

# ON THE MANIFOLD STRUCTURE OF THE SET OF UNPARAMETERIZED EMBEDDINGS WITH LOW REGULARITY

LUIS J. ALÍAS AND PAOLO PICCIONE

ABSTRACT. Given manifolds  $M$  and  $N$ , with  $M$  compact, we study the geometrical structure of the space of embeddings of  $M$  into  $N$ , having less regularity than  $C^\infty$ , quotiented by the group of diffeomorphisms of  $M$ .

## 1. INTRODUCTION

A very general class of geometrical variational problems can be formulated in terms of some action functional defined on the space  $\text{Emb}(M, N)$  of embeddings of a manifold  $M$  into some other manifold  $N$ . In many interesting examples, like for instance in the study of *minimal* or *constant mean curvature* embeddings  $x : M \rightarrow N$ , the functionals involved do not depend on the parameterization  $x$ , i.e., they are invariant by  $\text{Diff}(M)$  the diffeomorphism group of  $M$  that acts by right composition on the space of embeddings. Under these circumstances, given a solution  $x : M \rightarrow N$  of the variational problem, any embedding of the form  $x \circ \phi$ , with  $\phi \in \text{Diff}(M)$ , is also a solution of the problem, which is not geometrically distinct from  $x$ . This implies in particular, that typical compactness assumptions, like the Palais–Smale condition, obviously fail for parameterization invariant functionals. Namely, every critical level of a parameterization invariant functional is non compact. If one is interested in multiplicity results, like for instance Morse Theory or Bifurcation Theory, one has to identify solutions that are not geometrically different. There are several methods in the literature to get rid of the gauge invariance property in equivariant variational problems. One method is to impose a *gauge fixing condition*, in the language of [11], i.e., a smooth submanifold of the domain of the functional, which intersects all the orbits of the group action, and on which the variational problem has no invariance properties. A second method consists in determining an auxiliary functional, with *the same* critical points and which is no longer gauge invariant. This is illustrated well in the classical closed geodesic problem, originally formulated using the length functional in the space of immersions of the circle in a Riemannian manifold  $N$ . In this case, one replaces the length functional by a quadratic energy functional, which is no longer parameterization invariant, and has the same critical points. Nonetheless, the same technique may not be available for variational problems in higher dimension, and in this case the appropriate functional space to consider for the variational problem is the set of *unparameterized embeddings* of  $M$  into  $N$ . Two embeddings  $x_1, x_2 : M \rightarrow N$  are said to be equivalent if there exists a diffeomorphism  $\phi$  of  $M$  such that  $x_2 = x_1 \circ \phi$ ; an unparameterized embedding of  $M$  into  $N$  is an equivalence class of embedding of  $M$  into  $N$ . Actions of the diffeomorphism group of a manifold have been studied in several contexts, and one of the central question

---

*Date:* July 25th, 2010.

*2000 Mathematics Subject Classification.* 58J55, 35B32, 53C42.

This work was partially supported by MEC project MTM2009-10418, and Fundación Séneca project 04540/GERM/06, Spain. This research is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Regional Agency for Science and Technology (Regional Plan for Science and Technology 2007-2010). The research leading to this manuscript was carried out while the second author was visiting the *Departamento de Matemáticas* of the *Universidad de Murcia*, Spain, as a Visiting Professor. P.P. would like to thank faculties and staff of this institution for the warm hospitality and for the excellent working conditions provided to him.

is how to construct *slices* for these actions. The interested reader may look up [1] for the action on Riemannian metrics by pull-back, or [4] for gauge theory.

A crucial point is the choice of regularity for the embeddings. Namely, important properties of the variational problem, like for instance the Palais–Smale condition, or the Fredholmness condition for the second derivative, depend essentially on this choice. The  $C^\infty$  case has been extensively studied (see [7, 8, 9]), and a nice Frechet differentiable structure has been described for this set. The theory of manifolds modeled on general locally convex topological vector spaces has been recently developed in detail in [6]. Nevertheless, in view to applications in variational calculus, the Frechet structure of  $C^\infty$  embeddings is too weak, and it is desirable to have a geometry modeled on Banach or Hilbert spaces. Usually, a natural choice would be to consider embeddings of class  $C^k$ , or  $C^{k,\alpha}$ , with  $k < \infty$  and  $\alpha \in ]0, 1[$ , or some Sobolev regularity. However, when a regularity weaker than  $C^\infty$  is assumed for the embeddings, subtle obstructions arise when attempting to define a global differentiable structure on the quotient space of embeddings modulo diffeomorphisms. The problem is a consequence of the fact that, when  $k < \infty$ , the map of left-composition with a fixed diffeomorphism of class  $C^k$  is *not* a differentiable map in the space of  $C^k$ -maps. The transition maps of any natural atlas of charts for the space of unparameterized embeddings involves this type of operations.

The point we address in this paper is precisely an analysis of the local and global geometrical structure of the set of unparameterized embeddings having regularity weaker than  $C^\infty$ . We will show that, unlike the smooth case, such a set does not have a natural global differentiable structure, nonetheless local and global techniques from Calculus of Variations can be applied for parameterization invariant functionals. More precisely, we use Palais’ notion of Vector Bundle Neighborhood (VBN) for describing an atlas of charts for the set of unparameterized embeddings, whose transition functions are continuous. Using these charts, the set of unparameterized embeddings is “locally” a smooth submanifold of the space of embeddings. The restriction of any parameterization invariant smooth function on the space of embeddings defines a function on the space of unparameterized embeddings which is smooth in any local chart. Thus, one has a well defined notion of critical point, and we compute the first and the second variation at a critical point of a parameterization invariant smooth functional. In the last section we also analyze regularity properties of the action of the isometry group of the target manifold  $N$  on the space of unparameterized embeddings by left-composition. This action is also not smooth, but in local charts its orbits are smooth embedded submanifolds.

## 2. NOTATIONS AND PRELIMINARIES

Let us consider two smooth (i.e.,  $C^\infty$ ) manifolds  $M^m$  and  $N^n$ , with  $m < n$ . For simplicity, we will assume that  $M$  is compact, although an analogous theory can be developed also in the non compact case, along the lines of [12]. We will fix throughout an auxiliary Riemannian metric  $g$  on the target manifold  $N$ , and we will denote by  $\exp$  the corresponding exponential map. The metric  $g$  induces a norm on every vector bundle obtained by functorial construction from  $TN$  (like pull-backs, normal bundles of embeddings into  $N$ , etc.). The metric  $g$  will be used only for a more explicit description of the manifold charts; all the results of the present paper will not depend on the choice of such metric.

We will denote by  $\mathfrak{C}$  a regularity class of maps defined on  $M$ . More precisely, let  $\mathfrak{C}(M, \mathbb{R})$  be a Banach space of maps from  $M$  to  $\mathbb{R}$  such that

$$C^\infty(M, \mathbb{R}) \subset \mathfrak{C}(M, \mathbb{R}) \subset C^1(M, \mathbb{R}),$$

with *dense* inclusion  $C^\infty(M, \mathbb{R}) \hookrightarrow \mathfrak{C}(M, \mathbb{R})$  and *continuous* inclusion  $\mathfrak{C}(M, \mathbb{R}) \hookrightarrow C^1(M, \mathbb{R})$ . We require that  $\mathfrak{C}(M, \mathbb{R})$  be stable under composition from the right with functions  $f \in C^\infty(M, M)$  (this action is linear), and stable under composition from the

left with functions  $f \in C^\infty(\mathbb{R}, \mathbb{R})$ . We also assume that for all  $f \in C^\infty(\mathbb{R}, \mathbb{R})$ , the map  $\mathfrak{C}(M, \mathbb{R}) \ni g \mapsto g \circ f \in \mathfrak{C}(M, \mathbb{R})$  is smooth.

Typical examples of  $\mathfrak{C}$  are:

- $\mathfrak{C} = C^k$ , with  $k \geq 1$ ;
- $\mathfrak{C} = C^{k,\alpha}$ , with  $k \geq 1$  and  $\alpha \in ]0, 1[$  (Hölder type regularity);
- $\mathfrak{C} = W^{k,p}$ , with  $p(k-1) > m$  (Sobolev type regularity).

In several interesting examples, also non standard choices for the functor  $\mathfrak{C}$  may appear naturally, see Remark 2.1. Thus, treating the subject in such generality is not a useless abstraction.

A description of the differentiable structure of the set of maps  $f : M \rightarrow N$  of class  $\mathfrak{C}$  can be given as follows. Set  $\mathfrak{C}(M, \mathbb{R}^d) = \bigoplus_{i=1}^d \mathfrak{C}(M, \mathbb{R})$  and, given a subset  $S \subset \mathbb{R}^d$ , denote by  $\mathfrak{C}(M, S)$  the set of maps  $f \in \mathfrak{C}(M, \mathbb{R}^d)$  such that  $f(M) \subset S$ . Such set is endowed with the induced topology from  $\mathfrak{C}(M, \mathbb{R}^d)$ . If  $S$  is a submanifold of  $\mathbb{R}^d$ , then  $\mathfrak{C}(M, S)$  is a submanifold of  $\mathfrak{C}(M, \mathbb{R}^d)$ . Given a smooth embedding  $\phi : N \rightarrow \mathbb{R}^d$ , denote by  $\mathfrak{C}(M, N, \phi)$  the set of all maps  $f : M \rightarrow N$  such that  $\phi \circ f \in \mathfrak{C}(M, \phi(N))$ . The map  $\mathfrak{C}(M, N, \phi) \ni f \mapsto \phi \circ f \in \mathfrak{C}(M, \phi(N))$  is a bijection, and it induces a Banach manifold structure on  $\mathfrak{C}(M, N, \phi)$ . This differentiable structure is independent on  $\phi$ , i.e., given different embeddings  $\phi_i : N \rightarrow \mathbb{R}^d$ ,  $i = 1, 2$ , then  $\mathfrak{C}(M, N, \phi_1) = \mathfrak{C}(M, N, \phi_2)$ , and the differentiable structures induced by  $\mathfrak{C}(M, \phi_1(N))$  and  $\mathfrak{C}(M, \phi_2(N))$  coincide. We will therefore omit the symbol  $\phi$  in the notation of the set of maps  $f : M \rightarrow N$  of class  $\mathfrak{C}$ , and we will write  $\mathfrak{C}(M, N)$ . Given a smooth vector bundle  $\pi : E \rightarrow M$ , one also has a notion of *sections of class  $\mathfrak{C}$*  of  $\pi$ , defined in the obvious way.

*Remark 2.1.* When the Banach space  $\mathfrak{C}(M, \mathbb{R})$  is not separable, as in the case  $\mathfrak{C} = C^{k,\alpha}$ , with  $\alpha \in ]0, 1[$ , then  $\mathfrak{C}(M, N)$  is a non separable Banach manifolds. There are theories where separability is an important issue, especially when Sard's theorem needs to be invoked. A situation of this type is considered in [14], where the author proves a genericity result in the space of  $C^{k,\alpha}$  embeddings. As suggested in [13, § 1.5], a possible way of circumventing the problem is to consider rather than the space  $C^{k,\alpha}$ , the closed subspace  $C^{k,\alpha+}$  consisting of all  $C^{k,\alpha}$ -limits of functions of class  $C^{k+1}$ . This space is separable with respect to the  $C^{k,\alpha}$ -topology, and in fact it is second countable.

By the assumption that the inclusion  $\mathfrak{C}(M, \mathbb{R}) \hookrightarrow C^1(M, \mathbb{R})$  is continuous, it follows that the (possibly empty) subset of  $\mathfrak{C}(M, N)$  consisting of embeddings is open. In next Section we will describe an explicit set of local charts for such set, given intrinsically, i.e., without using embeddings of  $N$  into some Euclidean space.

### 3. THE MANIFOLD OF EMBEDDINGS

Classical references where the differentiable structure of  $\mathfrak{C}(M, N)$ , or more generally of spaces of  $\mathfrak{C}$ -sections of fiber bundles<sup>1</sup> with compact base, has been described explicitly are [2, 3, 10]; local charts of this differentiable structure are described by Palais using the notion of *vector bundle neighborhood* (VBN). When the base is non compact, restrictions on the space of sections are required in order to have a well defined Banach differentiable structure, see [12]. In order to get a better insight on our problem, let us recall how a global differentiable structure on  $\mathfrak{C}(M, N)$  is obtained, following the VBN approach of [10]. Given a Riemannian vector bundle  $E$  over  $M$  (i.e., a vector bundle endowed with a Riemannian structure on the fibers and a compatible connection), we will denote by  $\Gamma(E)$  the Banach space of all sections of class  $\mathfrak{C}$  of  $E$ . The essential property required for developing Palais' theory is the fact, proved in [10], that, given a compact manifold  $M$ , two Riemannian vector bundles  $E_1, E_2$  over  $M$ , and a smooth vector bundle morphism  $\Phi : E_1 \rightarrow E_2$ , the composition operator  $\Gamma(E_1) \ni s \mapsto \Phi \circ s \in \Gamma(E_2)$  is a smooth map.

<sup>1</sup>functions from  $M$  to  $N$  can be thought of as sections of the trivial fiber bundle  $M \times N$ .

The idea of vector bundle neighborhoods is that suitable small  $\mathfrak{C}$ -neighborhoods of a given map  $x : M \rightarrow N$  of class  $\mathcal{C}^\infty$  are parameterized by elements in neighborhoods of the zero section of the pull-back bundle  $x^*(TN)$  over  $M$ . More precisely, once a Riemannian metric  $g$  with Levi–Civita connection  $\nabla$  in  $N$  is fixed, a local chart  $\Phi$  of  $\mathfrak{C}(M, N)$  around a given smooth function  $x$  is obtained by associating to each section  $u$  of class  $\mathfrak{C}$  of the vector bundle  $x^*(TN)$  the map  $y : M \rightarrow N$  defined by  $y(p) = \exp_{x(p)}(u(p))$ , where  $\exp$  is the exponential map of  $\nabla$ . The inverse of the map that associates to each  $u$  the corresponding  $y$  defines a local chart from an open neighborhood of the zero section of  $x^*(TN)$  to an open neighborhood of  $x$ , that will be denoted by  $\Phi_x$ . The transition maps for charts in this atlas are computed as follows. Given smooth maps  $x_1, x_2 : M \rightarrow N$ , for  $i = 1, 2$  consider the map  $\text{EXP}_i : x_i^*(TN) \rightarrow M \times N$  defined by  $\text{EXP}_i(p, v) = (p, \exp_{x_i(p)}(v))$ ,  $v \in T_{x_i(p)}N$ . This gives a smooth diffeomorphism of an open subset containing the zero section of  $x_i^*(TN)$  onto an open neighborhood of the graph of  $x_i$ ; the composition  $\zeta = \text{EXP}_2^{-1} \circ \text{EXP}_1$  is a smooth diffeomorphism between two open neighborhoods of the zero sections of the vector bundles  $x_1^*(TN)$  and  $x_2^*(TN)$  that preserves the fibers. The transition map  $\Phi_{x_1}^{-1} \circ \Phi_{x_2}$  is given by left-composition with the smooth map  $\zeta$ , and thus it is differentiable (compare with the situation described in Remark 4.2). Moreover, when  $x$  varies in the set of smooth functions, the domain of these charts cover the entire  $\mathfrak{C}(M, N)$ , as we are assuming density of the inclusion  $\mathcal{C}^\infty(M, \mathbb{R}) \hookrightarrow \mathfrak{C}(M, \mathbb{R})$ . Hence, the collection of all such charts defines a differentiable atlas on  $\mathfrak{C}(M, N)$ . Given a smooth map  $x : M \rightarrow N$ , the tangent space  $T_x\mathfrak{C}(M, N)$  is identified, via the chart  $\Phi_x$ , with the space of all sections of class  $\mathfrak{C}$  of the pull-back bundle  $x^*(TN)$ .

The subset  $\text{Emb}(M, N)$  of  $\mathfrak{C}(M, N)$  consisting of all embeddings  $x : M \rightarrow N$  is open, and thus it inherits a natural Banach manifold structure from  $\mathfrak{C}(M, N)$ . One can consider the set  $\text{Diff}(M)$ , which is the set of all diffeomorphisms  $\phi : M \rightarrow M$  of class  $\mathfrak{C}$ ; observe that  $\text{Diff}(M)$  may fail to be closed under composition or inverse, so that in general it is not a group.  $\text{Diff}(M)$  is an open subset of  $\text{Emb}(M, M)$ , and thus it inherits a natural differentiable structure. However, even under the assumption that  $\text{Diff}(M)$  is closed under composition and inverse, *neither one of the two operations is differentiable*. Namely, the left-composition map  $\phi \mapsto x \circ \phi$  on  $\text{Diff}(M)$  in general is not of class  $\mathcal{C}^1$  (see [13, Appendix]). Similarly, the derivative of the map  $\phi \mapsto \phi^{-1}$  involves the derivative of  $\phi$ , and thus this is not differentiable at those points  $\phi$  whose derivative is not of class  $\mathfrak{C}$ .

#### 4. THE MANIFOLD OF UNPARAMETERIZED EMBEDDINGS

Two embeddings  $x, y : M \rightarrow N$  will be considered equivalent if there exists a  $\mathcal{C}^1$ -diffeomorphism  $\phi : M \rightarrow M$  such that  $y = x \circ \phi$ , i.e., if they are different parameterizations of the same submanifold of  $N$  diffeomorphic to  $M$ . If  $x$  and  $y$  are of class  $\mathcal{C}^k$ , then such diffeomorphism  $\phi$  will also be of class  $\mathcal{C}^k$ . For  $x \in \text{Emb}(M, N)$ , we will denote by  $[x]$  the class of all  $y \in \text{Emb}(M, N)$  that are equivalent to  $x$ .

**Definition 4.1.** The set of *unparameterized embeddings of class  $\mathfrak{C}$  of  $M$  into  $N$* , denoted by  $\widetilde{\text{Emb}}(M, N)$ , is the set:

$$\widetilde{\text{Emb}}(M, N) = \{[x] : x \in \text{Emb}(M, N)\}.$$

Thus,  $\widetilde{\text{Emb}}(M, N)$  can be thought as the set of all embedded submanifolds of class  $\mathfrak{C}$  of  $N$  that are  $\mathfrak{C}$ -diffeomorphic to  $M$ . We will now establish an infinite dimensional Banach *topological* structure on  $\widetilde{\text{Emb}}(M, N)$ , and we will describe suitable local charts of this structure.

Let  $x : M \rightarrow N$  be a smooth embedding; a local chart  $\tilde{\Phi} : \tilde{\mathcal{U}}_x \rightarrow \tilde{\mathcal{W}}_x$  in  $\widetilde{\text{Emb}}(M, N)$ , where  $\tilde{\mathcal{U}}_x$  is an appropriate neighborhood of  $[x]$  in  $\widetilde{\text{Emb}}(M, N)$ ,  $\mathcal{W}_x$  is an appropriate  $\mathfrak{C}$ -neighborhood of the zero section of the normal bundle of  $x$ , is given as follows. There exists an open subset  $U$  of the normal bundle  $x^\perp$  containing the zero section of this bundle,

and an open subset  $V$  of  $N$  containing the image  $x(M)$  such that the restriction of  $\exp$  to  $U$  gives a diffeomorphism from  $U$  to  $V$ . The space  $\Gamma(x^\perp)$  of all sections of class  $\mathfrak{C}$  of the normal bundle  $x^\perp$  is a Banach space, and the subset  $\Gamma(x^\perp; U)$  of  $\Gamma(x^\perp)$  consisting of all sections whose image is contained in  $U$  is open. A map  $\tilde{\Psi}_x : \Gamma(x^\perp; U) \rightarrow \widetilde{\text{Emb}}(M, N)$  is obtained by setting  $\tilde{\Psi}_x(u) = [y]$ , where  $y(p) = \exp_{x(p)}(u(p))$  for all  $p \in M$ . Clearly,  $y$  is an embedding of class  $\mathfrak{C}$  of  $M$  into  $N$ , since  $u$  is an embedding of class  $\mathfrak{C}$  of  $M$  into the normal bundle  $x^\perp$ , and  $\exp$  is a diffeomorphism from  $U$  to  $V$ . It is easy to see that  $\tilde{\Psi}_x$  is injective. In order to prove this, first observe that two embeddings  $x_1, x_2 \in \text{Emb}(M, N)$  are equivalent if and only if  $x_1(M) = x_2(M)$ . Now, observe that two distinct sections  $u_1, u_2 \in \Gamma(x^\perp; U)$  must have distinct images in  $U$ , and thus their composition with  $\exp$  are also different in  $V$ . This proves that  $\tilde{\Psi}_x$  is injective. The image of  $\tilde{\Psi}_x$  is the projection onto  $\widetilde{\text{Emb}}(M, N)$  of an open neighborhood of  $x$  in  $\text{Emb}(M, N)$ . If  $y \in \text{Emb}(M, N)$  is near  $x$ , in particular it has image contained in  $U$ , then  $\exp^{-1}(y(M))$  is the image of a section  $u$  of  $x^\perp$  of class  $\mathfrak{C}$ ; then,  $\tilde{\Psi}_x(u) = [y]$ . Thus, the map  $\tilde{\Psi}$  is a bijection from an open subset  $\mathcal{W}_x$  of  $\Gamma(x^\perp)$  containing the zero section, to a subset  $\mathcal{U}_x$  of  $\widetilde{\text{Emb}}(M, N)$  given by the projection onto  $\widetilde{\text{Emb}}(M, N)$  of an open neighborhood of  $x$  in  $\text{Emb}(M, N)$ . Its inverse will be denoted by  $\tilde{\Phi}_x$ , and the collection of such maps, as  $x$  varies in the set of all smooth embeddings of  $M$  into  $N$  is taken as an atlas of charts for  $\widetilde{\text{Emb}}(M, N)$ .

We note however that *there is no differentiable compatibility between two charts in this atlas*, i.e., the transition maps are in general *not* differentiable, but only continuous. Let us compute a transition map. Denote by  $x_1, x_2 : M \rightarrow N$  two smooth embeddings such that the classes  $[x_1]$  and  $[x_2]$  belong to the intersection of the domains  $\tilde{\mathcal{U}}_{x_1} \cap \tilde{\mathcal{U}}_{x_2}$  of the charts  $\tilde{\Phi}_{x_1}$  and  $\tilde{\Phi}_{x_2}$ . Denote by  $\exp_1, \exp_2$  the exponential map of  $g$  restricted to the normal bundles  $x_1^\perp$  and  $x_2^\perp$  respectively, that are diffeomorphisms between open subsets containing the zero section and tubular neighborhoods of the images  $x_1(M)$  and  $x_2(M)$  respectively. Thus, there are open subsets  $U_i \subset x_i^\perp$  containing the zero section such that the map  $\zeta : U_1 \rightarrow U_2$  given by  $\zeta = \exp_2^{-1} \circ \exp_1$  is a smooth diffeomorphism. Let  $u \in \tilde{\mathcal{W}}_{x_1} \cap \tilde{\mathcal{W}}_{x_2}$  be fixed and set  $u' = \tilde{\Phi}_{x_2}(\tilde{\Phi}_{x_1}^{-1}(u))$ .

*Remark 4.2.* The key observation here is that, in spite of the fact that the section  $u'$  of the normal bundle  $x_2^\perp$  has the same image of the map  $\zeta \circ u$ , the latter is *not* a section of  $x_2^\perp$ . This depends on the fact that the diffeomorphism  $\zeta$  is not a vector bundle morphism as in the case of the charts of  $\text{Emb}(M, N)$  (Section 3), i.e., it does not take fibres of  $x_1^\perp$  into fibres of  $x_2^\perp$ . In order to obtain the section  $u'$ , an *adjustment* needs to be done in the domain of  $\zeta \circ u$ , which is obtained by composition on the right with a diffeomorphism of the base  $M$  that depends on  $u$ ; it is precisely such adjustment that causes the loss of differentiability of the transition maps.

The following formula holds:

$$u' = \zeta \circ u \circ h_u^{-1},$$

where  $h_u : M \rightarrow M$  is the diffeomorphism:

$$h_u = \pi_2 \circ \zeta \circ u,$$

$\pi_2 : E_2 \rightarrow M$  being the projection of the vector bundle  $E_2$  over the base manifold  $M$ . Now, the maps  $u \mapsto \zeta \circ u$  and  $u \mapsto h_u$  are  $\mathcal{C}^\infty$ , but the function  $h \mapsto h^{-1}$  is not differentiable in  $\text{Diff}(M)$  where  $h$  is only of class  $\mathfrak{C}$ , as well as the function of composing on the left with  $\zeta \circ u$ , when  $u$  is only of class  $\mathfrak{C}$ . Thus, the map  $u \mapsto u'$  is continuous, but not differentiable.

We can then define a unique topology on  $\widetilde{\text{Emb}}(M, N)$  whose basis is the collection of the domains  $\tilde{\mathcal{U}}_x$  of the charts  $\tilde{\Phi}_x$ , as  $x$  varies in the set of smooth embeddings of  $M$  into

$N$ , and by requiring that each  $\widetilde{\Phi}_x$  is a homeomorphism onto its image. It is easy to see<sup>2</sup> that this topology is exactly the quotient topology induced by the canonical quotient map  $\widetilde{\pi} : \text{Emb}(M, N) \rightarrow \widetilde{\text{Emb}}(M, N)$ .

The reader should observe that the charts  $\Phi_x$  in  $\text{Emb}(M, N)$  and  $\widetilde{\Phi}_x$  in  $\widetilde{\text{Emb}}(M, N)$  look very much alike. The only difference is that  $\widetilde{\Phi}_x$  takes values in the space of sections of the normal bundle  $x^\perp$ , while  $\Phi_x$  takes values in the spaces of sections of  $x^*(TM)$ . If we identify<sup>3</sup>  $x^\perp$  with a subbundle of  $x^*(TN)$ , then this suggests that, roughly speaking, “locally  $\widetilde{\text{Emb}}(M, N)$  is a smooth submanifold of  $\text{Emb}(M, N)$ ”. Let us state this in a more precise way:

**Proposition 4.3.** *For  $x$  varying in the set of smooth embeddings of  $M$  into  $N$ , the family  $\left\{ (\widetilde{\mathcal{U}}_x, \widetilde{\Phi}_x) \right\}_x$  is an atlas of charts of  $\widetilde{\text{Emb}}(M, N)$ , whose domains form an open cover of  $\widetilde{\text{Emb}}(M, N)$ , and that makes  $\widetilde{\text{Emb}}(M, N)$  into an infinite dimensional topological manifold modeled on the Banach space  $\mathfrak{C}(M, \mathbb{R}^{n-m})$ .*

*The canonical projection  $\widetilde{\pi} : \text{Emb}(M, N) \rightarrow \widetilde{\text{Emb}}(M, N)$  is a quotient map.*

*For a given smooth embedding  $x : M \rightarrow N$ , by identifying the normal bundle  $x^\perp$  with a subbundle of the pull-back  $x^*(TN)$ , then the local chart  $\Phi_x$  of  $\text{Emb}(M, N)$  around  $x$  and the local chart  $\widetilde{\text{Emb}}(M, N)$  around  $[x]$  allow an identification of the neighborhood  $\widetilde{\mathcal{U}}(x)$  of  $[x]$  with the smooth submanifold of  $\text{Emb}(M, N)$  consisting of those  $\mathfrak{C}$ -embeddings in the domain of the chart  $\Phi_x$  for which  $\Phi_x$  takes values in the space of sections of the normal bundle  $x^\perp$ .  $\square$*

The local identification of  $\widetilde{\text{Emb}}(M, N)$  with submanifolds of  $\text{Emb}(M, N)$  is particularly useful for studying smooth maps.

**Corollary 4.4.** *Let  $\mathfrak{Z}$  be an arbitrary manifold and  $f : \text{Emb}(M, N) \rightarrow \mathfrak{Z}$  be a smooth function such that  $f(x) = f(y)$  for all pairs of equivalent embeddings  $x, y \in \text{Emb}(M, N)$ . Then, given any smooth embedding  $x : M \hookrightarrow N$ , considering the local chart  $(\widetilde{\mathcal{U}}(x), \widetilde{\Phi}_x)$  of  $\widetilde{\text{Emb}}(M, N)$ , the composition  $\widetilde{f}_x = f \circ \widetilde{\Phi}_x^{-1} : \widetilde{\Phi}_x(\widetilde{\mathcal{U}}_x) \rightarrow \mathfrak{Z}$  is smooth. If  $\mathfrak{Z} = \mathbb{R}$ , then  $u = \widetilde{\Phi}_x([y])$  is a critical point of  $\widetilde{f}_x$  if and only if  $y$  is a critical point of  $f$ .*

*Proof.* The map  $\widetilde{f}_x$  is the restriction to the subspace of  $\mathfrak{C}$ -sections of the normal bundle  $x^\perp$  of the smooth function  $f_x = f \circ \Phi_x^{-1}$ , thus  $\widetilde{f}_x$  is smooth. For  $u \in \widetilde{\Phi}_x(\widetilde{\mathcal{U}}_x)$ , the tangent space at  $u$  of the space of  $\mathfrak{C}$ -sections of the bundle  $x^*(TN)$  is identified with the space of sections of some vector subbundle  $E$  of  $x^*(TN)$  complementary to  $dx(TM)$  (if  $u \neq 0$ , then  $E$  will not necessarily be the normal bundle  $x^\perp$ ). The invariance property of  $f$  says that  $df_x$  vanishes on sections of the bundle  $dx(TM)$ , from which it follows easily that  $u = \widetilde{\Phi}_x([y])$  is a critical point of  $\widetilde{f}_x$  if and only if  $y$  is a critical point of  $f$ .  $\square$

*Remark 4.5.* Note that the result of Corollary 4.4 says in particular that, for a smooth function  $f$  on  $\text{Emb}(M, N)$  which is invariant by diffeomorphisms of  $M$ , one has a well defined notion of “critical point of  $f$  in  $\widetilde{\text{Emb}}(M, N)$ ”. We will say that  $[y]$  is a critical point of  $f$  in  $\widetilde{\text{Emb}}(M, N)$  if given  $x : M \rightarrow N$  smooth embedding such that  $[y]$  belongs to the domain  $\widetilde{\mathcal{U}}_x$  of the chart  $\widetilde{\Phi}_x$ , then  $\widetilde{\Phi}_x([y])$  is a critical point of the smooth function  $f \circ \widetilde{\Phi}_x^{-1}$ . Corollary 4.4 says that this notion does not depend on the choice of the chart around  $[y]$ ; of course, this conclusion could not be drawn using a change of charts argument.

<sup>2</sup>Consider the restriction of  $\widetilde{\pi}$  to the inverse image  $\widetilde{\pi}^{-1}(\widetilde{\mathcal{U}}_x)$  of the domain of some chart. Then, such restriction is continuous, open (because it admits continuous local sections with arbitrarily prescribed values at a given point), and surjective, hence it is a quotient map.

<sup>3</sup>We will identify the pull-back bundle  $x^*(TN)$  with the Whitney sum  $dx(TM) \oplus x^\perp$ .

When  $[x]$  is the class of a smooth embedding  $x : M \rightarrow N$ , then for all questions of differentiability at  $[x]$  it will be convenient to use the chart  $\tilde{\Phi}_x$ , centered at the point  $x$ . The tangent space at  $[x]$  is described in next:

**Lemma 4.6.** *Let  $x : M \rightarrow N$  be a smooth embedding. The tangent space at the point  $[x]$  to  $\widetilde{\text{Emb}}(M, N)$  is identified, via the chart  $\tilde{\Phi}_x$  with the Banach space  $\Gamma(x^\perp)$  of all  $\mathcal{C}$ -sections of the normal bundle  $x^\perp$ . If  $r \mapsto x_r \in \text{Emb}(M, N)$  is a  $\mathcal{C}^1$ -curve with  $x_{r_0} = x$  and  $\frac{d}{dr}|_{r=r_0} x_r = V \in \Gamma(x^*(TN))$  then  $r \mapsto \gamma_r = \tilde{\Phi}_x([x_r])$  is of class  $\mathcal{C}^1$ , and  $\frac{d}{dr}|_{r=r_0} \gamma_r = V^\perp \in \Gamma(x^\perp)$ , where  $V^\perp(p)$  is the orthogonal projection of  $V(p)$  onto the orthogonal space  $x^\perp(p)$ ,  $p \in M$ .*

*Proof.* The domain of  $\tilde{\Phi}_x$  is mapped by  $\tilde{\Phi}_x$  to an open neighborhood of the zero section of  $\Gamma(x^\perp)$ . The tangent space is therefore identified with the Banach space itself. Since  $r \mapsto x_r$  is  $\mathcal{C}^1$ , then  $r \mapsto \eta_r = \Phi_x(x_r)$  is  $\mathcal{C}^1$ ; now,  $\gamma_r = P^\perp(\eta_r)$ , where  $P^\perp : \Gamma(x^*(TN)) \rightarrow \Gamma(x^\perp)$  is the bounded linear map defined by  $P^\perp(W) = W^\perp$ . Thus,  $\gamma_r$  is of class  $\mathcal{C}^1$ , and its derivative at  $r_0$  is given by  $P(V) = V^\perp$ .  $\square$

**Proposition 4.7.** *Let  $f : \text{Emb}(M, N) \rightarrow \mathbb{R}$  be a smooth function invariant by diffeomorphisms of  $M$ , and assume that  $x : M \rightarrow N$  is a smooth embedding such that  $[x]$  is a critical point of  $f$  in  $\widetilde{\text{Emb}}(M, N)$  (in the sense of Remark 4.5). Then, the second variation  $d^2(f \circ \tilde{\Phi}_x^{-1})(0)$  coincides with the restriction of the second variation  $d^2(f \circ \Phi_x^{-1})(0)$  to the space of  $\mathcal{C}$ -sections of the normal bundle  $x^\perp$ .*

*Proof.* It follows immediately from the fact that, using the local charts  $\Phi_x$  and  $\tilde{\Phi}_x$  centered at  $x$ , then  $\widetilde{\text{Emb}}(M, N)$  ( $\cong$  sections of the normal bundle  $x^\perp$ ) is identified with a linear subspace of  $\text{Emb}(M, N)$  ( $\cong$  sections of the pull-back bundle  $x^*(TN)$ ).  $\square$

*Remark 4.8.* One may wonder whether the set  $\widetilde{\text{Emb}}(M, N)$  admits some other natural atlas of charts that are pairwise differentiably compatible, and that make it into a true Banach differentiable manifold. The existence of such a differentiable structure *fails* if one requires the natural property that the quotient map  $\tilde{\pi} : \text{Emb}(M, N) \rightarrow \widetilde{\text{Emb}}(M, N)$  be a smooth submersion. Namely, if such a differentiable structure existed, then the inverse image by this projection of points of  $\widetilde{\text{Emb}}(M, N)$ , i.e., equivalence classes of embeddings, would be embedded smooth submanifolds of  $\text{Emb}(M, N)$ . But as we have observed, equivalence classes of embeddings  $x$  that are only of class  $\mathcal{C}$  are *not* submanifolds, as they have the same regularity of the left-composition function  $\phi \mapsto x \circ \phi$ .

## 5. ACTION OF THE ISOMETRY GROUP

We will now study regularity questions concerning the action of the (connected component of the identity of the) isometry group  $G = \text{Iso}(N, g)$  of the Riemannian manifold  $(N, g)$  on the manifold  $\text{Emb}(M, N)$  given by composition on the left, and the corresponding action on  $\widetilde{\text{Emb}}(M, N)$ . It is well known (see for instance [5]) that  $G$  is a Lie group; if  $N$  is compact, then also  $G$  is compact.

**Proposition 5.1.** *The following regularity properties hold for the action of  $\text{Iso}(N, g)$ .*

- (1) *The action of  $\text{Iso}(N, g)$  on  $\text{Emb}(M, N)$  is by smooth diffeomorphisms.*
- (2) *The corresponding action on  $\widetilde{\text{Emb}}(M, N)$  is by homeomorphisms.*
- (3) *If  $x : M \rightarrow N$  is a smooth embedding, then the map*

$$\beta_x : \text{Iso}(N, g) \longrightarrow \widetilde{\text{Emb}}(M, N)$$

*defined by  $\beta_x(\psi) = \psi \cdot [x]$  is a smooth injective immersion on a neighborhood of the identity (when represented in any of the local charts described in Subsection 4).*

- (4) *The local charts of  $\widetilde{\text{Emb}}(M, N)$  described in Subsection 4, restricted to the orbit  $\text{Iso}(N, g) \cdot [x]$  of a smooth embedding  $x : M \rightarrow N$  are differentiably compatible, and they define a differentiable structure on the orbit of  $[x]$  in  $\widetilde{\text{Emb}}(M, N)$ . The action of  $\text{Iso}(N, g)$  on this orbit is smooth, and this orbit is diffeomorphic to the quotient  $\text{Iso}(N, g)/H_x$ , where  $H_x$  is the isotropy group of  $[x]$ .*

*Proof.* Isometries of  $(N, g)$  are smooth. Part (1) follows from the fact that left-composition with smooth maps is smooth on  $\text{Emb}(M, N)$ ; the inverse of left-composition by  $\psi$  is left-composition by  $\psi^{-1}$ . As to the map  $\widetilde{\text{Emb}}(M, N) \ni [y] \mapsto [\psi \circ y] \in \widetilde{\text{Emb}}(M, N)$ , this is continuous, but not smooth. Namely, given the  $\mathcal{C}$ -section  $u = \tilde{\Phi}_x(y)$  of  $x^\perp$ , then the map  $\exp^{-1} \circ \psi \circ \exp(u)$  is a map of class  $\mathcal{C}$  between open subsets of  $x^\perp$ , but it is not a section. Thus, when representing the composition  $[\psi \circ y]$  in local charts, a right composition with a diffeomorphism is needed, which as observed in Subsection 4 is not smooth, but only continuous. This proves part (2). For part (3), observe that the composition of  $\beta_x$  and the local charts  $\tilde{\Phi}_y$  applied to  $\psi$  involves only compositions of  $\psi$  with smooth diffeomorphism, and it is therefore a smooth injective immersion of (an open neighborhood of the identity in)  $\text{Iso}(N, g)$ . To prove part (4), observe that, by (3), the intersection of the orbit  $\text{Iso}(N, g) \cdot [x]$  with the domain of a chart  $\tilde{\Phi}_y$  is an immersed submanifold. Since the orbit of a smooth embedding consists only of classes of smooth embeddings, then the transition functions restrict to smooth maps at every point of the orbit. Smoothness of the action on this orbit also follows easily.  $\square$

It is an easy observation that, for all  $x \in \text{Emb}(M, N)$ , the stabilizer of  $[x]$  in  $\text{Iso}(N, g)$  is the subgroup consisting of all isometries  $\psi$  that preserve the image  $x(M)$ , i.e., such that  $\psi(x(M)) = x(M)$ .

## REFERENCES

- [1] D. G. EBIN, *The manifold of Riemannian metrics*, 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968) pp. 11–40 Amer. Math. Soc., Providence, R.I.
- [2] H. I. ELÍASSON, *Geometry of manifolds of maps*, J. Diff. Geom. **1** (1967), 169–194.
- [3] J. EELLS, *On the geometry of function spaces*, Symp. Inter. de Topologia Alg. Mexico, 1957, 303–308.
- [4] D. S. FREED, K. UHLENBECK, *Instantons and four-manifolds*, Second edition. Mathematical Sciences Research Institute Publications, 1. Springer-Verlag, New York, 1991.
- [5] S. KOBAYASHI, *Transformation groups in differential geometry*, Reprint of the 1972 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.
- [6] A. KRIEGL, P. MICHOR, *The convenient setting for Global Analysis*, M. S. M. vol. 53, Amer. Math. Soc., Providence, USA, 1997.
- [7] P. MICHOR, *Manifolds of smooth maps*, Cahiers Topologie Géom. Différentielle **19** (1978), no. 1, 47–78.
- [8] P. MICHOR, *Manifolds of smooth maps. II. The Lie group of diffeomorphisms of a noncompact smooth manifold*, Cahiers Topologie Géom. Différentielle **21** (1980), no. 1, 63–86.
- [9] P. MICHOR, *Manifolds of smooth maps. III. The principal bundle of embeddings of a noncompact smooth manifold*, Cahiers Topologie Géom. Différentielle **21** (1980), no. 3, 325–337.
- [10] R. PALAIS, *Foundations of Global Nonlinear Analysis*, W. A. Benjamin, 1968.
- [11] R. S. PALAIS, C.-L. TERNG, *Critical point theory and submanifold geometry*, Lecture Notes in Mathematics, 1353. Springer-Verlag, Berlin, 1988.
- [12] P. PICCIONE, D. V. TAUSK, *On the Banach differential structure for sets of maps on non-compact domains*, Nonlinear Anal. **46** (2001), no. 2, Ser. A: Theory Methods, 245–265.
- [13] B. WHITE, *The space of  $m$ -dimensional surfaces that are stationary for a parametric elliptic functional*, Indiana Univ. Math. J. **36** (1987), 567–602.
- [14] B. WHITE, *The space of minimal submanifolds for varying Riemannian metrics*, Indiana Univ. Math. J. **40** (1991), 161–200.



DEPARTAMENTO DE MATEMÁTICAS,  
UNIVERSIDAD DE MURCIA, CAMPUS DE ESPINARDO  
30100 ESPINARDO, MURCIA,  
SPAIN  
*E-mail address:* [ljalias@um.es](mailto:ljalias@um.es)

DEPARTAMENTO DE MATEMÁTICA,  
UNIVERSIDADE DE SÃO PAULO,  
RUA DO MATÃO 1010,  
CEP 05508-900, SÃO PAULO, SP, BRAZIL  
*E-mail address:* [piccione.p@gmail.com](mailto:piccione.p@gmail.com)