The closure of a sheet is not always a union of sheets, a short note

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Abstract

In this note we answer to a frequently asked question. If G is an algebraic group acting on a variety V, a G-sheet of V is an irreducible component of $V^{(m)}$, the set of elements of V whose G-orbit has dimension m. We focus on the case of the adjoint action of a semisimple group on its Lie algebra. We give two families of examples of sheets whose closure is not a union of sheets in this setting.

Let $\mathfrak{g}$ be a semisimple Lie algebra defined over an algebraically closed field $\Bbbk$ of characteristic zero. Let G be the adjoint group of $\mathfrak{g}$. For any integer m, one defines

$$\mathfrak{g}^{(m)} = \{x \in \mathfrak{g} \mid \dim G.x = m\}.$$

A *G-sheet* (or simply *sheet*) is an irreducible component of $\mathfrak{g}^{(m)}$ for some $m \in \mathbb{N}$. We refer to [TY, §39] for elementary properties of sheets. The most important one is that each sheet contains a unique nilpotent orbit.

There exists a well known subdivision of sheets which forms a stratification. The objects considered in this subdivision are Jordan classes and generalize the classical Jordan's block decomposition in $\mathfrak{gl}_n$. Those classes and their closures are widely studied in [Bo] (cf. also [TY, §39] for a more elementary viewpoint). Since sheets are locally closed, a natural question is then the following.

If S is a sheet, is $\overline{S}$ is a union of sheets?

The answer is negative in general. We give two families of counterexamples below.

1. A nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$ is said to be rigid if it is a sheet of $\mathfrak{g}$. Rigid orbits are key objects in the description of sheets given in [Bo]. They were classified for the first time in [Sp, §II.7&II.10]. The closure ordering of nilpotent orbits (or *Hasse diagram*) can be found in [Sp, §II.8&IV.2]. In the classical cases, a more recent reference for these lists is [CM]. One easily checks from these classifications that there may exists some rigid nilpotent orbit $\mathcal{O}_1$ that contains a non-rigid nilpotent orbit $\mathcal{O}_2$ in its closure. Then, we set $S = \mathcal{O}_1$ and we get $\mathcal{O}_2 \subset \overline{S} \subset \mathcal{N}(\mathfrak{g})$ where $\mathcal{N}(\mathfrak{g})$ is the set of nilpotent elements of $\mathfrak{g}$. Since $\mathcal{O}_2$ is not rigid, the sheets containing

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$\mathcal{O}_2$ are not wholly included in $\mathcal{N}(\mathfrak{g})$. Therefore, the closure of S is not a union of sheets.

Here are some examples of such nilpotent orbits. In the classical cases, we embed $\mathfrak{g}$ in $\mathfrak{gl}_n$ in the natural way. Then, we can assign to each nilpotent orbit $\mathcal{O}$, a partition of n, denoted by $\Gamma(\mathcal{O})$. This partition defines the orbit $\mathcal{O}$, sometimes up to an element of $\mathrm{Aut}(\mathfrak{g})$. In the case $\mathfrak{g} = \mathfrak{so}_8$ (type D_4), there is exactly one rigid orbit $\mathcal{O}_1$, such that $\Gamma(\mathcal{O}_1) = [3, 2^2, 1]$. It contains in its closure the non-rigid orbit $\mathcal{O}_2$ such that $\Gamma(\mathcal{O}_2) = [3, 1^5]$ (cf. [Mo, Table2, p.15]). Very similar examples can be found in types C and B.

In the exceptional cases, we denote nilpotent orbits by their Bala-Carter symbol as in [Sp]. Let us give some examples of the above described phenomenon.

- in type E_6 ($\mathcal{O}_1 = 3A_1$ and $\mathcal{O}_2 = 2A_1$),
- in type E_7 ($\mathcal{O}_1 = A_2 + 2A_1$ and $\mathcal{O}_2 = A_2 + A_1$),
- in type E_8 ($\mathcal{O}_1 = A_2 + A_1$ and $\mathcal{O}_2 = A_2$)
- and in type F_4 ($\mathcal{O}_1 = A_2 + A_1$ and A_2).

2. In the case $\mathfrak{g} = \mathfrak{sl}_n$ of type A, there is only one rigid nilpotent orbit, the null one. Hence the phenomenon depicted in 1 can not arise in this case. Let S be a sheet and let $\lambda_S = (\lambda_1 \geqslant \cdots \geqslant \lambda_{k(\lambda_S)})$ be the partition of n associated to the nilpotent orbit $\mathcal{O}_S$ of S. As a consequence of the theory of induction of orbits, cf. [Bo], we have

$$\overline{S} = \overline{G.\mathfrak{h}_S}^{reg} \tag{1}$$

where $\mathfrak{h}_S$ is the centre of a Levi subalgebra $\mathfrak{l}_S$. The size of the blocks of $\mathfrak{l}_S$ yield a partition of n, which we denote by $\tilde{\lambda}_S = (\tilde{\lambda}_1 \geqslant \cdots \geqslant \tilde{\lambda}_{p(\lambda_S)})$. In fact $\tilde{\lambda}$ is the dual partition of λ, i.e. $\tilde{\lambda}_i = \#\{j \mid \lambda_j \geqslant i\}$ (see, e.g., [Kr, §2.2]). In particular, the map sending a sheet S to its nilpotent orbit $\mathcal{O}_S$ is a bijection.

An easy consequence of (1) is the following (see [Kr, Satz 1.4]). Given any two sheets S and S' of $\mathfrak{g}$, we have $S \subset \overline{S'}$ if and only if $\mathfrak{h}_S$ is G-conjugate to a subspace of $\mathfrak{h}_{S'}$ or, equivalently, $\mathfrak{l}_{S'}$ is conjugate to a subspace of $\mathfrak{l}_S$. This can be translated in terms of partitions by defining a partial ordering on the set of partitions of n as follows. We say that $\lambda \preceq \lambda'$ if there exists a partition $(J_i)_{i \in [\![1, p(\lambda)]\!]}$ of $[\![1, p(\lambda')]\!]$ such that $\tilde{\lambda}_i = \sum_{j \in J_i} \tilde{\lambda}'_j$. Hence, we have the following characterization.

Lemma 1. *$S \subset \overline{S'}$ if and only if $\lambda_S \preceq \lambda_{S'}$.*

One sees that this criterion is strictly stronger than the characterization of inclusion relations of closures of nilpotent orbits (see, e.g., [CM, §6.2]). More precisely, one easily finds two sheets S and S' such that $\mathcal{O}_S \subset \overline{\mathcal{O}_{S'}}$ while $\lambda_S \not\preceq \lambda_{S'}$. Then, $\mathcal{O}_S \subset \overline{S'}$, S is the only sheet containing $\mathcal{O}_S$ and $S \not\subset \overline{S'}$. For instance, take $\lambda_{S'} = [3, 2]$, $\lambda_S = [3, 1, 1]$. Their respective dual partitions being $[2, 2, 1]$ and $[3, 1, 1]$, we have $\lambda_S \not\preceq \lambda_{S'}$.

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