mp\} &= -D \mp (J + R), \end{aligned} \quad (11)$$

and those for the bosonic-fermionic sector are

$$\begin{aligned}
[\mathcal{Q}_\pm, D] &= \mathcal{Q}_\pm, & [\mathcal{Q}_\pm, K] &= \mathcal{S}_\pm, & [D, \mathcal{S}_\pm] &= \mathcal{S}_\pm, & [H, \mathcal{S}_\pm] &= \mathcal{Q}_\pm \\
[\mathcal{Q}_\pm, G_\pm] &= 2\mathcal{X}_\pm, & [P_\pm, \mathcal{S}_\pm] &= 2\mathcal{X}_\pm, & [J, \mathcal{Q}_\pm] &= \mp\mathcal{Q}_\pm, & [J, \mathcal{S}_\pm] &= \mp\mathcal{S}_\pm, \\
[R, \mathcal{A}_\pm] &= \pm 2\mathcal{A}_\pm,
\end{aligned} \tag{12}$$

where $\mathcal{A} = \mathcal{Q}, \mathcal{S}, \mathcal{X}$. We remark that the R -parity operator for $\mathcal{N} = 2$ extensions also commute with all bosonic generators.

2.3. Triangular decomposition

The grading and triangular decomposition for $\mathfrak{s}(n)$ introduced in [31] are easily extended to the super Schrödinger algebras discussed in this paper. We define the following degree for $\mathfrak{s}(1/1)$:

$$\begin{aligned}
\deg K &= 2, & \deg G &= \deg \mathcal{S} = 1, & \deg D &= \deg M = \deg \mathcal{X} = 0, \\
\deg P &= \deg \mathcal{Q} = -1, & \deg H &= -2.
\end{aligned} \tag{13}$$

It is immediate to see that $\mathfrak{s}(1/1)$ is \mathbb{Z} -graded by this degree. According to the sign of the degree one may define the triangular decomposition of $\mathfrak{s}(1/1)$ as follow:

$$\begin{aligned}
\mathfrak{s}(1/1) &= \mathfrak{s}(1/1)^+ \oplus \mathfrak{s}(1/1)^0 \oplus \mathfrak{s}(1/1)^- \\
&= \{ K, G \} \oplus \{ D, M, \mathcal{X} \} \oplus \{ P, \mathcal{Q} \}.
\end{aligned} \tag{14}$$

Appropriate definitions of degree for other algebras also enable us to define the triangular decomposition. Below we give the decomposition. The algebras $\mathfrak{s}(1/2)$ and SS21 are \mathbb{Z}^2 -graded:

decomposition	generator (degree)
$\mathfrak{s}(1/2)^+$	$K(2, 0), G(1, 0), \mathcal{S}_+(1, 1), \mathcal{S}_-(1, -1), \mathcal{X}_+(0, 1)$
$\mathfrak{s}(1/2)^0$	$D(0, 0), R(0, 0), M(0, 0)$
$\mathfrak{s}(1/2)^-$	$H(-2, 0), P(-1, 0), \mathcal{Q}_+(-1, 1), \mathcal{Q}_-(-1, -1), \mathcal{X}_-(0, -1)$
$\mathfrak{s}(2/1)^+$	$K(2, 0), G_+(1, 1), G_-(1, -1), \mathcal{S}(1, 0), \mathcal{X}_+(0, 1)$
$\mathfrak{s}(2/1)^0$	$D(0, 0), J(0, 0), M(0, 0)$
$\mathfrak{s}(2/1)^-$	$H(-2, 0), P_+(-1, 1), P_-(-1, -1), \mathcal{Q}(-1, 0), \mathcal{X}_-(0, -1)$

While the algebras SS22 and $\hat{\mathfrak{s}}(2/2)$ are \mathbb{Z}^3 -graded:

decomposition	generator (degree)
$\mathfrak{s}(2/2)^+$	$K(2, 0, 0), G_+(1, 1, 0), G_-(1, -1, 0), \mathcal{S}_+(1, 0, 1),$ $\mathcal{S}_-(1, 0, -1), \mathcal{X}_{++}(0, 1, 1), \mathcal{X}_{+-}(0, 1, -1)$
$\mathfrak{s}(1/2)^0$	$D(0, 0, 0), J(0, 0, 0), M(0, 0, 0), R(0, 0, 0)$
$\mathfrak{s}(1/2)^-$	$H(-2, 0, 0), P_+(-1, 1, 0), P_-(-1, -1, 0), \mathcal{Q}_+(-1, 0, 1)$ $\mathcal{Q}_-(-1, 0, -1), \mathcal{X}_{-+}(0, -1, 1), \mathcal{X}_{--}(0, -1, -1)$
$\hat{\mathfrak{s}}(2/2)^+$	$K(2, 0, 0), G_+(1, 1, 0), G_-(1, -1, 0),$ $\mathcal{S}_+(1, -1, 2), \mathcal{S}_-(1, 1, -2), \mathcal{X}_+(0, 0, 2)$
$\hat{\mathfrak{s}}(1/2)^0$	$D(0, 0, 0), J(0, 0, 0), M(0, 0, 0), R(0, 0, 0)$
$\hat{\mathfrak{s}}(1/2)^-$	$H(-2, 0, 0), P_+(-1, 1, 0), P_-(-1, -1, 0),$ $\mathcal{Q}_+(-1, -1, 2), \mathcal{Q}_-(-1, 1, -2), \mathcal{X}_-(0, 0, -2)$

One can introduce an algebra anti-automorphism for the super Schrödinger algebras. It is defined for the bosonic generators of $\mathfrak{s}(1/\mathcal{N})$ by

$$\omega(P) = G, \quad \omega(H) = K, \quad \omega(D) = D, \quad \omega(M) = M, \quad \omega(R) = R, \quad (15)$$

and those for the algebras in $(2+1)$ dimension by

$$\omega(P_a) = G_{-a}, \quad \omega(H) = K, \quad \omega(D) = D, \quad \omega(M) = M, \quad \omega(J) = J, \quad \omega(R) = R. \quad (16)$$

The mapping of the fermionic elements of $\mathfrak{s}(1/1)$ is given by

$$\omega(\mathcal{Q}) = \mathcal{S}, \quad \omega(\mathcal{X}) = \mathcal{X}. \quad (17)$$

All the fermionic elements of the other algebras are obey the same transformation rule together with all the signature of the suffices being changed, e.g., $\omega(\mathcal{X}_{+-}) = \mathcal{X}_{-+}$. The mapping is involutive: $\omega^2 = id$. Let \mathfrak{g} be one of the superalgebras considered in this paper and let us denote its triangular decomposition by $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$. Then one see that $\omega(\mathfrak{g}^+) = \mathfrak{g}^-$, $\omega(\mathfrak{g}^0) = \mathfrak{g}^0$.

3. Verma modules and their reducibility

The algebras under consideration admit the triangular decomposition. This allows us to define the lowest weight modules for the algebras. The lowest weight vector v_0 is defined by

$$Xv_0 = 0 \quad \forall X \in \mathfrak{g}^-, \quad Xv_0 = \Lambda(X)v_0 \quad \forall X \in \mathfrak{g}^0, \quad (18)$$

where $\Lambda(X)$ is an eigenvalue of X . The Verma module over \mathfrak{g} is defined by

$$V^\Lambda = U(\mathfrak{g}^+)v_0, \quad (19)$$

where $U(\mathfrak{g}^+)$ is the enveloping algebra of \mathfrak{g}^+ . Reducibility of the Verma modules may be investigated by singular vectors as in the cases of semisimple Lie algebras and the bosonic Schrödinger algebras [31, 32]. A singular vector v_s is defined as a homogeneous element of V^Λ such that $v_s \neq \mathbb{C}v_0$ and $\mathfrak{g}^-v_s = 0$. In this section we provide explicit formulae of the singular vectors and reducibility of the Verma modules for *massive* case, that is, the generator M has nonvanishing eigenvalues. Here we give only the outline of the calculation. Detailed proof for $(1+1)$ dimensional algebras is found in [35] and for $(2+1)$ dimension will be published separately [36].

3.1. $(1+1)$ Dimensional algebras

We start with $\mathfrak{s}(1/1)$. The lowest weight condition (18) for $\mathfrak{s}(1/1)$ yields

$$\begin{aligned} \mathcal{Q}v_0 = Pv_0 &= 0, \\ Dv_0 = -dv_0, \quad Mv_0 = mv_0, \quad \mathcal{X}v_0 &= \chi v_0, \end{aligned} \quad (20)$$

where $d \in \mathbb{R}$ is the conformal weight and the minus sign is for later convenience. The variable χ is of odd parity relating to the mass eigenvalue by the relation $m = 2\chi^2$. The Verma module (19) for $\mathfrak{s}(1/1)$ is given by

$$V^d = \{ G^k K^\ell v_0, G^k K^\ell \mathcal{S}v_0 \mid k, \ell \in \mathbb{Z}_{\geq 0} \}. \quad (21)$$

It is easy to see that the generator D is diagonal on the basis given in (21). It follows that V^d is decomposed into a direct sum of the subspaces V_n^d spanned by the vectors v satisfying $Dv = nv$. The singular vector v_s belongs to V_n^d . One may set

$$v_s = \sum_k a_k v_k^{(n)}, \quad v_k^{(n)} \in V_n^d, \quad (22)$$

and impose the conditions

$$\mathcal{Q}v_s = Pv_s = 0. \quad (23)$$

This yields some relations for a_k then by solving the relations one may determine the singular vectors up to overall constant.

Proposition 1 *The Verma module V^d over $\mathfrak{s}(1/1)$ has precisely one singular vector iff $d+1/2 \in \mathbb{Z}_{\geq 0}$ and it is given by*

$$v_s = (G^2 - 2mK)^{d+1/2}(G - 2\chi\mathcal{S})v_0. \quad (24)$$

When the Verma module has singular vectors one find invariant subspaces constructed on the singular vector $\mathcal{I} = U(\mathfrak{g}^+)v_s$. Thus the module is reducible. To find irreducible modules one may consider the quotient module V^d/\mathcal{I} then seeks singular vectors in the quotient module. In the present case it turns out that there is no singular vector in the quotient module. Therefore V^d/\mathcal{I} is irreducible.

Proposition 2 *All irreducible lowest weight modules over $\mathfrak{s}(1/1)$ are listed as follows:*

- (i) *The Verma module V^d for $d + 1/2 \notin \mathbb{Z}_{\geq 0}$*
- (ii) *The quotient module V^d/\mathcal{I} for $d + 1/2 \in \mathbb{Z}_{\geq 0}$*

All irreducible modules given are infinite dimensional.

We now turn to the superalgebra $\mathfrak{s}(1/2)$. The lowest weight vector v_0 is defined by

$$\begin{aligned} \mathcal{Q}_{\pm}v_0 = Pv_0 = \mathcal{X}_{-}v_0 = 0, \\ Dv_0 = -dv_0, \quad Mv_0 = mv_0, \quad Rv_0 = rv_0. \end{aligned} \quad (25)$$

The Verma modules over $\mathfrak{s}(1/2)$ are given by

$$V^{d,r} = \{ G^k K^{\ell} \mathcal{S}_{+}^a \mathcal{S}_{-}^b \mathcal{X}_{+}^c v_0 \mid k, \ell \in \mathbb{Z}_{\geq 0}, a, b, c \in \{0, 1\} \}. \quad (26)$$

The same procedure as $\mathfrak{s}(1/1)$ leads us to the following propositions:

Proposition 3 *The Verma module $V^{d,r}$ over $\mathfrak{s}(1/2)$ has precisely one singular vector iff $d - 1/2 \in \mathbb{Z}_{+}$ and it is given by*

$$\begin{aligned} v_s^p &= (G^2 - 2mK)^{d-1/2}u_0, \\ u_0 &= (G\mathcal{S}_{-}\mathcal{X}_{+} + m\mathcal{S}_{+}\mathcal{S}_{-} + 2mK)v_0 + \frac{d+r+1}{2d+1}(G^2 - 2mK)v_0. \end{aligned} \quad (27)$$

Proposition 4 *All irreducible lowest weight modules over $\mathfrak{s}(1/2)$ are listed as follows:*

- (i) *The Verma module $V^{d,r}$ for $d - 1/2 \notin \mathbb{Z}_{+}$*
- (ii) *The quotient module $V^{d,r}/\mathcal{I}$ for $d - 1/2 \in \mathbb{Z}_{+}$*

where $\mathcal{I} \subset V^{d,r}$ is the invariant submodule constructed on the singular vector (27): $\mathcal{I} = U(\mathfrak{s}(1/2)^+)v_s$. All irreducible modules given are infinite dimensional.

3.2. (2 + 1) Dimensional algebras

More involved in this case, however, one may use the same procedure as (1 + 1) dimensional algebras to study the reducibility of the Verma modules.

We start with the superalgebra SS21. The lowest weight for SS21 is defined by

$$\begin{aligned} \mathcal{Q}v_0 &= P_{\pm}v_0 = \mathcal{X}_{-}v_0 = 0, \\ Dv_0 &= -dv_0, \quad Jv_0 = -jv_0, \quad Mv_0 = mv_0. \end{aligned} \quad (28)$$

The Verma modules are constructed on the lowest weight vector:

$$V^{d,j} = \{ G_+^k G_-^h K^\ell \mathcal{S}^a \mathcal{X}_+^b v_0 \mid k, h, \ell \in \mathbb{Z}_{\geq 0}, a, b \in \{0, 1\} \}. \quad (29)$$

The Verma modules contain a singular vector for discrete values of d :

Proposition 5 *The Verma module $V^{d,j}$ over SS21 has precisely one singular vector iff $d + 1 \in \mathbb{Z}_+$ and it is given by*

$$v_s = (G_+ G_- - 2mK)^{d+1} (G_- \mathcal{X}_+ - 2m\mathcal{S})v_0. \quad (30)$$

It turns out that the factor modules by the invariant submodule constructed on v_s do not contain any singular vectors. We thus have a classification of irreducible modules over SS21.

Proposition 6 *All irreducible lowest weight modules over SS21 are listed as follows:*

- (i) *The Verma module $V^{d,j}$ for $d + 1 \notin \mathbb{Z}_+$*
- (ii) *The quotient module $V^{d,j}/\mathcal{I}$ for $d + 1 \in \mathbb{Z}_+$*

where $\mathcal{I} \subset V^{d,r}$ is the invariant submodule constructed on the singular vector (30): $\mathcal{I} = U(\mathfrak{sl}(2/1)^+)v_s$. All irreducible modules given are infinite dimensional.

Next we investigate the standard $\mathcal{N} = 2$ extension SS22. The lowest weight vector for SS22 is defined by

$$\begin{aligned} \mathcal{Q}_{\pm}v_0 &= P_{\pm}v_0 = \mathcal{X}_{-+}v_0 = \mathcal{X}_{--}v_0 = 0, \\ Dv_0 &= -dv_0, \quad Jv_0 = -jv_0, \quad Rv_0 = rv_0, \quad Mv_0 = mv_0, \end{aligned} \quad (31)$$

and application of a element of $U(\mathfrak{sl}(2/2)^+)$ on v_0 generates a basis of a Verma module over SS21:

$$V^{d,j,r} = \{ G_+^k G_-^h K^\ell \mathcal{S}_+^{a_1} \mathcal{S}_-^{a_2} \mathcal{X}_{++}^{b_1} \mathcal{X}_{+-}^{b_2} v_0 \mid k, h, \ell \in \mathbb{Z}_{\geq 0}, a_i, b_i \in \{0, 1\} \}. \quad (32)$$

Reducibility of the Verma modules is similar to the other cases. We can show the following proposition on the existence of singular vectors.

Proposition 7 *The Verma module $V^{d,j,r}$ over SS22 has precisely one singular vector iff $d \in \mathbb{Z}_+$ and it is given by*

$$\begin{aligned} v_s &= (G_+ G_- - 2mK)^d (G_- \mathcal{X}_+ - 2m\mathcal{S})v_0, \\ u_0 &= \{ G_-^2 \mathcal{X}_{++} \mathcal{X}_{+-} - 2mG_- (\mathcal{S}_+ \mathcal{X}_{+-} - \mathcal{S}_- \mathcal{X}_{++}) + 4m^2 (\mathcal{S}_+ \mathcal{S}_- + K) \} v_0. \end{aligned} \quad (33)$$

Further search of singular vectors in the quotient modules conclude the following classification of irreducible modules:

Proposition 8 *All irreducible lowest weight modules over SS22 are listed as follows:*

- (i) *The Verma module $V^{d,j,r}$ for $d \notin \mathbb{Z}_+$*
- (ii) *The quotient module $V^{d,j,r}/\mathcal{I}$ for $d \in \mathbb{Z}_+$*

where $\mathcal{I} \subset V^{d,r}$ is the invariant submodule constructed on the singular vector (33): $\mathcal{I} = U(\mathfrak{sl}(2/2)^+)v_s$. All irreducible modules given are infinite dimensional.

We now turn to investigation of the exotic $\mathcal{N} = 2$ extension $\hat{\mathfrak{sl}}(2/2)$. The singular vectors and irreducible modules have been studied in [34]. Here we give a closed form expression of the singular vectors and more precise classification of irreducible modules. The lowest weight vector v_0 and the Verma modules $V^{d,j,r}$ are defined as usual:

$$\begin{aligned} \mathcal{Q}_\pm v_0 = P_\pm v_0 = \mathcal{X}_- v_0 = 0, \\ Dv_0 = -dv_0, \quad Jv_0 = -jv_0, \quad Rv_0 = 2rv_0, \quad Mv_0 = mv_0. \end{aligned} \quad (34)$$

$$V^{d,j,r} = \{ G_+^k G_-^h K^\ell \mathcal{S}_+^a \mathcal{S}_-^b \mathcal{X}_+^c v_0, \mid k, h, \ell \in \mathbb{Z}_{\geq 0}, a, b, c \in \{0, 1\} \}. \quad (35)$$

The classification of irreducible modules is summarized in the following two propositions.

Proposition 9 *The Verma module $V^{d,j,r}$ over $\hat{\mathfrak{sl}}(2/2)$ has precisely one singular vector iff $d \in \mathbb{Z}_+$ and $r = -(d - j + 2)/2$. It is given by*

$$\begin{aligned} v_s &= (G_+ G_- - 2mK)^d u_0, \\ u_0 &= (G_- \mathcal{S}_- \mathcal{X}_+ + M \mathcal{S}_+ \mathcal{S}_- + 2mK)v_0. \end{aligned} \quad (36)$$

Proposition 10 *All irreducible lowest weight modules over $\hat{\mathfrak{sl}}(2/2)$ are listed as follows:*

- (i) *The Verma module $V^{d,j,r}$ if $d \notin \mathbb{Z}_+$ nor $r \neq -(d - j + 2)/2$*
- (ii) *The quotient module $V^{d,j,r}/\mathcal{I}$ if $d \in \mathbb{Z}_+$ and $r = -(d - j + 2)/2$*

where $\mathcal{I} \subset V^{d,r}$ is the invariant submodule constructed on the singular vector (36): $\mathcal{I} = U(\hat{\mathfrak{sl}}(2/2)^+)v_s$. All irreducible modules given are infinite dimensional.

4. Concluding remarks

We investigate the lowest weight representations of the super Schrödinger algebras in $(1+1)$ and $(2+1)$ dimensional spacetime with $\mathcal{N} = 1, 2$ extensions. Our main result is a classification of irreducible lowest weight modules for nonvanishing mass. This was done by the use of singular vectors. All irreducible modules are infinite dimensional. Finite dimensional irreducible modules appear when the mass is set equal to zero [35]. One may introduce a bilinear form analogous to the Shapovalov form [37] of the semisimple Lie algebra to the Verma modules. Let \mathfrak{g} be a super Schrödinger algebra discussed in this paper and V be a Verma module over it. We define the bilinear form $(\ , \) : V \otimes V \rightarrow \mathbb{C}$ by the relations:

$$(Xv_0, Yv_0) = (v_0, \omega(X)Yv_0), \quad (v_0, v_0) = 1, \quad X, Y \in U(\mathfrak{g}) \quad (37)$$

By the bilinear form one may define unitary representations of the super Schrödinger algebras. It is also easy to see that if $v_m, v_n \in V$ have different weight of the generator D , then they are orthogonal with respect to this form:

$$(v_m, v_n) = 0. \quad (38)$$

It follows that a singular vector of \mathfrak{g} is orthogonal to any other vectors in V . This is the same property as semisimple case. Thus one may analyze the reducibility of the Verma modules also via the bilinear form.

One may use the singular vectors to obtain invariant partial differential equations. This is an analogy to the cases of semisimple Lie algebras [38, 39] and the bosonic Schrödinger algebras [31, 32, 40, 41]. In the latter case the obtained partial differential equations are a

hierarchy of free Schrödinger equations. To obtain the invariant equations we need a vector field representation of the algebra. Vector field representations for $\mathfrak{s}(1/1)$ and $\mathfrak{s}(1/2)$ are found in [35]. Another link of the present work to physical problem is a supersymmetric extension of the group theoretical approach to nonrelativistic holography discussed in [42]. It may also require vector field representations of the super Schrödinger algebras.

We restrict ourselves to the super Schrödinger algebras in this work. However there exist other algebras which may be regarded as nonrelativistic analogue of conformal algebras [19, 20] (see also [6]) and their supersymmetric extensions [43]. Physical importance of such algebras has been recognized widely [6, 7, 10–12, 18], however, representation theory of such algebras are far from completion. It is an important issue to develop representation theory for the algebras along the line of the present work. We announce that it has been done for one of the so-called conformal Galilei algebras in [44].

References

- [1] Niederer U 1972 *Helv. Phys. Acta.* **45** 802
- [2] Hagen C R 1972 *Phys. Rev.* **D5** 377
- [3] Henkel M 1994 *J. Stat. Phys.* **75** 1023 hep-th/9310081
- [4] Dunne G 1995 *Self-dual Chern-Simons theories* (Springer-Verlag)
- [5] Sachdev S 1999 *Quantum Phase Transitions* (Cambridge University Press)
- [6] Duval C and Horváthy P A 2009 *J. Phys. A:Math. Theor.* **42** 465206 arXiv:0904.0531 [math-ph]
- [7] Horváthy P A, Martina L and Stichel P C 2010 *SIGMA* **6** 060 arXiv:1002.4772 [hep-th]
- [8] Duval C, Gibbons G W and Horváthy P A 1991 *Phys. Rev.* **D43** 3907 hep-th/0512188
- [9] Mehen T, Stewart I W and Wise M B 2000 *Phys. Lett.* **B474** 145 hep-th/9910025
- [10] Stichel P C and Zakrzewski W J 2004 *Annals Phys.* **310** 158 arXiv:hep-th/0309038
- [11] Lukierski J, Stichel P C and Zakrzewski W J 2006 *Phys. Lett.* **A357** 1 hep-th/0511259
- [12] Lukierski J, Stichel P C and Zakrzewski W J 2007 *Phys. Lett.* **B650** 203 arXiv:hep-th/0702179
- [13] Nishida Y and Son D T 2007 *Phys. Rev.* **D76** 086004 arXiv:0706.3746 [hep-th]
- [14] Son D T 2008 *Phys. Rev.* **D78** 046003 arXiv:0804.3972 [hep-th]
- [15] Balasubramanian K and McGreevy J 2008 *Phys. Rev. Lett.* **101** 061601 arXiv:0804.4053 [hep-th]
- [16] Bagchi A and Gopakumar R 2009 *J. High Energy Phys.* JHEP07(2009)037 arXiv:0902.1385 [hep-th]
- [17] Alishahiha M, Davody A and Vahedi A 2009 *J. High Energy Phys.* JHEP08(2009)022 arXiv:0903.3953 [hep-th]
- [18] Martelli D and Tachikawa Y 2010 *J. High Energy Phys.* JHEP05(2010)091 arXiv:0903.5184 [hep-th]
- [19] Havas P and Plebański J 1978 *J. Math. Phys.* **19** 482
- [20] Negro J, del Olmo M A and Rodriguez-Marco A 1997 *J. Math. Phys.* **38** 3786, *ibid*, 3810
- [21] Beckers J and Hussin V 1986 *Phys. Lett.* **A118** 319
- [22] Beckers J, Dehin D and Hussin V 1987 *J. Phys. A:Math. Gen.* **20** 1137
- [23] Gauntlett J P, Gomis J and Townsend P K 1990 *Phys. Lett.* **B248** 288
- [24] Leblanc M, Lozano G and Min H 1992 *Ann. Phys. (N. Y.)* **219** 328 hep-th/9206039
- [25] Duval C and Horváthy P A 1994 *J. Math. Phys.* **35** 2516 hep-th/0508079
- [26] Nakayama Yu , Sakaguchi M and Yoshida K 2009 *J. High Energy Phys.* JHEP04(2009)096 arXiv:0902.2204 [hep-th]
- [27] Henkel M and Unterberger J 2006 *Nucl. Phys.* **B746**[FS] 155 arXiv:math-ph/0512024
- [28] Sakaguchi M and Yoshida K 2008 *J. Math. Phys.* **49** 102302 arXiv:0805.2661 [hep-th]
- [29] Sakaguchi M and Yoshida K 2008 *J. High Energy Phys.* JHEP08(2008)049 arXiv:0806.3612 [hep-th]
- [30] Perroud M 1977 *Helv. Phys. Acta* **50** 233
- [31] Dobrev V K, Doebner H D and Mrugalla Ch 1997 *Rep. Math. Phys.* **39** 201
- [32] Mrugalla Ch 1997 PhD Thesis, Technical University Clausthal
- [33] Feinsilver P, Kocik J and Schott R 2004 *Fort. Physik* **52** 343 math-ph/0008035.
- [34] Nakayama Yu 2008 *J. High Energy Phys.* JHEP10(2008)083 arXiv:0807.3344 [hep-th]
- [35] Aizawa N 2010 Lowest weight representations of super Schrödinger algebras in one dimensional space *Preprint* arXiv:1009.0085 [math-ph]
- [36] Aizawa N in preparation
- [37] Shapovalov N N 1972 *Funct. Anal. Appl.* **6** 307
- [38] Dobrev V K *Rep. Math. Phys.* **25** 159
- [39] Kostant B 1975 *Lecture Notes in Mathematics* **466** (Springer-Verlag: Berlin) p 101

- [40] Aizawa N, Dobrev V K and Doebner H D 2002 *Proc. 2nd Int. Symp Quantum Theory and Symmetries* ed E Kapuścik and A Horzela (World Scientific) p 222
- [41] Aizawa N, Dobrev V K, Doebner H D and Stoimenov S 2008 *Proc. VII Int. Workshop Lie Theory and Its Applications in Physics* ed H D Doebner and V K Dobrev (Heron Press) p 372
- [42] Aizawa N and Dobrev V K 2010 *Nucl. Phys.* **B828**[PM] 581 arXiv:0906.0257 [hep-th]
- [43] Lukierski J 2010 $D = 4$ Extended Galilei superalgebras with central charges *Preprint* arXiv:1009.0182 [hep-th]
- [44] Aizawa N and Isaac P S 2010 On irreducible representations of the exotic conformal Galilei algebra *Preprint* arXiv:1010.4075 [math-ph] to appear in *J. Phys. A:Math. Theor.*