A Liouville Theorem for the Axially-symmetric Navier-Stokes Equations

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November 24, 2010

Abstract

Let $v(x,t) = v^r e_r + v^\theta e_\theta + v^z e_z$ be a solution to the three-dimensional incompressible axially-symmetric Navier-Stokes equations. Denote by $b = v^r e_r + v^z e_z$ the radial-axial vector field. Under a general scaling invariant condition on b, we prove that the quantity $\Gamma = r v^\theta$ is Hölder continuous at $r = 0$, $t = 0$. As an application, we give a partial proof of a conjecture on Liouville property by Koch-Nadirashvili-Seregin-Sverak in [15] and Seregin-Sverak in [24]. As another application, we prove that if $b \in L^\infty([0,T], BMO^{-1})$, then v is regular. This provides an answer to an open question raised by Koch and Tataru in [14] about the uniqueness and regularity of Navier-Stokes equations in the axially-symmetric case.

Keyword: axially-symmetric Navier-Stokes equations, Liouville theorem, ancient solutions, Nash method

1   Introduction

In this paper we study the three-dimensional incompressible axially-symmetric Navier-Stokes equations. In cylindrical coordinates, the velocity field $v = v(x,t)$ is of the form

$$v(x,t) = v^r(r,z,t)e_r + v^\theta(r,z,t)e_\theta + v^z(r,z,t)e_z.$$

Here and throughout the paper, we write $x = (x_1, x_2, z)$, $r = r(x) = \sqrt{x_1^2 + x_2^2}$ and

$$e_r = e_r(x) = \begin{pmatrix} \frac{x_1}{r} \\ \frac{x_2}{r} \\ 0 \end{pmatrix}, \quad e_\theta = e_\theta(x) = \begin{pmatrix} -\frac{x_2}{r} \\ \frac{x_1}{r} \\ 0 \end{pmatrix}, \quad e_z = e_z(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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are the three orthogonal unit vectors along the radial, the angular, and the axial directions respectively. The radial, angular (or swirl) and axial components v^r, v^θ and v^z of the velocity field are governed by (see, for instance, [19])

$$\begin{cases} \partial_t v^r + b \cdot \nabla v^r - \frac{(v^\theta)^2}{r} + \partial_r p = \left(\Delta - \frac{1}{r^2}\right)v^r, \\ \partial_t v^\theta + b \cdot \nabla v^\theta + \frac{v^r v^\theta}{r} = \left(\Delta - \frac{1}{r^2}\right)v^\theta, \\ \partial_t v^z + b \cdot \nabla v^z + \partial_z p = \Delta v^z, \\ b = v^r e_r + v^z e_z, \quad \nabla \cdot b = \partial_r v^r + \frac{v^r}{r} + \partial_z v^z = 0. \end{cases} \tag{1.1}$$

Here without loss of generality, we have set the viscosity constant to be unit.

A special feature of the axially-symmetric Navier-Stokes equations is that the quantity $\Gamma = r v^\theta(x,t)$ satisfies an parabolic equation with singular drift terms:

$$\partial_t \Gamma + b \cdot \nabla \Gamma + \frac{2}{r} \partial_r \Gamma = \Delta \Gamma. \tag{1.2}$$

We remark that Γ enjoys the maximal principle. For this reason the axially-symmetric case appears more tractable than the full three-dimensional problem.

Nevertheless, it is well-known that global regularity of the three-dimensional incompressible Navier-Stokes equations is still wide open even in the axially-symmetric case. But if the swirl component of the velocity field v^θ is trivial, independently, Ladyzhenskaya [16] and Uchoviskii & Yudovich [28] proved that weak solutions are regular for all time (see also [18]). Recently, tremendous efforts and interesting progresses have been made on the regularity problem of the axially-symmetric Navier-Stokes equations with a general non-trivial swirl. For example, in [4, 5], Chen-Strain-Tsai-Yau proved, among other things, that the suitable weak solutions are smooth if the velocity field v satisfies $r|v| \le C_* < \infty$. Their method is based on the classical results by Nash [22], Moser [21] and De Giorgi [6]. In [15], Koch-Nadirashvili-Seregin-Sverak proved the same result using Liouville type theorem for ancient solutions of Navier-Stokes equations. See also [24] for a local version.

A velocity field is called an ancient solution if it exists in the time interval $(-\infty, 0]$, and it satisfies the Navier-Stokes equation in certain sense. A well known fact is that ancient solutions represent structures of singularity of evolution equations, which makes the study of ancient solutions an important topic.

In this paper, we study the axially-symmetric Navier-Stokes equations under a more general assumption on the radial-axial velocity vector b. To be precise, we consider b such that

$$b = b_1 + b_2 + b_3, \quad \nabla \cdot b_1 = \nabla \cdot b_2 = \nabla \cdot b_3 = 0, \tag{1.3}$$

where

$$\text{HSE}(b_1) \le C_*, \quad b_2 = \nabla \times B, \quad \sup_{-T<t<0} \|B\|_{\text{BMO}} \le C_*, \quad \sup_{-T<t<0, x\in\mathbb{R}^3} r|b_3| \le C_*. \tag{1.4}$$

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Some motivation and explanation for the condition and notations are in order. Here $[-T, 0]$ is the time interval where a solution exists. We often take $T = 1$ for convenience. The number C_* is an arbitrary positive constant and $\mathrm{HSE}(b_1)$ is called "the hollowed scaled energy", defined by

$$\mathrm{HSE}(b_1) = \sup_{0 < R < R_0} \dot{E}_R(b_1), \quad \dot{E}_R(b_1) = \sup_{-R^2 < t < 0} \frac{1}{R} \int_{B_{2R}/B_{R/8}} |b_1(\cdot, t)|^2 dx. \qquad (1.5)$$

Here R_0 is a positive number often taken as 1.

We use
$$\|b\|_E = \mathrm{HSE}(b_1) + \sup_{-T < t < 0} \|B\|_{\mathrm{BMO}} + \sup_{-T < t < 0} r|b_3| \qquad (1.6)$$

to denote the controlling quantity of b throughout the paper. Here $[-T, 0]$ is the time interval of concern, which may be shifted or scaled. The linear space consisted of those b such that $\|b\|_E < \infty$ is called space E. The results in this paper depends on b only in terms of $\|b\|_E$. We will use a positive function $K(\|b\|_E)$ to denote such a dependence, whose precise value may change from line to line. Notice that the space E contains BMO^{-1} which is the largest known space in which the Navier-Stokes equations are well-posed. See the interesting work by Koch and Tataru [14]. Another feature is that the condition on b_1 is imposed only on some subdomain of the space time cube. Outside of the subdomain, there is no restriction on b_1. With a little bit more efforts, we can also just impose conditions on part of the space time for b_2 and b_3 too. But here we do not pursue that.

Our first result states that $\Gamma = rv^\theta$ is Hölder continuous at $r = 0, t = 0$ if the radial-axial velocity field b satisfies (1.3) and (1.4). The Hölder continuity depends on b only through $\|b\|_E$.

Theorem 1.1. *Given a number $L > 0$, let $v = v(x, t)$, $(x, t) \in Q_L \equiv B(x, L) \times [-L^2, 0]$ be a $L^\infty_{\mathrm{loc}}(Q_L)$ weak solution to the three-dimensional axially-symmetric Navier-Stokes equations (1.1). Suppose that the radial-axial velocity field b satisfies (1.3)-(1.4). Then $\Gamma = rv^\theta$ is Hölder continuous at $(0,0)$ uniformly. i.e. There exist positive constants α and C, depending only on $\|b\|_E$, such that, for all $(x, t) \in Q_{L/2}$, it holds*

$$|\Gamma(x, t) - \Gamma(0, 0)| \leq C[(|x| + \sqrt{|t|})/L]^\alpha \sup_{Q_L} |\Gamma|.$$

Our proof is inspired by [4] where the authors had proved a version of the above theorem under the assumption $r|v| \leq C_*$ using an De Giorgi type argument (see also [5] for the method based on the direct estimation of an evolution kernel). Here we will treat the more general b using a Nash type method in a uniform way. We will first establish a local maximum estimate for solutions of (1.2) in terms of the controlling constant C_* for b in (1.4). This is done by using Moser's iteration method and De Giorgi type energy estimate, exploiting the structure of b. Similar argument has appeared in Zhang [29] and Chen-Strain-Tsai-Yau [4, 5] where b is some form bounded function or $|rb(x, t)| \leq C_*$. Then we apply the Nash type method to prove the Hölder continuity

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of Γ. One handy tool which allows to treat more general type of vector fields b is a simple two dimensional integration by parts argument (2.8). Another tool is the John-Nirenberg inequality for BMO functions, which was first employed in [8] and [25] to treat a linear heat equation with a drift term. We also utilize the role played the stream function, which helps to do integration by parts one more time. Let v be a velocity field. We recall that a function B is called a stream function of v if $v = \nabla \times B$.

The main significance of Theorem 1.1 is that it deduces the next two theorems. One of them gives a partial answer to an open question in [15] on Liouville properties. The other one establishes a condition on b such that solutions to axially-symmetric Navier-Stokes equations are regular. This regularity condition does not involve Lebesgue integral on b or absolute value of b, which may allow the capturing of more oscillatory functions.

Theorem 1.2. *Let $v = v(x,t)$ be a bounded, weak ancient solution to (1.1). Suppose also $r|v^\theta|$ is bounded and the stream function is a BMO function. Then $v \equiv 0$.*

Remark 1.3. The authors of [15] stated a conjecture on Liouville type theorem for the axially-symmetric Navier-Stokes equations: bounded, mild, ancient velocity fields are constants. The authors in [15] proved such kind of Liouville theorems in the three-dimensional axially-symmetric case without swirl, or under the condition $r|v|$ being bounded. The above theorem, under the conditions that $r|v^\theta|$ is bounded and the stream function is a BMO function, gives a proof of this conjecture.

Recall that rv^θ is scaling invariant and it also satisfies the maximum principle. Therefore its boundedness is a mild restriction. A bounded function is obviously a BMO function. Although a bounded velocity field may not have a bounded stream function in general, a boundedness assumption on the stream function is also very mild since one expects it to hold in most natural cases when the velocity is bounded.

In [14] p25, Koch and Tataru wrote that there has been a strong interest in obtaining well-posedness of Navier-Stokes equations assuming a BMO^{-1} space condition. They raised the question of uniqueness and regularity for solutions in the space $L^\infty([0,T], BMO^{-1})$. They also proved uniqueness for small solutions in such a space. Our Theorem 1.4 answers this question in the axially-symmetric case. We should mention that Miura [20] proved uniqueness in the smaller space $C([0,T], BMO^{-1})$.

Theorem 1.4. *Let $v = v(x,t)$ be a suitable weak solution to (1.1) in the space time region $\mathbf{R}^3 \times [0,T]$. Assume that the initial value satisfies $v(\cdot, 0) \in L^2$, $|rv^\theta(x,0)| < C$. Suppose also $v(\cdot, t) = \nabla \times B(\cdot, t)$ with $\sup_{0<t<T} \|B(\cdot, t)\|_{\mathrm{BMO}} \leq C_*$. Then v is smooth in $\mathbf{R}^3 \times (0,T]$. Here C and C_* are arbitrary positive constants.*

Remark 1.5. Note condition $|rv^\theta(x,0)| < C$ is only on the initial value. It can also be dropped by a approximation argument. We will not seek the full generality this time.

Remark 1.6. In [7], Escauriaza, Seregin and Sverak proved that $L_T^\infty L^3(Q)$ solutions to the Navier-Stokes equations are regular, which is the highly non-trivial borderline

case of Serrin's criterion. Their proof is based on the method of backward uniqueness and unique continuation together with a blowup argument. Since L^3 is imbedded into BMO^{-1}, our Theorem 1.4 also provides a new and simpler proof to such a criterion in the axially-symmetric case.

Before ending the introduction, let us mention some other related results on axially-symmetric Navier-Stokes equations. In the presence of swirl, there is the paper by J. Neustupa & M. Pokorny [23], proving the regularity of one component (either v^r or v^θ) implies regularity of the other components of the solution. Also proving regularity is the work of Q. Jiu & Z. Xin [12] under an assumption of sufficiently small zero dimension scaled norms. We would also like to mention the regularity results of D. Chae & J. Lee [3] who prove regularity results assuming finiteness of another zero dimensional integral. On the other hand, G. Tian & Z. Xin [27] constructed a family of singular axially symmetric solutions with singular initial data; T. Hou & C. Li [10] found a special class of global smooth solutions. See also a recent extension: T. Hou, Z. Lei & C. Li [9].

The paper is organized as follows: In section 2 we establish a local maximum estimate using De Giorgi type energy method and Moser's iteration method. Based on the local maximal estimate, we obtain the Hölder continuity of Γ and prove Theorem 1.1 by Nash's method in section 3. The argument is based on [4, 5]. Then in section 4 we prove our Theorem 1.2 and Theorem 1.4, using Theorem 1.1 and some new blow up arguments. The main idea is that a possible singularity falls only into two types. Type I singularity can be scaled into an axially-symmetric, bounded, ancient mild solution. Type II can be scaled to a two dimensional ancient solution. Then we show that either type leads to a contradiction with the assumption that the stream function is in the BMO space. In the process the two dimensional Liouville theorem in [15] plays an important role.

2   Local Maximum Estimate

In this section we prove a local maximum estimate of Γ using Moser's iteration method in proving the parabolic Harnack's inequality. These estimates will be used to obtain Hölder continuity of Γ in next section. The main idea is to exploit the divergence-free property of $b(x,t)$ and to construct a special cut-off function. We also learned from [4, 5] where the authors treated the term $\frac{2}{r}\partial_r\Gamma$ in the equation for Γ.

We first derive an energy estimate of De Giorgi type for (1.2). For this purpose we need a refined cut-off function. Set $\frac{1}{2} \leq \sigma_2 < \sigma_1 \leq 1$ and choose $\psi(y,s) = \phi(|y|)\eta(s)$ to be a smooth cut-off function satisfying:

$$\begin{cases} \text{supp}\phi \subset B(\sigma_1), \ \phi = 1 \text{ on } B(\sigma_2), \ 0 \leq \phi \leq 1, \\ \text{supp}\eta \subset \big(-(\sigma_1)^2, 0\big], \ \eta(s) = 1 \text{ on } \big(-(\sigma_2)^2, 0\big], \ 0 \leq \eta \leq 1, \\ |\eta'| \lesssim \frac{1}{(\sigma_1-\sigma_2)^2}, \ \big|\frac{\nabla\phi}{\sqrt{\phi}}\big| \lesssim \frac{1}{\sigma_1-\sigma_2}, \quad \big|\nabla\big(\frac{\nabla\phi}{\sqrt{\phi}}\big)\big| \lesssim \frac{1}{(\sigma_1-\sigma_2)^2}. \end{cases} \qquad (2.1)$$

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Here as usual we use $A \lesssim B$ to denote the inequality $A \leq CB$ for an absolute positive constant C. Such a cut-off function ϕ can be simply chosen as a square of a standard cut-off function. We will also use the following notations for domains. Let $R > 0$, we write $B_R = B(0, R)$ and

$$P(R) = B_R \times (-R^2, 0], \quad P(R_1, R_2) = B_{R_1}/B_{R_2} \times (-R_1^2, 0] \text{ for } R_1 > R_2.$$

Consider the functions $f = |\Gamma|^q$, $q > \frac{1}{2}$ and the cut-off functions $\psi_R(y, s) = \phi_R(y)\eta_R(s) = \phi(\frac{y}{R})\eta(\frac{s}{R^2})$. Testing (1.2) by $q|\Gamma|^{2q-2}\Gamma\psi_R^2$ gives

$$\frac{1}{2} \iint \left(\partial_s f^2 + (b \cdot \nabla)f^2 + \frac{2}{r}\partial_r f^2 \right)\psi_R^2 dyds = q \iint \Delta\Gamma|\Gamma|^{2q-2}\Gamma\psi_R^2 dyds. \qquad (2.2)$$

Using Cauchy-Schwarz's inequality and integration by parts, we compute that

$$q \iint (\Delta\Gamma)|\Gamma|^{2q-2}\Gamma\psi_R^2 dyds = q \iint (\Delta|\Gamma|)|\Gamma|^{2q-1}\psi_R^2 dyds$$

$$= -q \iint \left((2q-1)|\nabla\Gamma|^2\Gamma^{2q-2}\psi_R^2 + \nabla\psi_R^2 \cdot |\Gamma|^{2q-1}\nabla|\Gamma| \right) dyds$$

$$= - \iint \left((2 - \frac{1}{q})|\nabla f|^2\psi_R^2 + 2\psi_R\nabla\psi_R \cdot f\nabla f \right) dyds$$

$$= - \iint \left((2 - \frac{1}{q})|\nabla f|^2\psi_R^2 + 2f\nabla\psi_R \cdot \nabla(f\psi_R) - 2f^2|\nabla\psi_R|^2 \right) dyds$$

$$\lesssim - \iint |\nabla(f\psi_R)|^2 dyds + \iint f^2|\nabla\psi_R|^2 dyds$$

and

$$\frac{1}{2} \iint \psi_R^2\partial_s f^2 dyds$$

$$= \frac{1}{2} \int_{B(\sigma_1 R)} \psi_R^2 f^2(\cdot, t) dy - \frac{1}{2} \iint f^2\partial_s\psi_R^2 dyds.$$

Moreover, by the fact that $\Gamma = 0$ on the axis $r = 0$, we have

$$\iint \frac{1}{r}\partial_r f^2\psi_R^2 dyds = \iint \partial_r f^2\psi_R^2 drdzd\theta ds = \iint f^2\partial_r\psi_R^2 drdzd\theta ds.$$

Consequently, using (2.1), we have

$$\frac{1}{2} \int \psi_R^2 f^2(\cdot, t) dy + \iint |\nabla(f\psi_R)|^2 dyds \qquad (2.3)$$

$$\lesssim \frac{1}{(\sigma_1 - \sigma_2)^2 R^2} \iint_{P(\sigma_1 R)} f^2 dyds - \frac{1}{2} \iint (b \cdot \nabla f^2)\psi_R^2 dyds.$$

Now we start to treat the drift term involving $b = b_1 + b_2 + b_3$. For $R_1 > R_2$, let us denote that

$$\dot{E}(R_1, R_2, b) = \sup_{-R_1^2 \leq t \leq 0} \frac{1}{R_1 - R_2} \int_{B_{R_1} \backslash B_{R_2}} |b(\cdot, t)|^2 dx.$$

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By (2.1) and the divergence-free properties of the velocity field $b_1(x,t)$, we have

$$-\frac{1}{2}\iint(b_1\cdot\nabla f^2)\psi_R^2 dyds = \iint_{P(\sigma_1 R,\ \sigma_2 R)} b_1\cdot\frac{\nabla\phi_R}{\phi_R^{\frac{1}{2}}}(\psi_R f)^{\frac{3}{2}}(\eta_R f)^{\frac{1}{2}}dyds$$

$$\lesssim \frac{1}{(\sigma_1-\sigma_2)R}\int\|b_1\|_{L^2\left(B(\sigma_1 R,\ \sigma_2 R)\right)}\|\psi_R f\|_{L^6\left(B(\sigma_1 R)\right)}^{\frac{3}{2}}\|f\|_{L^2\left(B(\sigma_1 R)\right)}^{\frac{1}{2}}ds$$

$$\lesssim \left(\frac{\dot E(\sigma_1 R,\sigma_2 R,b_1)}{(\sigma_1-\sigma_2)R}\right)^{\frac{1}{2}}\|\psi_R f\|_{L_t^2 L^6\left(P(\sigma_1 R)\right)}^{\frac{3}{2}}\|f\|_{L_t^2 L^2\left(P(\sigma_1 R)\right)}^{\frac{1}{2}}.$$

Therefore

$$-\frac{1}{2}\iint(b_1\cdot\nabla f^2)\psi_R^2 dyds \tag{2.4}$$

$$\lesssim \frac{\dot E(\sigma_1 R,\sigma_2 R,b_1)^2}{(\sigma_1-\sigma_2)^2 R^2}\|f\|_{L_t^2 L^2\left(P(\sigma_1 R)\right)}^2 + \frac{1}{8}\iint_{P(\sigma_1 R)}|\nabla(\psi_R f)|^2 dyds.$$

Next we treat the term involving b_2. Let $\bar B = \bar B(t)$ be the average of $B(\cdot,t)$ in B_R. Then

$$-\frac{1}{2}\iint(b_2\cdot\nabla f^2)\psi_R^2 dyds$$

$$=\iint_{P(\sigma_1 R,\ \sigma_2 R)}(B-\bar B(t))\cdot\nabla\times\left(\frac{\nabla\phi_R}{\phi_R^{\frac{1}{2}}}(\psi_R f)^{\frac{3}{2}}(\eta_R f)^{\frac{1}{2}}\right)dyds$$

$$\lesssim \left\|\nabla\left(\frac{\nabla\phi_R}{\sqrt{\phi_R}}\right)\right\|_{L_t^\infty L^2\left(P(\sigma_1 R,\ \sigma_2 R)\right)}\|\psi_R f\|_{L_t^2 L^6\left(P(\sigma_1 R)\right)}^{\frac{3}{2}}\|(B-\bar B)f\|_{L_t^2 L^2\left(P(\sigma_1 R)\right)}^{\frac{1}{2}}$$

$$+\iint_{P(\sigma_1 R,\ \sigma_2 R)}(B-\bar B)\cdot\left[\frac{\nabla\phi_R}{\phi_R^{\frac{1}{2}}}\times\nabla\frac{(\psi_R f)^2}{\sqrt{\phi_R}}\right]dyds$$

$$\lesssim \left(\frac{1}{(\sigma_1-\sigma_2)R}\|(B-\bar B)f\|_{L_t^2 L^2\left(P(\sigma_1 R)\right)}\right)^{\frac{1}{2}}\|\nabla(\psi_R f)\|_{L_t^2 L^2\left(P(\sigma_1 R)\right)}^{\frac{3}{2}}$$

$$+\frac{1}{(\sigma_1-\sigma_2)R}\|(B-\bar B)f\|_{L_t^2 L^2\left(P(\sigma_1 R)\right)}\|\nabla(\psi_R f)\|_{L_t^2 L^2\left(P(\sigma_1 R)\right)}$$

$$+\left(\frac{1}{(\sigma_1-\sigma_2)R}\|(B-\bar B)f\|_{L_t^2 L^2\left(P(\sigma_1 R)\right)}\right)^2.$$

Hence

$$-\frac{1}{2}\iint(b_2\cdot\nabla f^2)\psi_R^2 dyds \lesssim \frac{1}{8}\iint_{P(\sigma_1 R)}|\nabla(\psi_R f)|^2 dyds$$

$$+\frac{1}{(\sigma_1-\sigma_2)^2 R^2}\|(B-\bar B)f\|_{L_t^2 L^2\left(P(\sigma_1 R)\right)}^2. \tag{2.5}$$

To control the last expression, we need to recall the well-known John-Nirenberg inequality for BMO functions (see [13] or [26]): for any $p\in(0,\infty)$,

$$\|B(\cdot,t)-\bar B(t)\|_{L^p(B_R)}\le C_p\|B(\cdot,t)\|_{BMO}|B_R|^{1/p}. \tag{2.6}$$

Taking $p = 6$ in the above inequality, we have

$$\|(B - \bar{B})f\|_{L_t^2 L^2 \left(P(\sigma_1 R)\right)} \leq \|f\|_{L_t^3 L^3 \left(P(\sigma_1 R)\right)} \|B - \bar{B}\|_{L_t^6 L^6 \left(P(\sigma_1 R)\right)}$$

$$\leq \|f\|_{L_t^3 L^3 \left(P(\sigma_1 R)\right)} \|B\|_{L_t^\infty BMO} |B_R|^{\frac{1}{6}} R^{\frac{1}{3}}.$$

Plug this into (2.5), we deduce

$$-\frac{1}{2} \iint (b_2 \cdot \nabla f^2) \psi_R^2 dy ds$$

$$\lesssim \frac{\|B\|_{L_t^\infty BMO}^2 R^{\frac{5}{3}}}{(\sigma_1 - \sigma_2)^2 R^2} \|f\|_{L_t^3 L^3 \left(P(\sigma_1 R)\right)}^2 + \frac{1}{8} \iint_{P(\sigma_1 R)} |\nabla(\psi_R f)|^2 dy ds. \tag{2.7}$$

The term involving b_3 has been treated in [5]. Here we give an alternative proof for completeness and simplicity.

$$|\frac{1}{2} \iint (b_3 \cdot \nabla f^2) \psi_R^2 dy ds| \lesssim |\frac{1}{2} \iint (b_3 \cdot \nabla(\psi_R^2) f^2) dy ds|$$

$$= |\iint (b_3 \cdot \nabla(\psi_R) \psi_R f^2) dy ds| \lesssim |\iint (\frac{1}{r} |\nabla \psi_R| \psi_R f^2) r dr d\theta ds| \tag{2.8}$$

$$= |\iint (|\nabla \psi_R| \psi_R f^2) dr d\theta ds|$$

$$= |\iint [\partial_r(|\nabla \psi_R| \psi_R) f^2 + |\nabla \psi_R| \psi_R \partial_r(f^2)] r \, dr d\theta ds|.$$

Using Young's inequality, we deduce

$$|\frac{1}{2} \iint (b_3 \cdot \nabla f^2) \psi_R^2 dy ds|$$

$$\lesssim \frac{1}{(\sigma_1 - \sigma_2)^2 R^2} \|f\|_{L_t^2 L^2 \left(P(\sigma_1 R)\right)}^2 + \frac{1}{8} \iint_{P(\sigma_1 R)} |\nabla(\psi_R f)|^2 dy ds. \tag{2.9}$$

Plugging in the above three estimates (2.4), (2.7) and (2.9) on terms involving b_i, $i = 1, 2, 3$ into (2.3), we arrive at

$$\sup_{-\sigma_1^2 R^2 \leq t \leq 0} \int_{B(\sigma_1 R)} \psi_R^2 f^2(\cdot, t) dy + \iint_{P(\sigma_1 R)} |\nabla(f \psi_R)|^2 dy ds \tag{2.10}$$

$$\lesssim \frac{1 + \dot{E}(\sigma_1 R, \sigma_2 R, b_1)^2}{(\sigma_1 - \sigma_2)^2 R^2} \iint_{P(\sigma_1 R)} f^2 dy ds + \frac{\|B\|_{L_t^\infty BMO}^2 R^{\frac{5}{3}}}{(\sigma_1 - \sigma_2)^2 R^2} \|f\|_{L_t^3 L^3 \left(P(\sigma_1 R)\right)}^2.$$

By Hölder inequality, this implies

$$\sup_{-\sigma_1^2 R^2 \leq t \leq 0} \int_{B(\sigma_1 R)} \psi_R^2 f^2(\cdot, t) dy + \iint_{P(\sigma_1 R)} |\nabla(f \psi_R)|^2 dy ds \tag{2.11}$$

$$\lesssim \frac{K(\|b\|_E) R^{\frac{5}{3}}}{(\sigma_1 - \sigma_2)^2 R^2} \|f\|_{L_t^3 L^3 \left(P(\sigma_1 R)\right)}^2.$$

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Here and later in the section, as has been mentioned in the introduction, $K = K(\cdot)$ is a one variable function which may change from line to line, and $\|b\|_E$ is defined in (1.6).

Our next step is to derive a mean value inequality based on (2.11) using Moser's iteration method. By Hölder inequality and Sobolev imbedding theorem, one has

$$\iint_{P(\sigma_1 R)} (\psi_R f)^{\frac{10}{3}} dyds \lesssim \int \left(\|f\psi_R(\cdot,s)\|^{\frac{4}{3}}_{L^2(B(\sigma_1 R))} \|\nabla(f\psi_R)\|^2_{L^2(B(\sigma_1 R))} \right) ds$$

$$\lesssim \sup_{-(\sigma_1 R)^2 \leq s < 0} \|f\psi_R(\cdot,s)\|^{\frac{4}{3}}_{L^2(B(\sigma_1 R))} \|\nabla(f\psi_R)\|^2_{L^2(P(\sigma_1 R))}.$$

Using (2.1) and (2.11), we obtain

$$\iint_{P(\sigma_2 R)} f^{\frac{10}{3}} dyds \lesssim \left\{ \frac{K(\|b\|_E)^{\frac{3}{2}}}{(\sigma_1 - \sigma_2)^3 R^{\frac{1}{2}}} \iint_{P(\sigma_1 R)} f^3 dyds \right\}^{\frac{10}{9}},$$

which implies that

$$\iint_{P(\sigma_2 R)} (|\Gamma|^{3q})^{\frac{10}{9}} dyds \leq \left\{ \frac{K(\|b\|_E)^{\frac{3}{2}}}{(\sigma_1 - \sigma_2)^3 R^{\frac{1}{2}}} \iint_{P(\sigma_1 R)} |\Gamma|^{3q} dyds \right\}^{\frac{10}{9}}. \qquad (2.12)$$

For integers $j \geq 0$ and a constant $\sigma = \frac{1}{3}$, set $\sigma_2 = \frac{1}{2}\left(1 + \sigma^{j+1}\right)$, $\sigma_1 = \frac{1}{2}\left(1 + \sigma^j\right)$, $q = (\frac{10}{9})^j$ in (2.12). Then we have

$$\left\{ \iint_{P\left(\frac{R}{2}\left(1+\frac{1}{\sigma^{j+1}}\right)\right)} |\Gamma|^{3(\frac{10}{9})^{j+1}} dyds \right\}^{\frac{1}{3}(\frac{9}{10})^{j+1}}$$

$$\leq \left\{ \frac{K(\|b\|_E)^{\frac{3}{2}}}{\sigma^{3j} R^{\frac{1}{2}}} \iint_{P\left(\frac{R}{2}\left(1+\frac{1}{\sigma^j}\right)\right)} |\Gamma|^{3(\frac{10}{9})^j} dyds \right\}^{\frac{1}{3}(\frac{9}{10})^j}.$$

By iteration, the above inequality gives

$$\left\{ \iint_{P\left(\frac{R}{2}\left(1+\frac{1}{\sigma^{j+1}}\right)\right)} |\Gamma|^{3(\frac{10}{9})^{j+1}} dyds \right\}^{\frac{1}{3}(\frac{9}{10})^{j+1}}$$

$$\leq \left\{ \frac{K(\|b\|_E)^{\frac{3}{2}}}{R^{\frac{1}{2}}} \right\}^{\sum_{k=0}^{j} \frac{1}{3}(\frac{9}{10})^k} \sigma^{-\sum_{l=0}^{j} l(\frac{9}{10})^l} \left\{ \iint_{P(R)} |\Gamma|^3 dyds \right\}^{\frac{1}{3}}$$

$$\lesssim \left\{ \frac{K(\|b\|_E)^{\frac{3}{2}}}{R^{\frac{1}{2}}} \right\}^{\frac{10}{3}\left(1-(\frac{9}{10})^j\right)} \left\{ \iint_{P(R)} |\Gamma|^3 dyds \right\}^{\frac{1}{3}}.$$

We take the limit $j \to \infty$ to yield that

$$\sup_{P\left(\frac{R}{2}\right)} |\Gamma| \lesssim \left(K(\|b\|_E)\right)^5 \left\{ \frac{1}{R^5} \iint_{P(R)} |\Gamma|^3 dyds \right\}^{\frac{1}{3}}. \qquad (2.13)$$

From this a well known algebraic trick (see p87 [11] e.g.) shows

$$\sup_{P\left(\frac{R}{2}\right)} |\Gamma| \lesssim \left(K(\|b\|_E)\right)\left\{\frac{1}{R^5} \iint_{P(R)} |\Gamma|^2 dyds\right\}^{\frac{1}{2}}. \tag{2.14}$$

Here the function $K(\cdot)$ may have changed at the last step.

3 Hölder Continuity of Γ

In this section we study the regularity of Γ using the local maximum estimates of (2.13) in section 2 and Nash type method for parabolic equations.

Let us first recall a Nash inequality, whose proof can be found in [5].

Lemma 3.1. *Let $M \geq 1$ be a constant and μ be a probability measure. Then for all $0 \leq f \leq M$, there holds*

$$\left|\ln \int f d\mu - \int \ln f d\mu\right| \leq \frac{M\|g\|_{L^2}}{\int f d\mu},$$

where $g = \ln f - \int \ln f d\mu$.

Let ζ be a smooth radial cut-off function such that

$$\begin{cases} \zeta = 1 \ \text{ on } \ B(\frac{1}{2}), \quad \zeta = 0 \ \text{ on } \ B(1)^c, \quad \int_{\mathbb{R}^3} \zeta^2(x)dx = 1, \\ \left|\frac{\nabla\zeta}{\sqrt{\zeta}}\right| < \infty, \quad \left|\nabla\left(\frac{\nabla\zeta}{\sqrt{\zeta}}\right)\right| < \infty, \end{cases} \tag{3.1}$$

and $\zeta_R(x) = \frac{1}{R^{\frac{3}{2}}}\zeta(\frac{x}{R})$. Let Φ be a positive solution to (1.2) in $P(R)$.

Lemma 3.2. *Let $\Phi \leq 2$ be a positive solution to (1.2) in $P(R)$ which is assumed to satisfy*

$$\|\Phi\|_{L^1\left(P(\frac{R}{2})\right)} \geq c_0 R^5. \tag{3.2}$$

Moreover, we assume that $\Phi(r = 0, z, t)$ is a constant bigger than 1. Then there holds

$$-\int \zeta_R^2(x) \ln \Phi(x, t)dx \leq M_0\left(1 + \|b\|_E^2\right) \tag{3.3}$$

for all $t \in [-\frac{c_0 R^2}{4}, 0]$ and some absolute positive constant M_0 depending only on c_0.

Proof. First of all, let us define $\widetilde{\Phi}(x, t) = \Phi(Rx, R^2t)$ and $\widetilde{b}(x, t) = Rb(Rx, R^2t)$. It is clear that $\widetilde{\Phi}$ solves the equation

$$\partial_t \widetilde{\Phi} + \widetilde{b} \cdot \nabla \widetilde{\Phi} + \frac{2}{r}\partial_r \widetilde{\Phi} = \Delta \widetilde{\Phi}$$

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on $P(1)$ and $0 \le \widetilde{\Phi} \le 2$, $\|\widetilde{\Phi}\|_{L^1\left(P(\frac{1}{2})\right)} \ge c_0$. The quantity we are going to control is $-\int \zeta_R^2(x) \ln \Phi(x,t)dx = -\int \zeta^2(x) \ln \widetilde{\Phi}(x, R^{-2}t)dx$ on a time internal $[-\frac{c_0 R^2}{4}, 0]$. Equivalently, we just need to estimate $-\int \zeta^2(x) \ln \widetilde{\Phi}(x,t)dx$ for $t \in [-\frac{c_0}{4}, 0]$.

Let $\Psi = -\ln \widetilde{\Phi}$. It is easy to see that Ψ solves the equation

$$\partial_t \Psi + \widetilde{b} \cdot \nabla \Psi + \frac{2}{r}\partial_r \Psi - \Delta \Psi + |\nabla \Psi|^2 = 0. \tag{3.4}$$

Hence, by testing (3.4) with ζ^2 and using integrating by parts and Cauchy-Schwarz's inequality, one has

$$\partial_t \int \Psi \zeta^2 dx + \int |\nabla \Psi|^2 \zeta^2 dx = \int \left(-\widetilde{b} \cdot \nabla \Psi - \frac{2}{r}\partial_r \Psi + \Delta \Psi\right)\zeta^2 dx$$

$$\le -\int \zeta^2 \widetilde{b} \cdot \nabla\left(\Psi - \bar{\Psi}(s)\right)dx - \int \frac{4\pi}{r}\partial_r\left(\Psi - \bar{\Psi}(s)\right)\zeta^2 r dr d\theta dz$$

$$+ \frac{1}{4}\int |\nabla \Psi|^2 \zeta^2 dx + \int |\nabla \zeta|^2 dx.$$

Here $\bar{\Psi}(s) = \int \Psi(\cdot, t)\zeta^2 dx$. Using the weighted Poincaré inequality

$$\int |\Psi - \bar{\Psi}(s)|^2 \zeta^2 dx \le C \int |\nabla \Psi|^2 \zeta^2 dx \tag{3.5}$$

and the divergence-free property of b, we can estimate

$$\left|\int \zeta^2 \widetilde{b_1} \cdot \nabla\left(\Psi - \bar{\Psi}(s)\right)dx\right| \le \frac{1}{8}\int |\nabla \Psi|^2 \zeta^2 dx + C\int |\widetilde{b_1}|^2 |\nabla \zeta|^2 dx,$$

and

$$\left|\int \zeta^2 \widetilde{b_2} \cdot \nabla\left(\Psi - \bar{\Psi}(s)\right)dx\right| = \left|\int \nabla \zeta^2 \nabla \times B\left(\Psi - \bar{\Psi}(s)\right)dx\right|$$

$$\lesssim \int |\zeta \nabla(\Psi - \bar{\Psi})||\nabla \zeta||B - \bar{B}|dx + \int \left|\nabla\left(\zeta^{\frac{3}{2}}\frac{\nabla \zeta}{\sqrt{\zeta}}\right)\right||B - \bar{B}||\Psi - \bar{\Psi}(s)|dx$$

$$\lesssim \int |\zeta \nabla(\Psi - \bar{\Psi})||\nabla \zeta||B - \bar{B}|dx + \int \left(\frac{|\nabla \zeta|^2}{\zeta} + \sqrt{\zeta}\nabla\frac{\nabla \zeta}{\sqrt{\zeta}}\right)|B - \bar{B}||\zeta||\Psi - \bar{\Psi}(s)|dx$$

$$\lesssim \frac{1}{8}\int |\nabla \Psi|^2 \zeta^2 dx + C\|B\|_{\text{BMO}}^2.$$

Here we just used the weighted Poincaré inequality and (2.6), with $p = 2$. Moreover

$$\left|\int \zeta^2 \widetilde{b_3} \cdot \nabla\left(\Psi - \bar{\Psi}(s)\right)dx\right| = \left|\int \nabla \zeta^2 \widetilde{b_3}\left(\Psi - \bar{\Psi}(s)\right)dx\right|$$

$$\lesssim \int |\nabla \zeta(\Psi - \bar{\Psi})|\zeta dr d\theta dz = \int \left|\frac{\nabla \zeta}{\sqrt{\zeta}}(\Psi - \bar{\Psi})\right|\zeta^{\frac{3}{2}} dr d\theta dz$$

$$\lesssim \int |\zeta \nabla(\Psi - \bar{\Psi})||\nabla \zeta|dx + \int \left(\frac{|\nabla \zeta|^2}{\zeta} + \sqrt{\zeta}\nabla\frac{\nabla \zeta}{\sqrt{\zeta}}\right)|\zeta||\Psi - \bar{\Psi}(s)|dx$$

$$\lesssim \frac{1}{8}\int |\nabla \Psi|^2 \zeta^2 dx + C.$$

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Here we also used the integration by parts. On the other hand, by recalling the assumption that $\Phi(r = 0, z, t)$ is a non-zero constant, one can estimate

$$- \int \frac{4\pi}{r} \partial_r \Psi \zeta^2 r dr d\theta dz = -4\pi \int_{-\infty}^{\infty} (\Psi - \bar{\Psi}) \zeta^2 dz \Big|_{r=0}^{r=\infty} + 4\pi \int (\Psi - \bar{\Psi}) \partial_r \zeta^2 dr d\theta dz$$

$$= 4\pi \int_{-\infty}^{\infty} \Psi \zeta^2 dz \Big|_{r=0} - 4\pi \bar{\Psi} \int_{-\infty}^{\infty} \zeta^2 dz \Big|_{r=0} + 4\pi \int (\Psi - \bar{\Psi}) \zeta \frac{\partial_r \zeta}{r} r dr d\theta dz$$

$$\leq C - C\bar{\Psi}(s) + \frac{1}{8} \int |\nabla \Psi|^2 \zeta^2 dx.$$

Here we also used the fact that the support of $\frac{1}{r}|\partial_r \zeta|$ is away from z- axis. Consequently, we obtain

$$\partial_t \int \Psi \zeta^2 dx + C \int \Psi \zeta^2 dx \leq -\frac{1}{2} \int |\nabla \Psi|^2 \zeta^2 dx + C(1 + \|b\|_E^2).$$

In order to proceed, we apply the Nash inequality in Lemma 3.1. Take $f = \widetilde{\Phi}$, $d\mu = \zeta^2(x)dx$. One has

$$\left| \ln \int \widetilde{\Phi} \zeta^2 dx + \int \Psi \zeta^2 dx \right|^2 \left(\int \widetilde{\Phi} \zeta^2 dx \right)^2 \leq M^2 \int \left| -\Psi + \int \Psi \zeta^2 dy \right|^2 \zeta^2 dx.$$

Here $M = 2$ is the upper bound of Φ. Using the weighted Poincaré inequality (3.5) once again, we have

$$\left| \ln \int \widetilde{\Phi} \zeta^2 dx + \int \Psi \zeta^2 dx \right|^2 \left(\int \widetilde{\Phi} \zeta^2 dx \right)^2 \leq C \int |\nabla \Psi|^2 \zeta^2 dx.$$

Hence, we finally obtain

$$\partial_t \bar{\Psi}(t) + C_0 \bar{\Psi}(t) \leq C(1 + \|b\|_E^2)$$

$$- (2C)^{-1} \left| \ln \int \widetilde{\Phi} \zeta^2 dx + \bar{\Psi} \right|^2 \left(\int \widetilde{\Phi} \zeta^2 dx \right)^2.$$

Let $\chi(s)$ be the characteristic function of the set

$$W = \left\{ s \in [-\frac{1}{4}, 0) : \|\widetilde{\Phi}(s)\|_{L^1(B_{\frac{1}{2}})} \geq \frac{c_0}{2} \right\}.$$

By the assumption (3.2) and hence $\|\widetilde{\Phi}\|_{L^1(P(\frac{1}{2}))} \geq c_0$, one has $|W| \geq \frac{3c_0}{4}$. In fact, if $|W| < \frac{3c_0}{4}$, then

$$\|\widetilde{\Phi}\|_{L^1(P(\frac{1}{2}))} < \int_W 2\big|B(\frac{1}{2})\big| ds + \int_{W^c} \frac{c_0}{2} ds \leq \frac{(2\pi + 1)c_0}{8} < c_0,$$

which contradicts with (3.2). Thus, we have

$$\partial_t \bar{\Psi}(t) + C_0 \bar{\Psi}(t) \leq C_0(1 + \|b\|_E^2) - 8C_0^{-1} c_0^2 \chi(s) \left| \ln \int \widetilde{\Phi} \zeta^2 dx + \bar{\Psi} \right|^2. \tag{3.6}$$

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Note that the obvious consequence of this inequality gives

$$\bar{\Psi}(s_2) \le \bar{\Psi}(s_1) + C_0 e^{C_0}(s_2 - s_1)(1 + \|b\|_E^2) \tag{3.7}$$

for $-\frac{1}{4} \le s_1 \le s_2 \le 0$. Hence, if for some $s_0 \in [-\frac{1}{4}, -\frac{c_0}{4})$ such that

$$\bar{\Psi}(s_0) \le \frac{4C_0}{c_0}(1 + \|b\|_E) + 2\left|\ln\frac{c_0}{2}\right|,$$

then we are done since

$$\bar{\Psi}(t) \le \bar{\Psi}(s_0) + \frac{C_0 e^{C_0}}{2}(1 + \|b\|_E^2)$$

for all $t \in [s_0, 0)$. Otherwise, one has

$$\bar{\Psi}(s) \ge \frac{4C_0}{c_0}(1 + \|b\|_E) + 2\left|\ln\frac{c_0}{2}\right|$$

for all $s \in [-\frac{1}{4}, -\frac{c_0}{4})$. For $s \in W \cap [-\frac{1}{4}, -\frac{c_0}{4})$, one has

$$\ln \int \tilde{\Phi}\zeta^2 dx \ge \ln \int_{B_{\frac{1}{2}}} \tilde{\Phi} dx \ge \ln\frac{c_0}{2}.$$

Hence, by (3.6), we have

$$\partial_t \bar{\Psi} + C_0 \bar{\Psi} \le -\frac{c_0^2 \chi(s)}{C_0}\bar{\Psi}^2, \quad -\frac{1}{4} \le s \le -\frac{c_0}{4}.$$

Solving the above inequality gives

$$\bar{\Psi}(-\frac{c_0}{4}) \le \frac{1}{\frac{c_0^2}{C_0}\int_{-\frac{1}{4}}^{-\frac{c_0}{4}} \chi(s)e^{-C_0 s}ds + \frac{1}{\bar{\Psi}(-\frac{1}{4})}} < \infty.$$

The bound of the $\bar{\Psi}(-\frac{c_0}{4})$ depends only on c_0 since $\overline{\Psi}(-\frac{1}{4}) > 0$ and $|W| \ge \frac{3c_0}{4}$. Starting from $s = -\frac{c_0}{4}$ and using (3.7), we have

$$\bar{\Psi}(s) \le \bar{\Psi}\left(-\frac{c_0}{4}\right) + C_0 e^{C_0}(1 + \|b\|_E^2)$$

for all $s \in [-\frac{c_0}{4}, 0]$, which completes the proof of the lemma. $\qquad \square$

As an corollary, Lemma 3.2 gives a lower bound of positive solutions of (1.2).

Corollary 3.3. *Let Φ, c_0 and M_0 be given in Lemma 3.2. Then there exists a constant $0 < \delta < 1$ depending only on $\|b\|_E$ such that*

$$\inf_{P\left(\frac{R}{8}\right)} \Phi \ge \frac{\delta}{2}. \tag{3.8}$$

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Proof. Using Lemma 3.2, we have

$$M_0\big(1 + \|b\|_E^2\big) \geq - \int \zeta_R^2(x) \ln \Phi(t,x) dx$$

$$= - \int_{\delta < \Phi \leq 1} \zeta_R^2(x) \ln \Phi(t,x) dx - \int_{\Phi \leq \delta} \zeta_R^2(x) \ln \Phi(t,x) dx$$

$$\quad - \int_{1 < \Phi \leq 2} \zeta_R^2(x) \ln \Phi(t,x) dx$$

$$\geq 0 - \int_{\Phi \leq \delta} \zeta_R^2(x) \ln \Phi(t,x) dx - \ln 2 \int_{1 < \Phi \leq 2} \zeta_R^2(x) dx$$

$$\geq - \int_{\Phi \leq \delta} \zeta_R^2(x) \ln \Phi(t,x) dx - \ln 2,$$

which implies that

$$- \int_{\Phi \leq \delta} \zeta_R^2(x) \ln \Phi(t,x) dx \lesssim 1 + \|b\|_E^2$$

for $-\frac{c_0 R^2}{4} \leq t \leq 0$. Consequently, we have

$$\Big|\Big\{x \in B\Big(\frac{R}{2}\Big)|\Phi(t,x) \lesssim \delta\Big\}\Big| \leq \frac{R^3}{-\ln \delta}\big(1 + \|b\|_E^2\big)$$

for $-\frac{R^2}{64} \leq t \leq 0$. Using the mean value inequality (2.14), one has

$$\sup_{P\left(\frac{R}{8}\right)} (\delta - \Phi)_+ \lesssim \left\{ \frac{K(\|b\|_E)}{R^5} \iint_{P\left(\frac{R}{2}\right)} (\delta - \Phi)_+^2 \, dy ds \right\}^{\frac{1}{2}}$$

$$\lesssim \frac{\delta}{\sqrt{|\ln \delta|}} K(\|b\|_E),$$

which gives

$$\inf_{P\left(\frac{\sqrt{c_0} R}{2}\right)} \Phi \geq \delta - \frac{C_0 \delta}{2\sqrt{|\ln \delta|}} K(\|b\|_E)$$

for some $C_0 > 0$ which is independent of δ and R. Then (3.8) follows by choosing a sufficiently small δ such that

$$\delta \leq \exp\big\{ - K(\|b\|_E)\big\}. \tag{3.9}$$

$\square$

Now we are ready to give the proof of Theorem 1.1.
Proof of Theorem 1.1.

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Without loss of generality we take $L = 1$.

For $0 < r \le 1$, we define

$$m_r = \inf_{P(r)} \Gamma, \quad M_r = \sup_{P(r)} \Gamma, \quad J_r = M_r - m_r.$$

As in [5], we define

$$\Phi = \begin{cases} \frac{2(M_1 - \Gamma)}{J_1} & \text{if } M_1 > -m_1, \\ \frac{2(\Gamma - m_1)}{J_1} & \text{if } M_1 \le -m_1. \end{cases}$$

It is clear that $0 \le \Phi \le 2$ is a non-negative solution of (1.2) in $P(1)$ and $a = \Phi(r = 0, z, t)$ is a constant bigger than 1. To verify that Φ satisfies the condition (3.2), we need the following lemma on the lower bound of $\|\Phi\|_{L^p}$ for $0 < p < 1$ as in [5].

Lemma 3.4. *Suppose that b satisfies (1.3) and (1.4). Then for arbitrary $p \in (0,1)$, Φ defined above satisfies*

$$\frac{1}{R^{\frac{5}{p}}} \|\Phi\|_{L^p(P(R, \frac{R}{2}))} \ge C^{-1} \big(K(\|b\|_E) \big)^{-\frac{2}{p}} a.$$

Proof. Since the lemma is scaling invariant, we just take $R = 1$ in the proof. Let $\psi = \phi(|x|)\eta(t)$, where $\phi \in C_0^\infty$ such that $\phi = 1$ on $B_{\frac{1}{2}}$, $\phi = 0$ on B_1^c, $\frac{\nabla\phi}{\sqrt{\phi}}$ and $\nabla\frac{\nabla\phi}{\sqrt{\phi}}$ are bounded, $\eta \in C_0^\infty$ such that $\eta = 1$ on $[-\frac{7}{8}, -\frac{1}{8}]$ and η is supported in $(-1, 0)$. Let us test (1.2) by $p\Phi^{p-1}\psi_R^2$, $p \in (0, \frac{1}{2})$, to derive that

$$\iint \Big(\partial_s \Phi^p + (b \cdot \nabla)\Phi^p + \frac{2}{r}\partial_r \Phi^p \Big)\psi^2 dy ds = p \iint \Delta\Phi\Phi^{p-1}\psi^2 dy ds. \tag{3.10}$$

Similarly as in [5], we have

$$-\iint \frac{2}{r}(\partial_r \Phi^p)\psi^2 dy ds \tag{3.11}$$

$$= \iint \Phi^p \frac{4}{|y'|}\psi(\partial_{|y'|}\psi)dy ds + \int_{-1}^0 ds \int 2\Phi^p\psi^2\big|_{r=0} dz$$

$$\ge -C \iint \Phi^p dy ds + \frac{3}{2}a^p.$$

Here $|y'| = \sqrt{y_1^2 + y_2^2}$ if $y = (y_1, y_2, y_3)$. Likewise

$$\iint \big(-\partial_s \Phi^p + p\Delta\Phi\Phi^{p-1} \big)\psi^2 dy ds \tag{3.12}$$

$$= \iint 2\Phi^p \big[\psi(\partial_s\psi) + |\nabla\zeta|^2 - \frac{p-2}{p}\zeta\Delta\zeta \big] dy ds - \frac{4(p-1)}{p} \iint |\nabla(\Phi^{\frac{p}{2}}\psi)|^2 dy ds$$

$$\ge -C \iint \Phi^p dy ds - \frac{4(p-1)}{p} \iint |\nabla(\Phi^{\frac{p}{2}}\psi)|^2 dy ds.$$

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Moreover, concerning the term involving b, we estimate it as follows:

$$-\iint \psi^2 (b_1 \cdot \nabla) \Phi^p dy ds = \iint \Phi^p b_1 \cdot \nabla \psi^2 dy ds \qquad (3.13)$$
$$\geq -C \|b_1\|_{L_t^\infty L^2(P(1,\frac{1}{2}))} \|\Phi^p\|_{L_t^1 L^2(P(1,\frac{1}{2}))},$$

and

$$-\iint \psi^2 (b_2 \cdot \nabla) \Phi^p dy ds = \iint (B - \bar{B}) \cdot \nabla \times (\Phi^p \nabla \psi^2) dy ds$$
$$= \iint (B - \bar{B}) \cdot \nabla \times [(\Phi^{\frac{p}{2}} \psi)^2 \frac{\nabla \psi}{\psi}] dy ds$$
$$\geq -\frac{C}{R} \|\Phi^{\frac{p}{2}} \psi\|_{L_t^2 L^2(P(1,\frac{1}{2}))} \|\nabla (\Phi^{\frac{p}{2}} \psi)\|_{L_t^2 L^2(P(1,\frac{1}{2}))} - C \iint |B - \bar{B}| \Phi^p dy ds$$
$$\geq \frac{2(p-1)}{p} \iint |\nabla (\Phi^{\frac{p}{2}} \psi)|^2 dy ds - C \iint \Phi^p dy ds$$
$$- C \left(\iint \Phi^{2p} dy ds \right)^{1/2} \left(\iint (B - \bar{B})^2 dy ds \right)^{1/2}.$$

By Hölder inequality and (2.6), we have

$$-\iint \psi^2 (b_2 \cdot \nabla) \Phi^p dy ds \geq \frac{2(p-1)}{p} \iint |\nabla (\Phi^{\frac{p}{2}} \psi)|^2 dy ds - K(\|b\|_E) \|\Phi\|^p_{L^{2p}(P(1,\frac{1}{2}))} \quad (3.14)$$

Just like (2.9), we also have

$$-\iint \psi^2 (b_3 \cdot \nabla) \Phi^p dy ds \geq \frac{(p-1)}{p} \iint |\nabla (\Phi^{\frac{p}{2}} \psi)|^2 dy ds - C \iint \Phi^p dy ds. \qquad (3.15)$$

Substituting (3.15), (3.14), (3.13), (3.12) and (3.11) into (3.10), we deduce

$$\frac{3}{2} a^p \leq C \iint \Phi^p dy ds + CK(\|b\|_E) \|\Phi^p\|^p_{L_t^2 L^2(P(1,\frac{1}{2}))}$$
$$\leq CK(\|b\|_E) \|\Phi\|^p_{L^{2p}(P(1,\frac{1}{2}))},$$

which completes the proof of the lemma, since $p \in (0, 1/2)$ is arbitrary. $\qquad \square$

Now we continue the proof of the theorem.

By Lemma 3.4, Φ satisfies the assumptions in Lemma 3.2 for $R = 1$. By Corollary 3.3, one has

$$\inf_{P\left(\frac{\sqrt{c_0}}{2}\right)} \Phi \geq \frac{\delta}{2}.$$

Noting that and $m_1 \leq \inf_{P\left(\frac{\sqrt{c_0}}{2}\right)} \Gamma \leq \sup_{P\left(\frac{\sqrt{c_0}}{2}\right)} \Gamma \leq M_1$, we have

$$J_{\frac{\sqrt{c_0}}{2}} = \mathrm{OSC}_{P\left(\frac{\sqrt{c_0}}{2}\right)} \Gamma \leq \left(1 - \frac{\delta}{4}\right) J_1. \qquad (3.16)$$

Iterating (3.16) immediately shows that Φ is Hölder continuous at $(0,0)$. $\qquad \square$

4 Applications to axially symmetric Navier-Stokes equation

This section is devoted to proving Theorems 1.2 and 1.4. We begin with

Proof of Theorem 1.2.

By the assumptions of the theorem, we can apply Theorem 1.1 to deduce that the function $\Gamma = rv^\theta$ is Hölder continuous at the space time point $(0,0)$. More precisely, for any fixed point $(x,t) \in \mathbf{R}^3 \times (-\infty, 0)$, there exist positive constants α and C such that for all sufficiently large $L > 0$, we have

$$|\Gamma(x,t) - \Gamma(0,0)| \le C[(|x| + \sqrt{|t|})/L]^\alpha \sup \Gamma.$$

Letting $L \to \infty$, we find that

$$rv^\theta(x,t) = \Gamma(x,t) = \Gamma(0,0).$$

Since v^θ is a bounded function, the only way this can happen is $v^\theta \equiv 0$. Hence v is a bounded, weak ancient solution without swirl. According to Theorem 5.2 in [15], the ancient solution $v = (0, 0, l(t))$ where $l = l(t)$ depends only on time. Therefore its stream function B is a harmonic function since $\Delta B = -\nabla \times v = 0$. Since the function $B = B(\cdot, t)$ is BMO, by (2.6) we know

$$\fint_{|x|<R} |B(x,t) - \bar{B}(t)|dx \le C\|B(\cdot,t)\|_{BMO}R^3.$$

Here $\bar{B}(t)$ is the average of $B(\cdot, t)$ in the ball B_R. Since $B(\cdot, t)$ is harmonic, the mean value theorem tells us that $\bar{B}(t) = B(0,t)$. Hence

$$\fint_{|x|<R} |B(x,t)|dx \le C\|B(\cdot,t)\|_{BMO}R^3 + B(0,t)R^3.$$

The mean value theorem then implies that $B(\cdot, t)$ is a bounded function since

$$|B(y,t)| = \left| \frac{1}{4\pi|y|^3} \int_{B(y,|y|)} B(z,t)dz \right|$$

$$\le \frac{1}{4\pi|y|^3} \int_{B(0,2|y|)} |B(z,t)|dz \lesssim \|B(\cdot,t)\|_{BMO} + |B(0,t)|.$$

The classical Liouville theorem shows that the stream function B, being a bounded function, is constant. Therefore $v = \nabla \times B = 0$. $\square$

Proof of Theorem 1.4. We use the method of contradiction. If there is a singularity to the axially-symmetric Navier-Stokes equations (1.1), then we can generate a nonzero, bounded, mild ancient solution as in [15]. Our Theorem 1.1 and a scaling argument

will then be used to show that such a bounded ancient solution is identically zero. This contradiction proves that singularity can not occur.

By time shifting, we assume that the solution v exists in the time interval $[-1, 0]$ and that $t = 0$ is a blow up time of v. The partial regularity theory in [2, 17] says that the Hausdorff measure of the singular space-time set of any suitable weak solution is zero. This implies that for axially-symmetric Navier-Stokes equations (1.1), suitable weak solutions can only develop singularities on the symmetric axis $r = 0$. Hence, without loss of generality, we may assume that $(0, 0)$ is the blowup point.

For $k \geq 1$, let (x_k, t_k) be a sequence of points such that

$$-1 < t_k \nearrow 0, \quad Q_k = |v(x_k, t_k)| = \gamma_k \max_{-1 < t < t_k} |v(x, t)| \nearrow \infty, \quad \gamma_k \longrightarrow 1. \qquad (4.1)$$

Define a sequence of functions $\{v^{(k)}\}$ by

$$v^{(k)}(x, t) = \frac{1}{Q_k} v(x_k + \frac{x}{Q_k}, t_k + \frac{t}{Q_k^2}), \quad -Q_k^2(1 + t_k) \leq t \leq 0. \qquad (4.2)$$

By [1], one can assume that $r_k = r(x_k)$ are uniformly bounded. It is clear that $\{v^{(k)}\}$ defined in (4.2) are mild solutions. Moreover, $\{v^{(k)}\}$ (up to a subsequence) converges to a bounded ancient weak solution $u(x, t)$ to the Navier-Stokes equations (for details, see [15]). By the construction, $|u(x, t)| \leq 1$ and $|u(0, 0)| = 1$.

We consider two cases.

Case 1 is when $r_k |v(x_k, t_k)| = r_k Q_k$ are uniformly bounded by some positive constant C. Then the functions $\{v^{(k)}\}$ are also axi-symmetric with respect to an axis which is parallel to the z-axis and is at distance at most C from it. Consequently, u is also axi-symmetric with respect to a suitable axis. Note that both the stream function and rv^θ are scaling invariant. Thus the stream function of u is in BMO and ru^θ is also bounded. Therefore we can apply Theorem 1.1 on u, which says that the swirl component of u vanishes. By Theorem 5.2 in [15], we conclude $u = (0, 0, l(t))$ with $l = l(t)$ being a function of time only. But this show $u = 0$ as in the proof of the previous theorem. This contradiction shows that Case 1 can not happen.

Case 2 is when $r_k |v(x_k, t_k)| = r_k Q_k$ is not uniformly bounded.

Hence, $r_k Q_k$ (up to a subsequence) goes to infinity as k tends to infinity. Due to Caffarelli-Kohn-Nirenberg's partial regularity theory, $\{x_k\}$ (up to a subsequence) converges to x_* which is a point on the z-axis such that $r_* = 0$. Due to the axis symmetry of v, x_k can be chosen so that $\theta(x_k) \to \theta_\infty$ for a θ_∞. Hence, $e_r(x_k) \to \nu$ and $e_\theta(x_k) \to \nu^\perp = (-\nu_2, \nu_1, 0)$ for an unit vector $\nu = (\nu_1, \nu_2, 0)$. Here $e_r(x)$ and $e_\theta(x)$ are defined as in the introduction and $(r(x), \theta(x))$ is the polar coordinate of (x_1, x_2).

It is clear that

$$\begin{cases} x_k + \frac{x}{Q_k} \in B(x_k, \frac{r_k}{\sqrt{r_k Q_k}}) & \text{for } x \in B(0, \sqrt{Q_k r_k}), \\ t_k - (\frac{r_k}{\sqrt{r_k Q_k}})^2 < t_k + \frac{t}{Q_k^2} \leq t_k < 0 & \text{for } -Q_k r_k < t \leq 0. \end{cases}$$

By the assumption on initial value and the maximum principle, we know

$$|v^\theta(t, y)| \lesssim \frac{1}{r_k} \quad \text{for } y \in B(x_k, \frac{r_k}{2}), \ t < 0,$$

which shows

$$\begin{cases} \left| v^{(k)}(x, t) e_\theta\left(x_k + \frac{x}{Q_k}\right) \right| = \frac{1}{Q_k} \left| v^\theta\left(x_k + \frac{x}{Q_k}, t_k + \frac{t}{Q_k^2}\right) \right| \lesssim \frac{1}{Q_k r_k}, \\ \quad \text{for } (x, t) \in B(0, \sqrt{r_k Q_k}) \times (-r_k Q_k, 0]. \end{cases} \tag{4.3}$$

Note that on $B(0, \sqrt{Q_k r_k}) \times (-Q_k r_k, 0]$, it is easy to see that $e_r\left(x_k + \frac{x}{Q_k}\right) \to \nu$ and $e_\theta\left(x_k + \frac{x}{Q_k}\right) \to \nu^\perp$ as $n \to \infty$. Moreover, for each k, $v^{(k)}$ is still a mild solution to the 3D Navier-Stokes equations. By (4.3), there exists a subsequence of $\{v^{(k)}\}$ (we will still denote it by $\{v^{(k)}\}$) and a bounded ancient solution $u(x, t)$ to the 3D Navier-Stokes equations on $\mathbb{R}^3 \times (-\infty, 0]$, which is mild in the sense of [15], such that

$$\begin{aligned} v^{(k)}(x, t) &= \frac{1}{Q_k} v^r\left(x_k + \frac{x}{Q_k}, t_k + \frac{t}{Q_k^2}\right) e_r\left(x_k + \frac{x}{Q_k}\right) \\ &+ \frac{1}{Q_k} v^\theta\left(x_k + \frac{x}{Q_k}, t_k + \frac{t}{Q_k^2}\right) e_\theta\left(x_k + \frac{x}{Q_k}\right) \\ &+ \frac{1}{Q_k} v^z\left(x_k + \frac{x}{Q_k}, t_k + \frac{t}{Q_k^2}\right) e_z\left(x_k + \frac{x}{Q_k}\right) \\ &\to u = u^r \nu + u^\theta \nu^\perp + u^z e_z \quad \text{in } L^\infty(\Omega) \end{aligned}$$

for any compact subset Ω of $\mathbb{R}^3 \times \mathbb{R}$ and $u(x, t) \cdot \nu^\perp = 0$. Hence,

$$u(x, t) = u^r(x, t)\nu + u^z(x, t)e_z. \tag{4.4}$$

On the other hand, for $(y, s) \in B(x_k, \frac{r_k}{\sqrt{r_k Q_k}}) \times \left[t_k - (\frac{r_k}{\sqrt{r_k Q_k}})^2, t_k\right]$, one has

$$\begin{aligned} &-\frac{1}{Q_k}[v^r(y, s)e_\theta(y) - v^\theta(y, s)e_r(y)] \\ &= \frac{1}{Q_k}\partial_\theta[v^r(y, s)e_r(y) + v^\theta(y, s)e_\theta(y) + v^z(y, s)e_z(y)] \\ &= \partial_\theta\left[v^{(k)}\left(Q_k(y - x_k), Q_k^2(s - t_k)\right)\right] \\ &= Q_k(\partial_\theta y \cdot \nabla)v^{(k)}\left(Q_k(y - x_k), Q_k^2(s - t_k)\right) \\ &= Q_k|y|\left(e_\theta(y) \cdot \nabla\right)v^{(k)}\left(Q_k(y - x_k), Q_k^2(s - t_k)\right), \end{aligned}$$

which gives that

$$\frac{1}{Q_k}\left[-v^r\left(x_k + \frac{x}{Q_k}, t_k + \frac{t}{Q_k^2}\right)e_\theta\left(x_k + \frac{x}{Q_k}\right)\right. \tag{4.5}$$

$$\left. +v^\theta\left(x_k + \frac{x}{Q_k}, t_k + \frac{t}{Q_k^2}\right)e_r\left(x_k + \frac{x}{Q_k}\right)\right]$$

$$= Q_k\left|x_k + \frac{x}{Q_k}\right|\left(e_\theta\left(x_k + \frac{x}{Q_k}\right) \cdot \nabla\right)v^{(k)}(x, t)$$

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for $(x, t) \in B(0, \sqrt{Q_k r_k}) \times (-Q_k r_k, 0]$. Since $r_k Q_k \to \infty$, we know $Q_k |x_k + \frac{x}{Q_k}| \to \infty$ for fixed x. But the left hand side of (4.5) is bounded by definition of Q_k. Hence, let $k \to \infty$, we have

$$(\nu^\perp \cdot \nabla) u(x, t) = 0. \tag{4.6}$$

Note that the Navier-Stokes equations are invariant under rotation. Without loss of generality, we set $\nu = e_1$ and $\nu^\perp = e_2$. Consequently, the limit function

$$u(x, t) = u^r(x_1, z, t) e_1 + u^z(x_1, z, t) e_z,$$

is a bounded ancient solution to the 2D Navier-Stokes equations. By Theorem 5.1 in [15], the limit

$$u(x, t) = u^r(t)\nu + u^z(t) e_z \tag{4.7}$$

depends only on t and that $|u(0, 0)| = 1$. By the argument in the proof of the previous theorem, the boundedness of the stream function of u in BMO norm implies that $u = 0$. This contradiction shows that Case 2 can not occur either. Therefore the assumption that v becomes singular at $(0, 0)$ is false, proving Theorem 1.4. $\square$

Acknowledgement

References

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