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A Gaussian estimate for the Heat Kernel on differential forms and application to the Riesz transform

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Abstract

Let (M^m, g) be a m-dimensional complete Riemannian manifold which satisfies the n-Sobolev inequality and on which the volume growth is comparable to the one of $\mathbb{R}^n$ for big balls; if the Hodge Laplacian on 1-forms is strongly positive and the Ricci tensor is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$, then we prove a Gaussian estimate on the Heat Kernel of the Hodge Laplacian on 1-forms. This allows us to prove that, under the same hypotheses, the Riesz transform $d\Delta^{-1/2}$ is bounded on L^p for all $1 < p < \infty$. Then, without the strong positivity assumption, we prove that the Riesz transform is bounded on L^p for all $1 < p < n$, when $n > 3$.

1 Introduction

1.1 A brief state of the art for the Riesz transform problem

In 1983 in [41], Strichartz raised the following question : when is the Riesz transform $d\Delta^{-1/2}$ bounded on $L^p(M)$, where M is a complete manifold? Since then, there have been a certain amount of papers which have studied this problem. We want to recall some of the results that have been so far obtained; we do not seek to be exhaustive, since there are now a lot of results on the Riesz transform problem in cases that include Cartan-Hadamard manifolds with a spectral gap (see [30]), Lie groups with polynomial growth (see [2]), or $\mathbb{R}^n$ for Riesz transform with potential $d(\Delta + V)^{-1/2}$ (c.f. the survey [3]). In this article, we are mostly interested in finding geometric hypotheses which ensure the boundedness of the Riesz transform, so let us now present some of the existing results concerning this aspect of the problem. Bakry proved in [5] that when M has non-negative Ricci curvature, the Riesz transform is bounded on L^p, for every $1 < p < \infty$. Carron, Coulhon and Hassell proved in [11] that on the connected sum of several copies of $\mathbb{R}^n$, the Riesz transform is bounded on L^p for all $1 < p < n$, and unbounded on L^p for $p \geq n$ if there is more than one end. Later in [10], Carron extended this result to a manifold M obtained as the connected sum of manifolds M_i, $1 \leq i \leq k$ all of which satisfying a Sobolev inequality of dimension $n > 3$: he proved that if for every $1 \leq i \leq k$, the Riesz transform on M_i is bounded on L^p and $\frac{n}{n-1} < p < n$, then the Riesz transform on M is bounded on L^p.

For manifolds satisfying a Sobolev inequality of dimension n, there are -at least- 3 different ranges of p that lead to different results for the Riesz transform: for $1 < p \leq 2$, the boundedness of the Riesz transform on L^p has been shown under a quite great generality (at least comparing to the other cases): Coulhon and Duong proved in [14] that if M satisfies the Doubling property together with a relative Faber-Krahn inequality, then the Riesz transform is bounded on L^p for $1 < p \leq 2$. For $2 \leq p < n$, it seems that the boundedness of the Riesz transform only depends on the geometry at infinity of the manifold, i.e. the geometry of each end of M considered *separately*. For this range of p's, Carron's perturbation argument [10] works: roughly, the boundedness of the Riesz transform is preserved under gluing and perturbation both of the metric and the topology on a compact set. For $n \leq p < \infty$, an obstruction to the boundedness of the Riesz transform can be the presence of more than one end (as we see for the exemple consisting of the connected sum of several Euclidean spaces). However, Carron's perturbation result [10] does not work for this range of p's, even if we assume that there is only one end. At this point, we mention another perturbation result that still works for this range of p's: Coulhon and Dungey in [13] proved that the boundedness of the Riesz transform is preserved with no restriction on p under a certain L^q perturbation of the metric (but the underlying manifold remaining the same). The problem of this result is that we must assume *a priori* that the manifold M is obtained by a change of metric from a manifold on which the Riesz transform is bounded (for example, $\mathbb{R}^n$ with a metric close to the Euclidean one); but if we want to generalize Bakry's result, we do not want to make this kind of assumptions: we would rather have intrinsic conditions in term of the Ricci curvature and other geometric quantities that would imply that the Riesz transform is bounded. The only known result for this range of p's, for manifolds that are not Lie groups, apart from the case of non-negative Ricci curvature (and apart from the examples obtained by the perturbation method of [13]), is the case of conical manifolds with compact basis, studied by Li in [29] (see also [16]). He has found in this case explicitely $p_0 > n$ depending on the geometry of the base, such that the Riesz transform is bounded on L^p if and only if $1 < p < p_0$. These manifolds have only one end, but in fact when $p_0 < \infty$ they may be too far from being perturbation of the Euclidean space $\mathbb{R}^n$ or of a manifold with non-negative Ricci curvature (their Ricci tensor may not be in $L^{\frac{n}{2}}$). Later, Guillarmou and Hassell have extended Li's results to the case of asymptotically conical manifolds (see [26] and [27]).

On the other hand, there are some results concerning sufficient or necessary conditions so that the Riesz transform be bounded. The point is that these results are analytical, in the sense that they reduce the problem of the boundedness of the Riesz transform to verifying some other assumption, typically on the Heat Kernel. One of these results is the following : under the assumptions that the Doubling property and the scaled Poincaré inequalities hold on M (which implies in a lot of cases that M has only one end), Auscher, Coulhon, Duong and Hofmann have shown in [4] that the boundedness of the Riesz transform on L^p is equivalent to the following L^p estimate of the gradient of the heat kernel:

$$||\nabla e^{-t\Delta}||_{p,p} \leq \frac{C_p}{\sqrt{t}},\ \forall t > 0.$$

However, we do not know any geometric caracterisation of these L^p estimates- it is only known that the non-negativity of the Ricci curvature is enough to get these for all p's. Let us also make the following related remark: so far, almost all the proofs that the Riesz transform is bounded rely on some hypotheses on the Heat Kernel; indeed, for $1 < p \leq 2$, what is used is a Gaussian upper estimate for the heat kernel. And to treat the case $p > 2$, many authors make assumptions on the gradient of the heat kernel, or, what turns out to be related, on the heat kernel on 1-forms $\vec{p_t}$. For example, Coulhon and Duong in [15] proved the boundedness of the Riesz transform on L^p for all $1 < p < \infty$ when a Gaussian upper estimate on the heat kernel on functions p_t holds, together with the domination condition:

$$||\vec{p_t}(x,y)|| \leq C p_t(x,y),\ \forall t > 0, \forall x,y \in M.$$

This implies that a Gaussian estimate holds for the heat kernel on 1-forms. Moreover, they show that a Gaussian estimate for the heat kernel on 1-forms is enough to get this domination. Examples of manifolds that satisfy this domination condition are the manifolds with non-negative Ricci curvature (and conversely, if we have the domination with constant $C = 1$, the Ricci curvature is non-negative). To our knowledge, it is the only case where a Gaussian estimate for the Heat Kernel on 1-forms has been proven. If one wants to prove Gaussian estimates for the Heat Kernel on 1-forms in more general cases, a certain amount of negative Ricci curvature must be allowed. At each point $p \in M$, the Ricci tensor:

$$Ric_p : \Lambda^1 T^*M \to \Lambda^1 T^*M,$$

is symmetric (with respect to g_p, the metric in p). Therefore, we can write, at each point:

$$Ric_p = (Ric_+)_p - (Ric_-)_p,$$

with $(Ric_+)_p$ and $(Ric_-)_p$ non-negative.

Definition 1.1 *$Ric_+ \in End(\Lambda^1 T^*M)$ and $Ric_- \in End(\Lambda^1 T^*M)$ are respectively the positive and negative part of the Ricci tensor.*

One wants to extend the Gaussian estimate for the Heat Kernel on 1-forms to the case where Ric_- is "small" in an appropriate sense. In this direction, in a recent paper, Coulhon and Zhang ([19]) prove that under some strong positivity of the operator $\vec{\Delta}$ (the Hodge Laplacian on 1-forms), there is an estimate of the form:

$$||\vec{p_t}(x,y)|| \leq \frac{Ct^\alpha}{V(x,\sqrt{t})} exp(-cd^2(x,y)/t),$$

for $t \geq 1$: it differs from the Gaussian estimate by a polynomial factor t^α, where $\alpha > 0$ depends on q such that Ric_-, the negative part of the Ricci tensor (see Definition (1.1)) is in L^q. Unfortunately, such estimates with the additionnal polynomial factor are not enough to get the boundedness of the Riesz transform, even for small values of α. Another recent tool to obtain Gaussian estimates for the heat kernel on forms is a result of Sikora (see [38]) which generalises to the Hodge Laplacian the well-known fact (due to Davies) for the Laplacian on functions that (Gaussian) off-diagonal estimates follow from suitable on-diagonal estimates.

In this article, we prove Gaussian estimates for the heat kernel on 1-forms for a class of manifolds larger than that of non-negative Ricci curvature. We were inspired by ideas developped by Barry Simon in [39] for operators of the form $L = \Delta + V$ on $\mathbb{R}^n$, where V is a potential whose negative part is in $L^{\frac{n}{2}\pm\epsilon}$; he obtained that $||e^{-tL}||_{\infty,\infty} \leq C$, $\forall t > 0$ if and only if V is a sub-critical potential (i.e. a potential such that L is strongly positive). The uniform upper bound $||e^{-tL}||_{\infty,\infty} \leq C$, $\forall t > 0$ may be seen, by applying the Nash argument (this uses the sub-criticality of V), to be an on-diagonal upper bound of the Heat Kernel associated to L of the form:

$$|p_t^V(x,x)| \leq \frac{C}{t^{n/2}}, \ \forall t > 0, \ \forall x \in M$$

where p_t^V is the Heat Keernel associated to L.

1.2 Our results

Let us describe roughly the structure of the article: it is essentially divided in two parts. In the first one, we prove a Gaussian estimate on the Heat Kernel on 1-forms for short-range Ricci potential: we assume that $Ric_- \in L^{\frac{n}{2}\pm\epsilon}$, together with some strong positivity assumption comparable to the one of Coulhon-Zhang, and a Sobolev inequality as well as a volume growth assumption (to be discussed below). This improves the result of Coulhon-Zhang ([19]): under our conditons, we get rid of the polynomial term. As a corollary, we obtain the boundedness of the Riesz transform on L^p for all $1 < p < \infty$. In a second part, we remove the hypothesis of strong positivity, and we show that the Riesz transform is still bounded on L^p if we restrict ourselves to the range $1 < p < n$.

Now let us present more precisely our results. For the first part, in addition to $Ric_- \in L^{\frac{n}{2}\pm\epsilon}$, the hypotheses that we make is that the Hodge Laplacian is "strongly positive", which roughly means that its kernel in $L^{\frac{2n}{n-2}}$ is zero (for a precise definition, see Definition (2.2)), and we also assume that the volume growth of M is "compatible with the Sobolev dimension"; this means that the volume of big balls $B(x,R)$ is comparable to R^n, where n is the dimension in the Sobolev inequality that we assume to hold on M (see Definition (3.2)). The hypothesis that the Ricci curvature is bounded from below is to ensure that the local geometry of the manifold is bounded. Our result is the following :

Theorem 1.1 *Let (M^m, g) be an m-dimensional complete Riemannian manifold which satisfies the n-Sobolev inequality, and whose negative part of the Ricci tensor Ric_- is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$. We also assume that the Hodge Laplacian on 1-forms $\vec{\Delta}$ is strongly positive, that the volume growth of M is compatible with the Sobolev dimension, and that Ric is bounded from below;*
Then the Gaussian estimate is valid for $e^{-t\vec{\Delta}}$: for all $\delta > 0$, there exists a constant C such that

$$||\vec{p_t}(x,y)|| \leq \frac{C}{V(x,t^{1/2})} \exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \ \forall x,y \in M, \ \forall t > 0.$$

Thus, the Riesz transform is bounded on L^p for all $1 < p < \infty$.

Let us recall the definition of L^2-cohomology. The first space of reduced cohomology is defined to be:

$$H_2^1(M) := \frac{\{\alpha \in L^2 : d\alpha = 0\}}{\overline{\{dC_0^\infty(M)}} = \{\alpha \in L^2 : \vec{\Delta}\alpha = 0\},$$

where $\vec{\Delta}$ is the Hodge Laplacian acting on 1-forms.
A consequence of Proposition 1.3 in [8] is that, when $n > 4$, $\vec{\Delta}$ is strongly positive if and only if $H_2^1(M) = \{0\}$.
Thus we get the following:

Corollary 1.1 *Assume $n > 4$;*
Let (M^m, g) be a complete Riemannian manifold which satisfies the n-Sobolev inequality, and whose negative part of the Ricci tensor Ric_- is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$. We also assume that $H_2^1(M) = \{0\}$, that the volume growth of M is compatible with the Sobolev dimension, and that Ric is bounded from below;
Then the Gaussian estimate is valid for $e^{-t\vec{\Delta}}$: for all $\delta > 0$, there exists a constant C such that

$$||\vec{p_t}(x, y)|| \leq \frac{C}{V(x, t^{1/2})} \exp\left(-\frac{d^2(x, y)}{(4 + \delta)t}\right), \forall x, y \in M, \forall t > 0.$$

Thus, the Riesz transform is bounded on L^p for all $1 < p < \infty$.

Remark 1.1 *$H_2^1(M) = \{0\}$ implies in particular that M has only one end.*

This applies in particular to conical manifolds with non-negative Ricci curvature outside a compact set: if we take a compact $(n-1)$-dimensional manifold Σ which satisfies $Ric_\Sigma \geq n - 2$, the conical manifold $c(\Sigma)$ with base Σ satisfies $Ric_{c(\Sigma)} \geq 0$. Let us remark that by the Lichnerowicz-Obata Theorem, we have $\lambda_1(\Sigma) \geq n - 1 = \lambda_1(S^{n-1})$ (with equality if and only if Σ is isometric to S^{n-1}). So this is coherent with Li's result (see [29]), which asserts that the Riesz transform on $c(\Sigma)$ is bounded on L^p for all $1 < p < \infty$ when $\lambda_1(\Sigma) \geq n - 1$, since by Bakry we know that $Ric \geq 0$ implies the boundedness of the Riesz transform on L^p for all $1 < p < \infty$. It is known that a conical manifold of dimension n satisfies the n-Sobolev inequality, and that its volume growth is compatible with the Sobolev dimension. Thus we get the following result:

Corollary 1.2 *Let M be a complete Riemannian manifold such that there exist $K_1 \subset M$, $K_2 \subset c(\Sigma)$ compact sets such that $M \setminus K_1$ is isometric to $c(\Sigma) \setminus K_2$;*
Assume that $Ric_\Sigma \geq n - 2$ and $H_2^1(M) = \{0\}$;
Then the Gaussian estimate is valid for $e^{-t\vec{\Delta}}$ on M: for all $\delta > 0$, there exists a constant C such that

$$||\vec{p_t}(x, y)|| \leq \frac{C}{V(x, t^{1/2})} \exp\big(-\frac{d^2(x, y)}{(4 + \delta)t}\big), \forall x, y \in M, \forall t > 0.$$

Moreover, the Riesz transform is bounded on L^p for all $1 < p < \infty$ (we already knew that from the work of Guillarmou-Hassell [26], [27]).

In a second part, we remove the hypothesis of strong positivity. The perturbation method of Carron in [10] allows us to prove:

Theorem 1.2 *Assume that $n > 3$;*
Let (M^m, g) be an m-dimensional complete Riemannian manifold which satisfies the n-Sobolev inequality, and whose negative part of the Ricci tensor Ric_- is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$.
We also assume that the volume growth of M is compatible with the Sobolev dimension, and that the Ricci curvature is bounded from below.
Then for every $1 < p < n$, the Riesz transform is bounded on L^p on M.

Remark 1.2 *Let us note that in all these results, the Sobolev exponent n is not necessarily equal to the dimension m of the manifold M; in fact, n must only satisfy $n \geq m$.*

2 Preliminaries

Throughout the text, M will denote a complete non-compact manifold which satisfies the **n-Sobolev inequality**: there is a constant C such that

$$||f||_{\frac{2n}{n-2}} \leq C||\nabla f||_2,\ \forall f \in C_0^\infty(M).$$

We consider an operator of the form $\nabla^*\nabla + R_+ - R_-$, acting on a Riemannian fiber bundle $E \to M$, where ∇ is a connection on $E \to M$ compatible with the metric, and for $p \in M$, $R_+(p)$, $R_-(p)$ are non-negative symmetric endomorphism acting on the fiber E_p. Let us denote $\bar{\Delta} := \nabla^*\nabla$, the "rough Laplacian", and $C^\infty(E)$ (resp. $C_0^\infty(E)$) the set of smooth sections of E (resp., of smooth sections of E which coincide with the zero section outside a compact set).
We define $H := \bar{\Delta} + R_+$. We will consider the L^2-norm on sections of E:

$$||\omega||_2^2 = \int_M |\omega|^2(p)dvol(p),$$

where $|\omega|(p)$ is the norm of the evaluation of ω in p. We will denote $L^2(E)$, or simply L^2 when there is no confusion possible for the set of sections of E with finite L^2 norm.
We have in mind the case of $\vec{\Delta} = d^*d + dd^*$, the Hodge Laplacian acting on 1-forms, for which we have the **Bochner decomposition**:

$$\vec{\Delta} = \bar{\Delta} + Ric,$$

where $\bar{\Delta} = \nabla^*\nabla$ is the rough Laplacian on 1-forms, and $Ric \in End(\Lambda^1 T^*M)$ is canonically identified - using the metric - to the Ricci tensor.
From classical results in spectral analysis (an obvious adaptation to $\bar{\Delta}$ of Strichartz's proof that the Laplacian is self-adjoint on a complete manifold, see Theorem 3.13 in [31]), we know that if R_- is bounded and in L^1_{loc}, then $\bar{\Delta} + R_+ - R_-$ is essentially self-adjoint on $C_0^\infty(E)$.

2.1 Consequences of the Sobolev inequality

We will see in this section that $H = \bar{\Delta} + R_+$ shares with the usual Laplacian acting on functions a certain amount of functionnal properties, among which:

Proposition 2.1 *H satisfies the n-Sobolev inequality: there is a constant C such that*

$$||\omega||^2_{\frac{2n}{n-2}} \leq C\langle H\omega, \omega\rangle,\ \forall \omega \in C_0^\infty(E).$$

Proof of Proposition (2.1):
For the reader's convenience, we rewrite the proof of the Appendix of [6], written by G. Besson. Since R_+ is non-negative, it is enough to prove that $\bar{\Delta}$ satisfies the n-Sobolev inequality. If $\omega \in C_0^\infty(\Gamma(E))$, $\langle \bar{\Delta}\omega, \omega\rangle = \int_M |\nabla\omega|^2$. Let us define $f_\epsilon := \sqrt{|\omega|^2 + \epsilon^2} - \epsilon \in C_0^\infty(M)$. By the n-Sobolev inequality on M,

$$||f_\epsilon||^2_{\frac{2n}{n-2}} \leq C \int_M |\nabla f_\epsilon|^2.$$

But

$$\nabla f_\epsilon = \frac{\nabla(|\omega|^2)}{2\sqrt{|\omega|^2 + \epsilon}}.$$

If $|X| = 1$, $|\nabla_X(|\omega|^2)| = 2|\langle \nabla_X\omega, \omega\rangle| \leq 2|\nabla_X\omega||\omega| \leq 2|\nabla\omega||\omega|$ (indeed, $|\nabla\omega|^2(p) = \sum_i |\nabla_{X_i}\omega|^2(p)$ if $(X_i)_i$ is an orthonormal basis of T_pM), hence

$$|\nabla(|\omega|^2)| = \sup_{|X|=1} |\nabla_X(|\omega|^2)| \leq 2|\nabla\omega||\omega|,$$

so that

$$|\nabla f_\epsilon| \leq |\nabla \omega|.$$

We obtain

$$||f_\epsilon||^2_{\frac{2n}{n-2}} \leq C||\nabla\omega||_2^2.$$

Furthermore, $f_\epsilon \to |\omega|$ uniformly when $\epsilon \to 0$, so we can let $\epsilon \to 0$ in the preceeding inequality, obtaining:

$$||\omega||_{\frac{2n}{n-2}} \leq C||\nabla\omega||_2^2.$$

But by integration by parts, if $\omega \in C_0^\infty(E)$, $\langle\bar{\Delta}\omega, \omega\rangle = \int_M |\nabla\omega|^2$, so we have:

$$||\omega||_{\frac{2n}{n-2}} \leq C\langle\bar{\Delta}\omega, \omega\rangle.$$

□

Remark 2.1 *In fact, the preceeding proof shows the Kato inequality (also proved in [6]): for all $\omega \in C_0^\infty(E)$, $|\omega| \in W^{1,2}$ and*

$$q_{\bar{\Delta}}(\omega) \geq q_\Delta(|\omega|),$$

where $q_{\bar{\Delta}}$ (resp. q_Δ) is the quadratic forms associated to the self-adjoint operator $\bar{\Delta}$ (resp. to Δ). In the usual terminology of [6], we say that Δ dominates $\bar{\Delta}$; in fact, it is obvious that Δ dominates $H+\lambda$, for all $\lambda \geq 0$.

A consequence of this domination is:

Proposition 2.2 *For all $\omega \in C_0^\infty(E)$,*

$$|e^{-tH}\omega| \leq e^{-t\Delta}|\omega|, \, \forall t \geq 0,$$

and

$$|H^{-\alpha}\omega| \leq \Delta^{-\alpha}|\omega|, \, \forall \alpha > 0.$$

Proof of Proposition (2.2):
The first part comes directly from [6].
The second domination is a consequence of the first one and of the following formulae:

$$H^{-\alpha} = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-tH} t^{\alpha-1} dt,$$

$$\Delta^{-\alpha} = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-t\Delta} t^{\alpha-1} dt.$$

□

This domination property, together with the fact that $e^{-t\Delta}$ is a contraction semigroup on L^p for all $1 \leq p \leq \infty$, gives at once that e^{-tH} is also a contraction semigroup on all the L^p spaces.
From the ultracontractivity estimate:

$$||e^{-t\Delta}||_{1,\infty} \leq \frac{C}{t^{n/2}}, \, \forall t > 0,$$

valid since M satisfies a n-Sobolev inequality (see [37]), and the domination of Proposition (2.2), we deduce that we also have:

$$||e^{-tH}||_{1,\infty} \leq \frac{C}{t^{n/2}}, \, \forall t > 0.$$

By interpolation with $||e^{-tH}||_{\infty,\infty} \leq 1$, we deduce that for all $1 \leq p \leq \infty$, there exists C such that

$$||e^{-tH}||_{p,\infty} \leq \frac{C}{t^{n/2p}}, \, \forall t > 0.$$

Interpolating with $||e^{-tH}||_{p,p} \leq 1$, we obtain:

$$||e^{-tH}||_{p,q} \leq \frac{C}{t^{\frac{n}{2p}\left(1-\frac{p}{q}\right)}}.$$

Furthermore, the domination property also yields that e^{-tH} is a contraction semigroup on L^p, $1 \leq p \leq \infty$, so by Stein's Theorem (Theorem 1 p.67 in [40]), e^{-tH} is analytic bounded on L^p, for all $1 < p < \infty$. Hence we have proved:

Corollary 2.1 *e^{-tH} is a contraction semigroup on L^p, for all $1 \leq p \leq \infty$.*
For all $1 \leq p \leq \infty$, there exists C such that:

$$||e^{-tH}||_{p,q} \leq \frac{C}{t^{\frac{n}{2p}\left(1-\frac{p}{q}\right)}}, \, \forall t > 0, \, \forall q > p.$$

Moreover, e^{-tH} is analytic bounded on L^p with sector of angle $\frac{\pi}{2}\left(1 - \left|\frac{2}{p} - 1\right|\right)$, for all $1 < p < \infty$.

We recall the following consequences of the analyticity of a semigroup, which come from the Dunford-Schwarz functionnal calculus (see [34], p.249):

Corollary 2.2 *Let e^{-zA} an analytic semigroup on a Banach space X. Then there exists a constant C such that for all $\alpha > 0$:*

1. $$||A^{\alpha}e^{-tA}|| \leq \frac{C}{t^{\alpha}}, \, \forall t > 0.$$

2. $$||(I + tA)^{\alpha}e^{-tA}|| \leq C, \, \forall t > 0.$$

Furthermore, the domination property also yields that e^{-tH} is a contraction semigroup on L^p, $1 \leq p \leq \infty$, so by Stein's Theorem (Theorem 1 p.67 in [40]), e^{-tH} is analytic bounded on L^p, for all $1 < p < \infty$. Thus:

Theorem 2.1 *H satisfies the following properties:*

1. *The **mapping properties**:*
   For all $\alpha > 0$,

   $$H^{-\alpha/2} : L^p \longrightarrow L^q$$

   is bounded whenever $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $p < q < \infty$ (in particular we must have $p < \frac{n}{\alpha}$).

2. *The **Gagliardo-Nirenberg inequalities**:*
   For all $r > \nu$ and all $s > 0$,

   $$||\omega||_{L^{\infty}} \leq C(n, r, s)||H\omega||_{r/2}^{\theta}||\omega||_{s/2}^{1-\theta}, \, \forall \omega \in C_0^{\infty}(E),$$

   where $\theta = \frac{n/s}{1-(n/r)+(n/s)}$, under the condition $r > n$.

Proof of Theorem (2.1:
The mapping properties for H are the consequence of the domination of Proposition (2.2) and of the mapping properties for Δ, which hold since M satisfies a n-Sobolev inequality (cf [42], Theorem 1 and [18], Theorem II.4.1). The Gagliardo-Nirenberg inequalities are extracted from [12], Theorems 1 and 2, given the ultracontractivity of e^{-tH} and its analiticity on L^p for $1 < p < \infty$ (Corollary (2.1)).

□

Furthermore, we have the following important fact:

Proposition 2.3 *All the results of this section are also valid if we replace H by $H+\lambda$ with $\lambda>0$ (since $H+\lambda$ is dominated by Δ, for all $\lambda\geq 0$), and moreover the constants in the Sobolev inequality, in the Gagliardo-Nirenberg inequality and also the norms of the operators $(H+\lambda)^{-\alpha}: L^p \to L^q$, are all bounded* ***independantly*** *of λ.*

This will be intensively used later.

2.2 Strong positivity

As in the previous section, denote $H := \bar{\Delta} + R_+$. We assume - as it is the case for the Laplacian on 1-forms - that $H - R_-$ is a *non-negative* operator:

Assumption 1 *$H - R_-$ is a non-negative operator*

It is equivalent to the following inequality : if $\omega \in C_0^\infty(E)$,

$$0 \leq \langle Ric_- \omega, \omega \rangle \leq \langle H\omega, \omega \rangle.$$

Simon and Davies first ([39] and [20]), in the Euclidean setting, and later Coulhon and Zhang ([19]) for the case of manifolds, have studied the large time behaviour of a semigroup $e^{-t(\Delta-V)}$ generated by a Schrödinger operator with non-positive potential, assuming that $\Delta - V$ is a positive operator. They find that if $V \in L^q$ for some q, then $||e^{-t(\Delta-V)}||_{\infty,\infty} \leq Ct^\alpha$, where α depends on q. Moreover, if we make the supplementary assumption that the operator $\Delta - V$ is "strongly positive" (or "V strongly subcritical"), then the exponent α can be lowered. In particular, on $\mathbb{R}^n$, under the assumptions that $V \in L^{\frac{n}{2}\pm\epsilon}$ is strongly subcritical, Simon obtains in [39] that $||e^{-t(\Delta-V)}||_{\infty,\infty} \leq C$. What we do here is to generalize this to the case of operators of the form $\nabla^*\nabla + R_+ - R_-$ acting on the sections of a vector bundle $E \to M$ over M. Let us begin by recalling the following classical definition:

Definition 2.1 *The Hilbert space H_0^1 is the completion of $C_0^\infty(E)$ for the norm given by the quadratic form associated to the self-adjoint operator H.*

We recall some of the properties of this space H_0^1 associated to H:

Proposition 2.4

1. *$H_0^1 \hookrightarrow L^{\frac{2n}{n-2}}(E)$. In particular, it is a space of sections of $E \to M$.*
2. *$H^{1/2}$, defined on $C_0^\infty(E)$, extends uniquely to a bijective isometry from H_0^1 to $L^2(E)$. Thus we can consider $H^{-1/2}: L^2(E) \to H_0^1$.*
3. *If we consider the operator $H^{1/2}$ given by the Spectral Theorem - denote it $H_s^{1/2}$ to avoid confusion with the one we have just defined from H_0^1 to L^2 - then $\mathcal{D}om(H_s^{1/2}) = H_0^1 \cap L^2(E)$, and moreover $H^{1/2}$ coincide with $H_s^{1/2}$ on $H_0^1 \cap L^2(E)$.*

Sketch of proof of Proposition (2.4):
(1) is a consequence of the n-Sobolev inequality of Proposition (2.1). The Sobolev inequality implies that H is non-parabolic, and (2) can be obtained by the same method as in [21]. (3) can also be obtained by the techniques developped in [21] in the context of Schrödinger operators acting on functions, which adapts to the case of Schrödinger operators acting on sections of a vector bundle.

□

In what follows, we assume that $R_- \in L^{\frac{n}{2}}$.

Definition 2.2 *We say that $L := H - R_-$ is* ***strongly positive*** *if one of the following equivalent -at least when $R_- \in L^{\frac{n}{2}}$- conditions is satisfied :*

1. *There exists $\epsilon > 0$ such that:*

$$0 \leq \langle R_- \omega, \omega \rangle \leq (1-\epsilon)\langle H\omega, \omega\rangle, \; \forall \omega \in C_0^\infty(E).$$

2. $$Ker_{H_0^1}(L) = \{0\}.$$

3. *The (non-negative, self-adjoint compact if $R_- \in L^{\frac{n}{2}}$) operator $A := H^{-1/2} R_- H^{-1/2}$ acting on $L^2(E)$ satisfies:*

$$||A||_{2,2} \leq (1-\epsilon),$$

where $\epsilon > 0$.

Remark 2.2 *In general, we have the equivalence between 1) and 3) and the implication 3)⇒2), under the sole hypothesis that A is self-adjoint (which is the case if $R_- \in L^{\frac{n}{2}}$, but can be true under more general conditions). The fact that 2)⇒3) is true as soon as A is self-adjoint compact.*

Proof of the equivalence:
We can write:

$$L = H - R_- = H^{1/2}(I-A)H^{1/2}.$$

First, let us prove that 1) ⇔ 3′), where 3′) is defined to be:

$$3') : \langle Au, u\rangle \leq (1-\epsilon)\langle u, u\rangle, \; \forall u \in L^2.$$

Remark that 3′) is equivalent to 3) when A is self-adjoint. Let $\omega \in C_0^\infty(\Gamma(E))$, and set $u = H^{1/2}\omega \in L^2(\Gamma(E))$. Then

$$\begin{aligned} \langle Au, u\rangle \leq (1-\epsilon)\langle u, u\rangle \quad &\Leftrightarrow \quad \langle H^{-1/2}R_-\omega, H^{1/2}\omega\rangle \leq (1-\epsilon)\langle H^{1/2}\omega, H^{1/2}\omega\rangle \\ &\Leftrightarrow \quad \langle R_-\omega, \omega\rangle \leq (1-\epsilon)\langle H\omega, \omega\rangle \end{aligned}$$

Then we show that 3) ⇒ 2). This is a consequence of the following Lemma:

Lemma 2.1 *If A is self-adjoint, then*

$$H^{1/2} : Ker_{H_0^1}(L) \to Ker_{L^2}(I-A)$$

is an isomorphism (and it is of course an isometry).

Proof of Lemma (2.1):
Let $u \in H_0^1$; we can write $u = H^{-1/2}\varphi$, where $\varphi \in L^2(\Gamma(E))$. By definition, $Lu = 0$ means that for every $v \in C_0^\infty(\Gamma(E))$,

$$\langle u, Lv\rangle = 0.$$

This equality makes sense, because since H satisfies a n-Sobolev inequality, $H_0^1 \hookrightarrow L^1_{loc}$. The Spectral Theorem then implies, since $C_0^\infty \subset \mathcal{D}om(H)$, that given $v \in C_0^\infty(\Gamma(E))$ the following equality holds in $L^2(\Gamma(E))$:

$$Hv = H^{1/2}H^{1/2}v.$$

Hence

$$Lv = (H - R_-)v = H^{1/2}(I-A)H^{1/2}v.$$

Let $w := (I - A)H^{1/2}v$; then the preceeding equality shows that $w \in \mathcal{D}om(H^{1/2}) = H_0^1 \cap L^2(\Gamma(E))$. Furthermore, $H^{1/2}w = Hv$ is compactly supported, so we have:

$$\langle u, H^{1/2}w\rangle = \langle H^{1/2}u, w\rangle.$$

Indeed, if $u \in H_0^1 \cap L^2$ it is a consequence of Lemma 3.1 in [21], and a limiting argument plus the fact that $H_0^1 \hookrightarrow L^2_{loc}$ shows that it is true for all $u \in H_0^1$.

$$Lu = 0 \Longleftrightarrow \forall v \in C_0^\infty,\ \langle H^{1/2}u, (I-A)H^{1/2}v\rangle = 0.$$

But since $H^{1/2}C_0^\infty(E)$ is dense in $L^2(E)$, we get, using the fact that A is self-adjoint:

$$\begin{array}{rcl} Lu = 0 & \Longleftrightarrow & \forall v \in L^2,\ \langle H^{1/2}u, (I-A)v\rangle = 0 \\ & \Longleftrightarrow & H^{1/2}u \in Ker_{L^2}(I-A) \end{array}$$

□

It remains to prove that 2) ⇒ 3) ; this is a consequence of Lemma (2.1) and of the following Lemma, which is extracted from Proposition 1.2 in [9]:

Lemma 2.2 *Assume $R_- \in L^{\frac{n}{2}}$. Then $A := H^{-1/2}R_-H^{-1/2}$ is a non-negative, self-adjoint compact operator on $L^2(\Gamma(E))$. Moreover,*

$$||A||_{2,2} \leq C||R_-||_{\frac{n}{2}},$$

where C depends only on the Sobolev constant for H.

□

We will also need the following Lemma, which is an easy consequence of the definition of strong positivity:

Lemma 2.3 *Let H be of the form: $H = \bar{\Delta} + R_+$, with R_+ non-negative. Let $R_- \in End(\Lambda^1 T^*M)$ be symmetric, non-negative, in $L^{\frac{n}{2}}$ such that $L := H - R_-$ is strongly positive. Then the n-Sobolev inequality is valid for L, i.e.*

$$||\omega||^2_{\frac{2n}{n-2}} \leq C\langle L\omega, \omega\rangle,\ \forall \omega \in C_0^\infty(E).$$

Proof of Lemma (2.3):
By definition of strong positivity,

$$\langle R_-\omega, \omega\rangle \leq (1-\epsilon)\langle H\omega, \omega\rangle.$$

Therefore:

$$\begin{array}{rcl} \langle L\omega, \omega\rangle & = & \langle H\omega, \omega\rangle - \langle R_-\omega, \omega\rangle \\ & \geq & (1-(1-\epsilon))\langle H\omega, \omega\rangle \\ & \geq & \epsilon C||\omega||^2_{\frac{2n}{n-2}}, \end{array}$$

where we have used in the last inequality the fact that H satisfies a n-Sobolev inequality.

□

3 Gaussian upper-bound for the Heat Kernel on 1-forms

3.1 Estimates on the resolvent of the Schrödinger-type operator

In this section, we will show how to obtain bounds on the resolvent of $L := \nabla^*\nabla + R_+ - R_- = H - R_-$. In order to do this, we first have to estimate the resolvent of the operator $H = \bar{\Delta} + R_+$. Recall from Corollary (2.1) that e^{-tH} is a contraction semigroup on L^p, for all $1 \leq p \leq \infty$. Using the formula:

$$(L+\lambda)^{-1} = \int_0^\infty e^{-tL} e^{-t\lambda} dt,$$

we get:

Proposition 3.1 *Let (M^m, g) be an m-dimensional complete Riemannian manifold. Then for all $\lambda > 0$ and for all $1 \leq p \leq \infty$,*

$$||(H+\lambda)^{-1}||_{p,p} \leq \frac{1}{\lambda}.$$

Remark 3.1 *The case $p = \infty$ is by duality, for $(H+\lambda)^{-1}$ is defined on L^∞ by duality. Indeed, for $g \in L^\infty$, we define $(H+\lambda)^{-1}g$ so that:*

$$\langle (H+\lambda)^{-1}g, f\rangle := \langle g, (H+\lambda)^{-1}f\rangle, \forall f \in L^1$$

(recall that $(L^1)' = L^\infty$). It is then easy to see that $||(H+\lambda)^{-1}||_{1,1} \leq \frac{1}{\lambda}$ implies $||(H+\lambda)^{-1}||_{\infty,\infty} \leq \frac{1}{\lambda}$.

We now estimate the resolvent of the operator $L := \nabla^*\nabla + R_+ - R_-$; as before, L acts on the sections of a vector bundle $E \to M$ (see the beginning of the Preliminaries for the general context). The key result is the following:

Theorem 3.1 *Let (M, g) be a complete Riemannian manifold which satisfies the n-Sobolev inequality, and suppose that R_- is in $L^{\frac{n}{2}\pm\epsilon}$ for some $\epsilon > 0$. We also assume that $L := H - R_-$, acting on the sections of $E \to M$, is strongly positive. Then for all $1 \leq p \leq \infty$, there exists a constant $C(p)$ such that*

$$||(L+\lambda)^{-1}||_{p,p} \leq \frac{C(p)}{\lambda}, \forall \lambda > 0.$$

Proof of Theorem (3.1):
In this proof, we write L^q for $L^q(E)$. Let us denote $H_\lambda := H + \lambda$. So

$$(L+\lambda)^{-1} = (I - T_\lambda)^{-1} H_\lambda^{-1},$$

where $T_\lambda := H_\lambda^{-1} R_-$. If we can prove that $(I - T_\lambda)^{-1}$ is a bounded operator on L^p, with norm independant of λ, then by Proposition (3.1) we are done. To achieve this, we will show that the series $\sum_{n\geq 0} T_\lambda^n$ converges on $\mathcal{L}(L^p, L^p)$, with a bound of the norm independant of $\lambda \geq 0$.
The aim of the next two Lemmas is to prove that T_λ acts on all the L^q spaces. We single out the case $q = \infty$, for it requires a different ingredient for its proof:

Lemma 3.1 *$T_\lambda : L^\infty \longrightarrow L^\infty$ and is bounded as a linear operator, with a bound of the norm independant of $\lambda \geq 0$.*

Proof of Lemma (3.1):
We have seen that e^{-tH_λ} satisfies the mapping properties and the Gagliardo-Nirenberg inequalities of Theorem (2.1) with constants independant of $\lambda \geq 0$. Let $u \in L^\infty$. We apply the Gagliardo-Nirenberg inequality for H_λ:

$$||T_\lambda u||_\infty \leq C||R_- u||_{n/2+\epsilon}^\theta ||T_\lambda u||_p^{1-\theta}, \forall p,$$

with C independant of λ (see Remark (1.2)). We have $||R_-u||_{\frac{n}{2}+\epsilon} \leq ||R_-||_{\frac{n}{2}+\epsilon}||u||_\infty$. By the mapping properties of Theorem (2.1), $H_\lambda^{-1} : L^{\frac{n}{2}-\epsilon} \to L^s$ for a certain s, with a norm bounded independantly of λ. So we get :

$$||T_\lambda u||_\infty \leq C||R_-||^\theta_{n/2+\epsilon}(||H_\lambda^{-1}||_{n/2-\epsilon,s}||R_-||_{n/2-\epsilon})^{1-\theta}||u||_\infty \leq C||u||_\infty$$

□

Lemma 3.2 *1. For all $1 \leq \beta \leq \infty$,*

$$R_- : L^\beta \to L^{\frac{n\beta}{n+2\beta}}$$

is bounded.

2. There exists $\nu > 0$ (small and independant of $\lambda \geq 0$), such that for all $\beta < \infty$, and for all $\lambda \geq 0$,

$$T_\lambda : L^\beta \to L^r \cap L^s,$$

where $\frac{1}{r} = \max(\frac{1}{\beta} - \nu, 0^+))$ and $\frac{1}{s} = \min(\frac{1}{\beta} + \nu, 1)$, is bounded, with a bound of the norm independant of λ (here 0^+ denotes any positive number).

3. For $\beta = \infty$,

$$T_\lambda : L^\infty \to L^\infty \cap L^p$$

is bounded with a norm bounded independantly of λ, if p big enough.

4. For β large enough,

$$T_\lambda : L^\beta \to L^\beta \cap L^\infty$$

is bounded with a norm bounded independantly of λ.

Proof of Lemma (3.2):
If $u \in L^\beta$ and $v \in L^\gamma$, then

$$||uv||_{\frac{\gamma\beta}{\gamma+\beta}} \leq ||u||_\beta ||v||_\gamma.$$

Therefore, $R_- : L^\beta \to L^q$ is bounded, where $\frac{1}{q} = \frac{1}{\beta} + \frac{1}{p}$, for all $p \in [\frac{n}{2} - \epsilon, \frac{n}{2} + \epsilon]$. Taking $p = \frac{n}{2}$, we find the first result of the Lemma.
Applying the mapping property (2.1), we deduce that:

$$H_\lambda^{-1} R_- : L^\beta \to\to L^r \cap L^s$$

is bounded independantly of β, and also independantly of $\lambda \geq 0$ by Remark (1.2), where

$$\frac{1}{r} = \max\left(\left(\frac{2}{n+2\epsilon} - \frac{2}{n}\right) + \frac{1}{\beta}, 0^+\right) = \max\left(\frac{1}{\beta} - \mu, 0^+\right),$$

and

$$\frac{1}{s} = \min\left(\left(\frac{2}{n-2\epsilon} - \frac{2}{n}\right) + \frac{1}{\beta}, 1\right) = \min\left(\frac{1}{\beta} + \mu', 1\right),$$

hence the second part of the Lemma with $\nu = \min(\mu, \mu')$.

For the case $\beta = \infty$, we have $s = \frac{1}{\mu'} = p$ large, and we already know from Lemma (3.1) that T_λ send L^∞ to L^∞.

For the case β large enough : since $R_- \in L^{\frac{n}{2}+\epsilon}$, if β is large enough and $u \in L^\beta$, then $R_- u \in L^{\frac{n}{2}+\alpha}$ for an $\alpha > 0$. We may apply Gagliardo-Nirenberg's inequality: for such a β,

$$||H_\lambda^{-1} R_- u||_\infty \leq C ||R_- u||^\theta_{n/2+\alpha} ||H_\lambda^{-1} R_- u||_\beta^{1-\theta}.$$

This yields the result.

□

As a corollary of Lemma (3.2), we obtain:

Proposition 3.2 *For all $1 \leq \beta \leq \infty$ and $1 \leq \alpha \leq \infty$, there exists an $N \in \mathbb{N}$ (which depends only on β and α, and not on λ), such that for all $\lambda \geq 0$,*

$$T_\lambda^N : L^\alpha \to L^\beta$$

is bounded with a bound of its norm independant of λ.

Thus, if we can prove that there is a $\beta \in [1, \infty]$ and a $\mu \in (0,1)$ such that

$$||T_\lambda^k||_{\beta,\beta} \leq C(1-\mu)^k, \ k \in \mathbb{N},$$

with C independant of $\lambda \geq 0$, we will obtain that the series $\sum_{n \geq 0} T_\lambda^n$ converges in $\mathcal{L}(L^p, L^p)$ for all $1 \leq p \leq \infty$, uniformly with respect to $\lambda \geq 0$. It is the purpose of the next Lemma :

Lemma 3.3 *Let $\beta := \frac{2n}{n-2}$. Then $||T_\lambda^k||_{\beta,\beta} \leq C(1-\mu)^k$ for all $k \in \mathbb{N}$ with constants C and $0 < \mu < 1$ independant of $\lambda \geq 0$.*

Proof of Lemma (3.3):
We write :

$$T_\lambda = H_\lambda^{-1/2} [H_\lambda^{-1/2} R_- H_\lambda^{-1/2}] H_\lambda^{1/2},$$

and we define $A_\lambda := H_\lambda^{-1/2} R_- H_\lambda^{-1/2}$. Let us define the Hilbert space $H^1_{0,\lambda}$ to be the closure of $C_0^\infty(E)$ for the norm:

$$\omega \mapsto \left(\int_M |\nabla \omega|^2 + \lambda |\omega|^2 \right)^{1/2} = Q_\lambda(\omega)^{1/2},$$

where Q_λ is the quadratic form associated to the self-adjoint operator H_λ. If $\lambda > 0$, it is the space $H_0^1 \cap L^2 = \mathcal{D}om(H^{1/2})$, but with a different norm. The choice of the norm is made so that $H_\lambda^{1/2} : H^1_{0,\lambda} \to L^2$ is an isometry. Since $A_\lambda : L^2 \to L^2$, and given that $T_\lambda = H_\lambda^{-1/2} A_\lambda H_\lambda^{1/2}$, we deduce that :

$$T_\lambda : H^1_{0,\lambda} \to H^1_{0,\lambda} \text{ with } ||T_\lambda||_{H^1_{0,\lambda}, H^1_{0,\lambda}} = ||A_\lambda||_{2,2}.$$

But by the equivalence 1) ⇔ 3) in the Definition (2.2), the existence of $\mu \in (0,1)$ such that $||A_\lambda||_{2,2} \leq 1 - \mu$ is equivalent to:

$$\langle R_- \omega, \omega \rangle \leq (1-\mu) \langle (H_\lambda)\omega, \omega \rangle, \ \forall \omega \in C_0^\infty(\Gamma(E)).$$

Since $\langle (H+\lambda)\omega, \omega \rangle = \langle H\omega, \omega \rangle + \lambda ||\omega||_2^2 \geq \langle H\omega, \omega \rangle$, we obtain that the existence of some $\mu \in (0,1)$ such that for all $\lambda \geq 0$, $||A_\lambda||_{2,2} \leq 1-\mu$ is equivalent to the strong positivity of L. Therefore $||T_\lambda||_{H^1_{0,\lambda}, H^1_{0,\lambda}} \leq (1-\mu)$. Moreover, by the functionnal consequence of Sobolev's inequality (Theorem (2.1)),

$$H_\lambda^{-1/2} : L^{\frac{2n}{n+2}} \to L^2,$$

with norm bounded independantly of $\lambda \geq 0$ (by Remark (1.2), and by Lemma (3.2),

$$R_- : L^{\frac{2n}{n-2}} \to L^{\frac{2n}{n+2}},$$

so that, using that $H_\lambda^{-1/2} : L^2 \to H^1_{0,\lambda}$ is an isometry and that we can write $T_\lambda = H_\lambda^{-1/2}[H_\lambda^{-1/2} R_-]$, we get that

$$T_\lambda : L^\beta \to H^1_{0,\lambda},$$

is bounded with a bound of the norm independant of $\lambda \geq 0$. Furthermore, $H^1_{0,\lambda} \hookrightarrow H^1_0$ is continuous of norm less than 1, and the n-Sobolev inequality for H says precisely that:

$$H^1_0 \hookrightarrow L^\beta$$

continuously. Therefore, $H^1_{0,\lambda} \hookrightarrow L^\beta$ continuously, with a bound of the norm independant of λ. Then we write $T_\lambda^k = T_\lambda^{k-1} T_\lambda$, with

$$T_\lambda : L^\beta \to H^1_{0,\lambda}$$

and

$$T_\lambda^{k-1} : H^1_{0,\lambda} \to H^1_{0,\lambda} \hookrightarrow L^\beta,$$

so that we get:

$$||T_\lambda^k||_{\beta,\beta} \leq C(1-\mu)^k.$$

□

As a byproduct of the proof (more precisely, of Proposition (3.2) and Lemma (3.3)), we get:

Corollary 3.1

$$(L+\lambda)^{-1} = (I - T_\lambda)^{-1} H_\lambda^{-1},$$

with $(I - T_\lambda)^{-1} : L^p(E) \to L^p(E)$ *bounded with a bound of the norm independant of* λ*, for all* $1 \leq p \leq \infty$*.*

We could hope to deduce from Theorem (3.1) that e^{-tL} is uniformly bounded on all the L^p spaces, by an argument similar to the Hille-Yosida Theorem. In particular, the Hille-Yosida-Phillips Theorem tells us that the bound

$$||(L+\lambda)^{-k}|| \leq \frac{C}{\lambda^k}, \forall k \in \mathbb{N},$$

with C independant of λ and k, is necessary and sufficient to obtain e^{-tL} uniformly bounded. The issue here is that applying Theorem (3.1) directly yields:

$$||(L+\lambda)^{-k}|| \leq \frac{C^k}{\lambda^k}, \forall k \in \mathbb{N},$$

i.e. the constant is not independant of k. In fact, applying the method of Theorem (3.1) in a less naïve way would in fact yield:

$$||(L+\lambda)^{-k}|| \leq \frac{Ck}{\lambda^k}, \forall k \in \mathbb{N},$$

i.e. the growth of the constant is linear in k and not exponential.

We do not know if we can obtain that e^{-tL} is uniformly bounded in the general setting of Theorem (3.1). However, the operator that we want to consider *in fine* is $\vec{\Delta} = \bar{\Delta} + Ric$, and this additional information will allow us to use the idea, coming from Sikora [38], that a Gaussian off-diagonal estimate for $e^{-t\vec{\Delta}}$ follows from a suitable on-diagonal estimate. Therefore, our goal is to obtain first on-diagonal bounds for e^{-tL}, i.e. estimates for $||e^{-tL}||_{2,\infty}$, and for this, following an idea of Sikora in [38], we can try to prove bounds for $||(L+\lambda)^{-k}||_{2,\infty}$. The point is that the bound needed on $||(H-R_-+\lambda)^{-k}||_{2,\infty}$ need not be independant of k, so Theorem (3.1) should be enough to prove it! We follow this path in the next section.

Remark 3.2 *Of course, at the end, if we succeed in proving the Gaussian estimate for $e^{-t\vec{\Delta}}$, $e^{-t\vec{\Delta}}$ will be uniformly bounded on all the L^p spaces.*

3.2 On-diagonal upper bounds

The next Proposition is a slight generalisation of Sikora's idea:

Proposition 3.3 *Let X be a measurable metric space. Let L be a self-adjoint, positive unbounded operator on $L^2(X)$, and let $1 < p < \infty$. Assume that the semigroup e^{-tL} is analytic bounded on $L^p(X)$ (it is necessarily the case if $p = 2$). The following statements are equivalent:*

1. *There exists a constant C such that for all $t > 0$,*

$$||e^{-tL}||_{p,\infty} \leq \frac{C}{t^{n/2p}}.$$

2. *For an (for all) $\alpha > n/2p$, there exists a constant $C(p,\alpha)$ such that*

$$||(L+\lambda)^{-\alpha}||_{p,\infty} \leq C(p,\alpha)\lambda^{-\alpha+n/2p},\ \forall\lambda > 0.$$

Proof of Proposition (3.3):
First, notice that

$$||(L+\lambda)^{-\alpha}||_{p,\infty} \leq C(p,\alpha)\lambda^{-\alpha+n/2p},\ \forall\lambda > 0$$

can be rewritten as

$$||(I+tL)^{-\alpha}||_{p,\infty} \leq C(p,\alpha)t^{-n/2p},\ \forall t > 0.$$

2) $\Rightarrow$ 1): since e^{-tL} is analytic bounded on L^p, by Proposition (2.2) there is a constant C such that:

$$||(I+tL)^{\alpha}e^{-tL}||_{p,p} \leq C,\ \forall t > 0.$$

We then write $e^{-tL} = (I+tL)^{-\alpha}\big((I+tL)^{\alpha}e^{-tL}\big)$ to obtain the result.

1) $\Rightarrow$ 2): we have

$$(L+\lambda)^{-\alpha} = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty e^{-\lambda t}e^{-tL}t^{\alpha-1}dt,$$

so that

$$||(L+\lambda)^{-\alpha}||_{p,\infty} \leq \frac{1}{\Gamma(\alpha+1)}\int_0^\infty e^{-\lambda t}||e^{-tL}||_{p,\infty}t^{\alpha-1}dt.$$

Using the hypothesis, we obtain:

$$||(L+\lambda)^{-\alpha}||_{p,\infty} \leq \frac{1}{\Gamma(\alpha+1)}\int_0^\infty e^{-\lambda t}t^{\alpha-n/2p-1}dt = \frac{1}{\Gamma(\alpha+1)}\lambda^{-\alpha+n/2p}\int_0^\infty e^{-u}u^{\alpha-n/2p-1}du.$$

Since $\alpha - n/2p > 0$, the integral $\int_0^\infty e^{-u}u^{\alpha-n/2p-1}du$ converges, hence the result.

$\square$

We will use both sides of the equivalence. First, we apply this to H (which, by Corollary (2.1), satisfies $||e^{-tH}||_{p,\infty} \leq \frac{C}{t^{n/2p}}$ and which is analytic bounded on L^p for $1 < p < \infty$ by Corollary (2.1)), to get:

Corollary 3.2 *For all $1 \leq p \leq \infty$ and $\alpha > n/2p$, there exists a constant $C(p, \alpha)$ such that*

$$||H_\lambda^{-\alpha}||_{p,\infty} \leq C(p,\alpha)\lambda^{-\alpha+n/2p},\ \forall \lambda > 0.$$

We now use the other side of the equivalence in Proposition (3.3) (i.e. a bound on the resolvent implies a bound on the semigroup) to prove the following Theorem, which is our main result in this section:

Theorem 3.2 *Let (M^n, g) be an complete Riemannian manifold which satisfies the n-Sobolev inequality, and assume that R_- is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$. We also assume that $L := H - R_- = \nabla^*\nabla + R_+ - R_-$, acting on the sections of a fibre bundle $E \to M$, is strongly positive. Then we have the following on-diagonal estimate: there is a constant C such that*

$$||e^{-tL}||_{2,\infty} \leq \frac{C}{t^{n/4}},\ \forall t > 0.$$

Proof of Theorem (3.2):
In this proof, we write L^q for $L^q(E)$. By Proposition (3.3), it is enough to prove the estimate:

$$||(L+\lambda)^{-N}||_{2,\infty} \leq \frac{C_N}{\lambda^{N-n/4}},\ \forall \lambda > 0, \tag{1}$$

for an $N > n/4$. We use the fact that for all $1 \leq p \leq \infty$, we have $(L+\lambda)^{-1} = (I - T_\lambda)^{-1}H_\lambda^{-1}$ on L^p, where $(I - T_\lambda)^{-1}$ is bounded on all the L^p spaces, with a bound for the norm independant of $\lambda \geq 0$ (c.f. Corollary (3.1)). Let $k = \lfloor n/4 \rfloor = \lfloor \frac{1}{2}/\frac{2}{n} \rfloor$. We will show the estimate (1) for $N = k+1$.

First case: $\frac{n}{4} \notin \mathbb{N}$
We want to show the estimate $||(L+\lambda)^{-k-1}||_{2,\infty} \leq \frac{C}{\lambda^{(k+1)-n/4}}$, $\forall \lambda > 0$. Define $p > \frac{n}{2}$ by:

$$\frac{1}{p} = \frac{1}{2} - k\frac{2}{n}.$$

By the mapping property (Theorem (2.1)),

$$H_\lambda^{-1} : L^r \longrightarrow L^s,\ \frac{1}{s} = \frac{1}{r} - \frac{2}{n},\ \forall r < \frac{n}{2},$$

with a norm bounded independantly of λ. Using the fact that $(I - T_\lambda)^{-1}$ is bounded on all the L^p spaces, with a bound for the norm independant of $\lambda \geq 0$, we get that

$$(L+\lambda)^{-k} : L^2 \longrightarrow L^p$$

is bounded uniformly in $\lambda \geq 0$. Since $\frac{n}{2p} < 1$, we have by Corollary (3.2):

$$H_\lambda^{-1} : L^p \longrightarrow L^\infty,$$

with

$$||H_\lambda^{-1}||_{p,\infty} \leq C\lambda^{-1+\frac{n}{2p}},$$

so that:

$$||(L+\lambda)^{-k-1}||_{2,\infty} \leq C(k)\lambda^{-1+\frac{n}{2p}} = \frac{C(k)}{\lambda^{k+1-n/4}}.$$

Second case: $\frac{n}{4} \in \mathbb{N}$ hence $k = \frac{n}{4}$. We write $H_\lambda^{-1} = H_\lambda^{-\alpha}H^{-1+\alpha}$, where $\alpha \in (0,1)$. Then by Proposition (3.1), $||H_\lambda^{-1+\alpha}||_{2,2} \leq \frac{1}{\lambda^{1-\alpha}}$, and

$$H_\lambda^{-\alpha} : L^2 \longrightarrow L^q,\ \frac{1}{q} = \frac{1}{2} - \alpha\frac{2}{n}$$

is bounded with a norm bounded independantly of $\lambda > 0$. This time, we define $p > \frac{n}{2}$ by:

$$\frac{1}{p} = \frac{1}{2} - (k-1+\alpha)\frac{2}{n}.$$

We get:

$$||(L+\lambda)^{-k}||_{2,p} \leq ||(L+\lambda)^{-(k-1)}||_{q,p}||(I-T_\lambda)^{-1}||_{q,q}||H_\lambda^{-\alpha}||_{2,q}||H_\lambda^{-(1-\alpha)}||_{2,2} \leq \frac{C}{\lambda^{1-\alpha}},$$

Therefore, using that $||H_\lambda^{-1}||_{p,\infty} \leq C\lambda^{-1+\frac{n}{2p}}$ and $||(I-T_\lambda)^{-1}||_{\infty,\infty} \leq C$ independant of λ, we obtain:

$$||(L+\lambda)^{-k-1}||_{2,\infty} \leq \frac{C_k}{\lambda^{(1-n/2p)+(1-\alpha)}}.$$

But $\frac{n}{2p} = \frac{n}{4} - (k-1+\alpha)$, which yields what we want.

□

3.3 Pointwise estimates of the Heat Kernel on 1-forms

Let us denote by $\vec{p_t}$ the Heat Kernel of the Hodge Laplacian on 1-forms; for all $x, y \in M$, $\vec{p_t}(x,y)$ is a linear morphism from T_y^*M to T_x^*M. A consequence of Sikora's work (Theorem 4 in [38]) is:

Theorem 3.3 *Let M be a complete Riemannian manifold. If we have the on-diagonal estimates:*

$$||e^{-t\vec{\Delta}}||_{2,\infty} \leq \frac{C}{t^{n/4}},\ \forall t > 0,$$

then we have a Gaussian-type estimate for $e^{-t\vec{\Delta}}$: for all $\delta > 0$, there exists a constant C such that

$$||\vec{p_t}(x,y)|| \leq \frac{C}{t^{n/2}} \exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right),\ \forall x, y \in M,\ \forall t > 0.$$

We can generalize a little bit this result: we can add a non-negative potential V to the Hodge Laplacian. Theorem 6 in Sikora generalizes in the following way (which is almost contained in Sikora's paper, and is easy to prove):

Definition 3.1 *Let L be a self-adjoint positive definite operator acting on a metric measured space X. We say that the* ***Gaffney-Davies estimates*** *hold for L if:*

$$|\langle e^{-tL} f_1, f_2\rangle| \leq Ce^{-\frac{r^2}{4t}}||f_1||_2||f_2||_2,\ \forall t > 0,$$

whenever f_n is continuous and supp$(f_n) \subset B(x_n, r_n)$ for $n = 1, 2$, and $0 \leq r < d(x_1, x_2) - (r_1 + r_2)$.

Proposition 3.4 *Let M be a complete Riemannian manifold, and let $L := \vec{\Delta} + V$, with $V \in C^\infty$ a non-negative potential. Then the Gaffney-Davies estimates hold for L.*

We have the following slight generalisation of Theorem (3.3), consequence of Theorem (3.3), together with Proposition (3.4):

Corollary 3.3 *Let M be a complete Riemannian manifold, and let $V \in C^\infty$ be a non-negative potential. Denote by $\vec{p_t}^V$ the kernel of $e^{-t(\vec{\Delta}+V)}$. If we have the on-diagonal estimate:*

$$||e^{-t(\vec{\Delta}+V)}||_{2,\infty} \leq \frac{C}{t^{n/4}},\ \forall t > 0,$$

then we have a Gaussian-type estimate for $e^{-t(\vec{\Delta}+V)}$: for all $\delta > 0$, there exists a constant C such that

$$||\vec{p_t}^V(x,y)|| \leq \frac{C}{t^{n/2}} \exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right),\ \forall x, y \in M,\ \forall t > 0.$$

The bound that we obtain is not exactly what is usually called a Gaussian estimate on $\vec{p_t}$; indeed, a Gaussian estimate on $\vec{p_t}$ is a bound of the following type:

$$||\vec{p_t}(x,y)|| \leq \frac{C}{V(x,t^{1/2})} \exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \forall x,y \in M, \forall t > 0.$$

The problem comes from the term $V(x,t^{1/2})$, which may not behave like $t^{-n/2}$. Indeed, when M satisfies a n-Sobolev inequality, we only have the lower bound (proved in [7] and [1]):

$$V(x,R) \geq CR^n, \forall R > 0, \forall x \in M,$$

which implies by the way that $n \geq dim(M)$. For example the Heisenberg group $\mathbb{H}_1$ is a manifold of dimension 3 which satisfies a 4-Sobolev inequality but whose volume of geodesic balls satisfies:

$$V(x,R) \approx R^3 \text{ if } R \leq 1,$$

and

$$V(x,R) \approx R^4 \text{ if } R \geq 1.$$

Definition 3.2 *Let M be a complete Riemannian manifold of dimension m, which satisfies a n-Sobolev inequality. We say that the volume growth of M is* ***compatible with the Sobolev dimension*** *if there is a constant C such that:*

$$V(x,R) \leq CR^n, \forall x \in M, \forall R \geq 1.$$

Definition 3.3 *We say that M satisfies a* ***relative Faber-Krahn inequality*** *of exponent n if there is a constant C such that for every $x \in M$ and $R > 0$, and every non-empty subset $\Omega \subset B(x,R)$,*

$$\lambda_1(\Omega) \geq \frac{C}{R^2}\left(\frac{|B(x,R)|}{|\Omega|}\right)^{2/n},$$

where $\lambda_1(\Omega)$ is the first eigenvalue of Δ on Ω for the DIrichlet boudary conditions.

We also recall two classical definitions:

Definition 3.4 *For $x \in M$ and $R > 0$, denote by $V(x,R)$ the volume of the geodesic ball centered in x, of radius R. We say that M satisfies the* ***Doubling Property*** *if there is a constant C such that*

$$V(x,2R) \leq CV(x,R), \forall x \in M, \forall R > 0.$$

Definition 3.5 *We say that M satisfies the* ***scaled Poincaré inequalities*** *if there exists a constant C such that for every ball $B = B(x,r)$ and for every function f with f, ∇f locally square integrable,*

$$\int_B |f - f_B|^2 \leq Cr^2 \int_B |\nabla f|^2.$$

It is proved in [24] that the relative Faber-Krahn inequality is equivalent to the Doubling Property together with the following Gaussian upper bound of the heat kernel:

$$p_t(x,y) \leq \frac{C}{V(x,\sqrt{t})} exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \forall t > 0, \forall x,y \in M.$$

We have the following property, which is not new but whose proof is given for the reader's convenience:

Proposition 3.5 *Let M be a complete Riemannian manifold of dimension m, which satisfies a n-Sobolev inequality, and whose Ricci curvature is bounded from below. If the volume growth of M is compatible with the Sobolev dimension, then M satisfies a relative Faber-Krahn inequality of exponent n.*

Proof of Proposition (3.5):
We first explain why we have a relative Faber-Krahn inequality for balls of small radius. The hypothesis that the Ricci curvature is bounded from below implies by Theorem (3.1) in [36] that we can find a constant μ_m such that each ball $B(x,R)$ with $R \leq 1$ satisfies the m-Sobolev inequality with Sobolev constant $\leq \mu_m$, and $V(x,R) \leq CR^m$ for $R \leq 1$. Now, it is proved in [7] that the m-Sobolev inequality for $B(x,R)$ is equivalent to a Faber-Krahn inequality for $B(x,R)$, with Faber-Krahn constant Γ_m that can be estimated in term of μ_m. Thus, each ball $B(x,R)$ satisfies a Faber-Krahn inequality with a constant Γ_m that does depend neither on $x \in M$ nor on $R \leq 1$, i.e. for all open subset $\Omega \subset B(x,R)$,

$$\lambda_1(\Omega) \geq \frac{\Gamma_m}{|\Omega|^{\frac{2}{m}}}.$$

But since $m \leq n$ and $\frac{|B(x,R)|}{|\Omega|} \geq 1$, $\left(\frac{|B(x,R)|}{|\Omega|}\right)^{\frac{2}{n}} \leq \left(\frac{|B(x,R)|}{|\Omega|}\right)^{\frac{2}{m}}$. Thus, using that $|B(x,R)| \leq CR^m$ for $R \leq 1$, we get :

$$\lambda_1(\Omega) \geq \frac{C}{|\Omega|^{\frac{2}{m}}} \geq \frac{C}{R^2}\left(\frac{|B(x,R)|}{|\Omega|}\right)^{\frac{2}{m}} \geq \frac{C}{R^2}\left(\frac{|B(x,R)|}{|\Omega|}\right)^{\frac{2}{n}},$$

thus we have a relative Faber-Krahn inequality of exponent n for balls of radius ≤ 1.

For balls of radius ≥ 1: again, by [7], since a n-Sobolev inequality holds on M, M satisfies a Faber-Krahn inequality of exponent n, i.e. for all open subset $\Omega \subset M$,

$$\lambda_1(\Omega) \geq \frac{C}{|\Omega|^{\frac{2}{n}}}.$$

The hypothesis $|B(x,R)| \leq CR^n$, for $R \geq 1$ then implies:

$$\lambda_1(\Omega) \geq \frac{C}{|\Omega|^{\frac{2}{n}}} \geq \frac{C}{R^2}\left(\frac{|B(x,R)|}{|\Omega|}\right)^{\frac{2}{n}}.$$

□

Example 3.1 *The Heisenberg group $\mathbb{H}_1$ satisfies a relative Faber-Krahn inequality of exponent 4; in fact, it even satisfies the scaled Poincaré inequalities and the Doubling Property, which is equivalent (by the work of Grigor'yan [23] and Saloff-Coste [35]) to the conjunction of a Gaussian upper and lower bound for the heat kernel.*

Every manifold with $Ric \geq 0$ (or more generally, with $Ric \geq 0$ outside a compact set, finite first Betti number and only one end, c.f. Theorem 1.1 in [25]) satisfies the scaled Poincaré inequalities, and thus a relative Faber-Krahn inequality of exponent $dim(M)$.

Taking into account what we have obtained in Theorem (3.2), we get one of the main results of our paper:

Theorem 3.4 *Let (M,g) be a complete Riemannian manifold which satisfies the n-Sobolev inequality, and whose negative part of the Ricci tensor Ric_- is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$. We also assume that the Hodge Laplacian on 1-forms $\vec{\Delta}$ is strongly positive, that the volume growth of M is compatible with the Sobolev dimension, and that Ric is bounded from below.*
Then the Gaussian estimate is valid for $e^{-t\vec{\Delta}}$: for all $\delta > 0$, there exists a constant C such that

$$||\vec{p_t}(x,y)|| \leq \frac{C}{V(x,t^{1/2})}\exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \forall x,y \in M, \forall t > 0.$$

Proof of Theorem (3.4):

By the result of Theorem (3.2), we can apply Theorem (3.3) to the Hodge Laplacian; given that the volume growth of M is compatible with the Sobolev dimension, the estimate of Theorem (3.3) writes:

$$||\vec{p_t}(x,y)|| \leq \frac{C}{V(x,t^{1/2})}\exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \forall x,y \in M, \forall t \geq 1.$$

The fact that M satisfies a relative Faber-Krahn inequality implies:

$$|p_t(x,y)| \leq \frac{C}{V(x,t^{1/2})}\exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \forall x,y \in M, \forall t > 0.$$

But the hypothesis that the Ricci curvature is bounded from below then implies the Gaussian estimate of $||\vec{p_t}(x,y)||$ for small times:

$$||\vec{p_t}(x,y)|| \leq \frac{C}{V(x,t^{1/2})}\exp\big(-\frac{d^2(x,y)}{(4+\delta)t}\big), \forall x,y \in M, \forall t \leq 1.$$

Indeed, this comes from the fact that we have the domination (proved in [28]):

$$||p_t(x,y)|| \leq e^{-t(\Delta - Ric_-)}(x,y) \leq e^{-t(\Delta - C)}(x,y)$$

if $Ric \geq -C$.

□

From [8], we know that if $n > 4$, $Ker_{H_0^1}(\vec{\Delta}) = Ker_{L^2}(\vec{\Delta})$. Recall that by the Sobolev inequality, $Ker_{H_0^1}(\vec{\Delta}) \hookrightarrow Ker_{L^{\frac{2n}{n-2}}}(\vec{\Delta})$. Taking this into account, we get the following:

Corollary 3.4 *Assume $n > 4$.*
Let (M^m, g) be an m-dimensional complete Riemannian manifold which satisfies the n-Sobolev inequality, and whose negative part of the Ricci tensor Ric_- is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$. We also assume that for some $k \in [2, \frac{2n}{n-2}]$, $Ker_{L^k}(\vec{\Delta}) = \{0\}$, and that the volume growth of M is compatible with the Sobolev dimension.
Then the Gaussian estimate is valid for $e^{-t\vec{\Delta}}$: for all $\delta > 0$, there exists a constant C such that

$$||\vec{p_t}(x,y)|| \leq \frac{C}{V(x,t^{1/2})}\exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \forall x,y \in M, \forall t > 0.$$

Also, for the case with a potential, if we apply Theorem (3.2) and Corollary (3.3), we get:

Corollary 3.5 *Let (M^m, g) be an m-dimensional complete Riemannian manifold which satisfies the $Sobolev_n$ inequality, and whose negative part of the Ricci tensor Ric_- is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$. Let $V \in C^\infty$ be a non-negative potential such that $\vec{\Delta} + V$ is strongly positive. Assume also that the volume growth of M is compatible with the Sobolev dimension and that Ric is bounded from below. Denote by $\vec{p_t}^V$ the kernel of $e^{-t(\vec{\Delta}+V)}$.*
Then we have a Gaussian estimate for $e^{-t(\vec{\Delta}+V)}$: for all $\delta > 0$, there exists a constant C such that

$$||\vec{p_t}^V(x,y)|| \leq \frac{C}{V(x,\sqrt{t})}\exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \forall x,y \in M, \forall t > 0.$$

4 Applications

The Gaussian estimate on the Heat Kernel on 1-forms has a certain number of consequences, which we decribe now.

4.1 Estimates on the gradient of the Heat kernel on functions and scaled Poincaré inequalities

First, Coulhon and Duong (p. 1728-1751 in [15]) have noticed that a Gaussian estimate on the heat kernel on 1-forms -in fact, a Gaussian estimate on the heat kernel on exact 1-forms is enough- leads to the following estimate for the gradient of the heat kernel on functions:

$$|\nabla_x p_t(x,y)| \leq \frac{C}{\sqrt{t}V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \forall t>0, \forall x,y \in M,$$

which, when the on-diagonal Gaussian upper bound for the Heat Kernel on functions and the Doubling Property hold, yields the Gaussian lower bound for the Heat Kernel on functions:

$$p_t(x,y) \geq \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \forall t>0, \forall x,y \in M.$$

In addition, if M satisfies a n-Sobolev inequality and if the volume growth of M is compatible with the Sobolev dimension, we know from Proposition (3.5) that M satisfies a relative Faber-Krahn inequality of exponent n, and this implies by the work of Grigor'yan ([24]) that we have the corresponding upper-bound for the heat kernel on functions:

$$p_t(x,y) \leq \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \forall t>0, \forall x,y \in M$$

But we know from the work of Saloff-Coste and Grigor'yan ([35] and [23]) that the two-sided Gaussian estimates for the Heat Kernel on functions are equivalent to the conjonction of the scaled Poincaré inequalities and the Doubling Property.
Thus we have proved the following theorem, which extends similar results for manifolds with $Ric \geq 0$:

Theorem 4.1 *Let (M^m, g) be an m-dimensional complete Riemannian manifold which satisfies the n-Sobolev inequality, and whose negative part of the Ricci tensor Ric_- is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$. We assume that the Hodge Laplacian on 1-forms $\vec{\Delta}$ is strongly positive. We also assume that the volume growth of M is compatible with the Sobolev dimension, and that the Ricci curvature is bounded from below.*
Then we have the following estimates on the heat kernel on functions:

$$|\nabla_x p_t(x,y)| \leq \frac{C}{\sqrt{t}V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \forall t>0, \forall x,y \in M,$$

$$\frac{c}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right) \leq p_t(x,y) \leq \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right), \forall t>0, \forall x,y \in M,$$

and on M the scaled Poincaré inequalities hold.

4.2 Boundedness of the Riesz transform

In [38], Sikora shows that when a Gaussian estimate holds for a semigroup e^{-tH}, where H is a self-adjoint operator, then for every local operator A such that $AL^{-\alpha}$ is bounded on L^2, $\alpha > 0$, then $AL^{-\alpha}$ is bounded on L^p for all $1 < p \leq 2$. Given this, we obtain the following result and its corollary, which are consequences of Theorem 10 in [38] (or Theorem 5.5 in Coulhon-Duong [15]), and of Theorem (3.4):

Theorem 4.2 *Let (M^m, g) be an m-dimensional complete Riemannian manifold which satisfies the n-Sobolev inequality, and whose negative part of the Ricci tensor Ric_- is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$. We assume that the Hodge Laplacian on 1-forms $\vec{\Delta}$ is strongly positive. We also assume that the volume growth of M is compatible with the Sobolev dimension, and that the Ricci curvature is bounded from below.*
Then $d^*\vec{\Delta}^{-1/2}$ is bounded on L^p for $1 < p \leq 2$.

Corollary 4.1 *Let (M^m, g) be an m-dimensional complete Riemannian manifold which satisfies the n-Sobolev inequality, and whose negative part of the Ricci tensor Ric_- is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$. We assume that the Hodge Laplacian on 1-forms $\vec{\Delta}$ is strongly positive. We also assume that the volume growth of M is compatible with the Sobolev dimension, and that the Ricci curvature is bounded from below.*
Then the Riesz transform $d\Delta^{-1/2}$ is bounded on L^p, for all $1 < p < \infty$.

Given Corollary (4.1), and using the analyticity of $e^{-t\Delta}$ on L^p for $1 < p < \infty$, we obtain the following estimates on the gradient of the Heat Kernel on functions (which is also a consequence of the gradient estimate that we have proven above):

Corollary 4.2 *There exists a constant C such that for all $1 < p < \infty$,*

$$||\nabla e^{-t\Delta}||_{p,p} \leq \frac{C}{\sqrt{t}}, \forall t > 0.$$

Remark 4.1 *One can also deduce from the estimate $|\nabla_x p_t(x,y)| \leq \frac{C}{\sqrt{t}V(x,\sqrt{t})} exp\left(-\frac{d^2(x,y)}{(4+\delta)t}\right)$ of the gradient of the heat kernel, that for all $2 \leq p \leq \infty$, the L^p estimate of the gradient holds, i.e. $||\nabla e^{-t\Delta}||_{p,p} \leq \frac{C}{\sqrt{t}}$ (see Proposition 1.10 in [4]). Given that by Theorem (4.1) the scaled Poincaré inequalities hold, we can also deduce directly from the main result of [4] that the Riesz transform is bounded on L^p for every $1 < p < \infty$.*

With our method, we also recover one of Bakry's result (Theorem 4.1 in [5]) and an slight extension of it in our context:

Theorem 4.3 *Let (M^m, g) be an m-dimensional complete manifold with Ricci curvature bounded from below by $-a$, $a \geq 0$, satisfying the n-Sobolev inequality, and whose volume growth is compatible with the Sobolev dimension.*
Then for all $1 < p < \infty$, there exists a constant $C(p)$ such that:

$$||df||_p \leq C(p)(||\Delta^{1/2} f||_p + a||f||_p), \forall f \in C_0^\infty(M).$$

If furthermore $Ric_- \in L^{n/2\pm\epsilon}$ with $\epsilon > 0$, then for all $b > 0$ and all $1 < p < \infty$, there exists a constant $C(p)$ such that:

$$||df||_p \leq C(p)(||\Delta^{1/2} f||_p + b||f||_p), \forall f \in C_0^\infty(M).$$

Proof of Theorem (4.2):
The first part can be obtained directly from Sikora's work, but the proof that we give is instructive for the rest of the paper. We have:

$$\vec{\Delta} + a = \bar{\Delta} + R_+ + (a - R_-),$$

and by hypothesis $a - R_- \geq 0$, so that $\vec{\Delta} + a$ is strongly positive. By Corollary (3.5), $e^{-t(\vec{\Delta}+a)}$ satisfies a Gaussian upper-estimate. Since we obviously have the commutation:

$$de^{-t(\Delta+a)} = e^{-t(\vec{\Delta}+a)}d,$$

using the formula: $H^{-1/2} = \frac{1}{\Gamma(3/2)} \int_0^\infty e^{-tH} \frac{dt}{\sqrt{t}}$,
we deduce that

$$d(\Delta + a)^{-1/2} = (\vec{\Delta} + a)^{-1/2} d.$$

We can then apply Sikora's argument (the proof of Theorem 10 in [38]), to get that $d(\Delta + a)^{-1/2}$ is bounded on L^p. But by an argument of Bakry, it is equivalent to the inequality:

$$||df||_p \leq C(p)(||\Delta^{1/2} f||_p + a||f||_p), \forall f \in C_0^\infty(M).$$

For the case where $Ric_- \in L^{n/2\pm\epsilon}$: for every $b > 0$, $\vec{\Delta} + b$ is strongly positive. Indeed, let us define $H = \bar{\Delta} + Ric_+ + b$, so that $\vec{\Delta} + b = H - Ric_-$. Then if H_0^1 is the space associated to H, we have $Ker_{H_0^1}(\vec{\Delta} + b) = \{0\}$. In fact, since $b \neq 0$, $H_0^1 \hookrightarrow L^2$ continuously. But $\vec{\Delta} + b$ has a spectral gap, so $Ker_{L^2}(\vec{\Delta} + b) = \{0\}$, which proves that $\vec{\Delta} + b$ is strongly positive. We can thus apply Corollary (3.5) to get that $e^{-t(\vec{\Delta}+b)}$ satisfies a Gaussian estimate. We then conclude as in the first case.

□

5 Boundedness of the Riesz transform in the range $1 < p < n$

As announced in the introduction, we now remove the hypothesis of strong positivity. We are mainly inspired by the perturbation technique developped by Carron in [10]. This section is devoted to the proof of the following result:

Theorem 5.1 *Assume $n > 3$.*
Let (M^m, g) be an m-dimensional complete Riemannian manifold which satisfies the n-Sobolev inequality, and whose negative part of the Ricci tensor Ric_- is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$. We also assume that the Ricci curvature is bounded from below, and that the volume growth of M is compatible with the Sobolev dimension.
Then for every $1 < p < n$, the Riesz transform is bounded on L^p on M.

Remark 5.1 *The hypotheses that we have made imply (by Proposition (3.5)) that M satisfies the relative Faber-Krahn inequality of exponent n, which is equivalent to the conjunction of the Doubling Property and of the Gaussian upper-estimate on p_t, i.e.*

$$p_t(x,y) \leq \frac{C}{V(x,\sqrt{t})} e^{-\frac{d^2(x,y)}{(4+\delta)t}}, \, \forall t > 0, \forall x, y \in M.$$

And we know by [14] that all this implies that the Riesz transform is bounded on M for all $1 < p \leq 2$. What we prove below is that the Riesz transform is bounded on L^p for every $\frac{n}{n-1} < p < n$, which is thus enough to get the result.

The proof of this result is by a perturbation argument: using ideas of [10], we will show that if $V \in C_0^\infty$ is non-negative, then $d(\Delta + V)^{-1/2} - d\Delta^{-1/2}$ is bounded on L^p for $\frac{n}{n-1} < p < n$. Then we will prove that if V is chosen such that $\vec{\Delta} + V$ be strongly positive, $d(\Delta + V)^{-1/2}$ is bounded on L^p for $\frac{n}{n-1} < p < n$. The following Lemma will then conclude the proof of Theorem (5.1):

Lemma 5.1 *Let (M, g) be a complete Riemannian manifold which satisfies the n-Sobolev inequality, and whose negative part of the Ricci tensor is in $L^{n/2}$.*
Then we can find a non-negative potential $V \in C_0^\infty$ such that $\vec{\Delta} + V$ is strongly positive.

Proof of Lemma (5.1):
If we write $\vec{\Delta} + V = (\bar{\Delta} + W_+) - W_- = H - W_-$, and $A := H^{-1/2} W_- H^{-1/2}$, then by the definition of strong positivity, $\vec{\Delta} + V$ is strongly positive if and only if $||A||_{2,2} < 1$. Moreover, by Lemma (2.2), we have $||A||_{2,2} \leq C||W_-||_{n/2}$, where C is independant of the chosen potential $V \leq 0$. Therefore, it is enough to take V such that $||(V - Ric_-)_-||_{\frac{n}{2}} < \frac{1}{C}$, which is possible since $Ric_- \in L^{n/2}$.

□

5.1 A perturbation result

Our aim here is to prove:

Theorem 5.2 *Assume $n > 3$.*
Let (M^m, g) be an m-dimensional complete Riemannian manifold which satisfies the $Sobolev_n$ inequality, whose Ricci curvature is bounded from below. Let $V \in C_0^\infty$ be a non-negative potential.
Then for every $\frac{n}{n-1} < p < n$, $d(\Delta + V)^{-1/2} - d\Delta^{-1/2}$ is bounded on L^p on M.

The proof is an adaptation of the proof in [10]. To adapt these ideas to the case of Schrödinger operator with non-negative potential, we will need some preliminary results. First, we recall a "classical" result:

Proposition 5.1 *Let $V \in C_0^\infty$ be non-negative, and let Ω be a smooth, open, relatively compact subset. Let Δ_D be the Laplacian with Dirichlet conditions on Ω.*
Then the Riesz transforms $d(\Delta_D + V)^{-1/2}$ and $d\Delta_D^{-1/2}$ are bounded on L^p for $1 < p < \infty$.

We also recall the next Lemma and its proof from [10] :

Lemma 5.2 *Let (M, g) be a complete Riemannian manifold with Ricci curvature bounded from below, and $V \in C_0^\infty$ be a non-negative potential.*
Then for all $1 < p < \infty$, there is a constant C such that

$$||df||_p \leq C(||\Delta f||_p + ||f||_p),\ \forall f \in C_0^\infty(M),$$

and

$$||df||_p \leq C(||(\Delta + V)f||_p + ||f||_p),\ \forall f \in C_0^\infty(M).$$

Proof of Lemma (5.2):
By Theorem 4.1 in [5], the local Riesz transform is bounded on the L^p for $1 < p < \infty$, i.e. we have the following inequality for $a \geq 0$ sufficently large:

$$||df||_p \leq C(||\Delta^{1/2} f||_p + a||f||_p),\ \forall f \in C_0^\infty(M).$$

We then use the fact that for all $1 < p < \infty$, there exists a constant C such that:

$$||\Delta^{1/2} f||_p \leq C\sqrt{||\Delta f||_p ||f||_p} \leq \frac{C}{2}(||\Delta f||_p + ||f||_p).$$

A proof of this inequality can be found in [17].
For the case with a potential, we have $||(\Delta + V)f||_p + a||f||_p \geq ||\Delta f||_p - ||V||_\infty ||f||_p + a||f||_p$. Taking $a > ||V||_\infty$, we get the result.

□

Proof of Theorem (5.2):
Let $p \in (\frac{n}{n-1}, n)$.
We follow the proof of Carron in [10]. We define $L_0 := \Delta + V$, $L_1 := \Delta$; we take K_1 smooth, compact containing the support of V, and K_2, K_3 smooth, compact such that $K_1 \subset\subset K_2 \subset\subset K_3$. We also denote $\Omega := M \setminus K_1$. Let (ρ_0, ρ_1) a partition of unity such that $supp \rho_0 \subset \Omega$ and $supp \rho_1 \subset K_2$. We also take ϕ_0 and ϕ_1 to be C^∞ non-negative functions such that $supp \phi_0 \subset \Omega$, $supp \phi_1 \subset K_3$ and such that $\phi_i \rho_i = \rho_i$. Moreover, we assume that $\phi_1|_{K_2} = 1$.
We define $H_0 := \Delta + V$ with Dirichlet boundary conditions on K_3, and $H_1 := \Delta$ with Dirichlet boundary conditions on K_3. Then, following Carron, we construct parametrices for $e^{-t\sqrt{L_1}}$ and $e^{-t\sqrt{L_0}}$: the one for $e^{-t\sqrt{L_1}}$ is defined by

$$E_t^1(u) := \phi_1 e^{-t\sqrt{H_1}}(\rho_1 u) + \phi_0 e^{-t\sqrt{L_1}}(\rho_0 u),$$

and the one for $e^{-t\sqrt{L_0}}$ is defined by

$$E_t^0(u) := \phi_1 e^{-t\sqrt{H_0}}(\rho_1 u) + \phi_0 e^{-t\sqrt{L_1}}(\rho_0 u).$$

Let us note that for $e^{-t\sqrt{L_0}}$, we approximate by $e^{-t\sqrt{L_1}}$ outside the compact K_3, and not by $e^{-t\sqrt{L_0}}$. Let us also remark that $E_0^1(u) = E_0^0(u) = u$, as it should. We then have:

$$e^{-t\sqrt{L_i}}(u) = E_t^i(u) - G_i\big[(-\frac{\partial^2}{\partial t^2} + L_i)E_t^i(u)\big],$$

where G_i is the Green operator on $\mathbb{R}_+ \times M$ with Dirichlet boundary condition, associated to $-\frac{\partial^2}{\partial t^2} + L_i$.

Next we have to show that the error term can be well-controled. We compute:

$$(-\frac{\partial^2}{\partial t^2} + L_1)E_t^1(u) = [L_1, \phi_0]e^{-t\sqrt{L_1}}(\rho_0 u) + [L_1, \phi_1]e^{-t\sqrt{H_1}}(\rho_1 u),$$

and

$$(-\frac{\partial^2}{\partial t^2} + L_0)E_t^0(u) = [L_0, \phi_1]e^{-t\sqrt{H_0}}(\rho_1 u) + [L_1, \phi_0]e^{-t\sqrt{L_1}}(\rho_0 u) + (L_0 - L_1)\phi_0 e^{-t\sqrt{L_1}}(\rho_0 u).$$

But $L_0 - L_1 = V$ is supported in K_1, therefore $(L_0 - L_1)\phi_0 e^{-t\sqrt{L_1}}(\rho_0 u) = 0$. Moreover, we have $[\Delta + V, \phi_i] = [\Delta, \phi_i]$, therefore $[L_0, \phi_i]e^{-t\sqrt{H_0}}(\rho_i u) = (\Delta\phi_i)(e^{-t\sqrt{H_0}}(\rho_i u)) - 2\langle d\phi_i, \nabla e^{-tH_0}(\rho_i u)\rangle$. Define $S_t^i(u) := (-\frac{\partial^2}{\partial t^2} + L_i)E_t^i(u)$. We get:

$$S_t^1(u) = [\Delta, \phi_0]e^{-t\sqrt{L_1}}(\rho_0 u) + [\Delta, \phi_1]e^{-t\sqrt{H_1}}(\rho_1 u),$$

and

$$S_t^0(u) = [\Delta, \phi_0]e^{-t\sqrt{L_1}}(\rho_0 u) + [\Delta, \phi_1]e^{-t\sqrt{H_0}}(\rho_1 u).$$

Lemma 2.4 in [10] implies:

$$||[\Delta, \phi_0]e^{-t\sqrt{\Delta}}(\rho_0 u)||_1 + ||[\Delta, \phi_0]e^{-t\sqrt{\Delta}}(\rho_0 u)||_p \leq \frac{C}{(1+t)^{n/p}}.$$

Furthermore, if $f_1(u) := [\Delta, \phi_1]e^{-t\sqrt{H_1}}(\rho_1 u) = (\Delta\phi_1)e^{-t\sqrt{H_1}}(\rho_1 u) - 2\langle d\phi_1, \nabla e^{-tH_1}(\rho_1 u)\rangle$,

and $f_0(u) := [\Delta, \phi_1]e^{-t\sqrt{H_0}}(\rho_1 u) = (\Delta\phi_1)e^{-t\sqrt{H_0}}(\rho_1 u) - 2\langle d\phi_1, \nabla e^{-tH_0}(\rho_1 u)\rangle$, we have as in [10]:

$$||f_i(u)||_1 + ||f_i(u)||_p \leq \frac{C}{(1+t)^{n/p}}||u||_p,\ \forall t > 0.$$

Indeed, if we denote $p_i^D(t, x, y)$ the heat kernel of H_i, then for F_1, F_2 disjoint compact subsets,

$$\lim_{t \to 0} p_i^D(t, ., .)|_{F_1 \times F_2} = 0 \text{ in } C^1$$

(cf [22] Lemma 3.2 and [32], Proposition 5.3). But by our hypotheses, the supports of ρ_1 and of $\Delta\phi_1$ are compact and disjoints, as are the ones of ρ_1 and $d\phi_1$. Therefore the kernels of the operators $S^i(t) := [\Delta, \phi_1]e^{-t\sqrt{H_i}}\rho_1$ are uniformly bounded as $t \to 0$. So we get:

$$||S^i(t)||_{p,\infty} \leq C,\ \forall t \in [0, 1].$$

Now, the operators H_i have a spectral gap, so $||e^{-t\sqrt{H_i}}||_{2,2} \leq e^{-ct}$, where $c > 0$. If $v \in W^{1,2}(K_3)$ is a non-negative solution of $\frac{\partial v}{\partial t} + (\Delta + V)v = 0$, then $\frac{\partial v}{\partial t} + \Delta v \leq 0$, and therefore by the parabolic maximum principle, v attains its maximum on $\{t = 0\} \cup \partial K_3$. If we take $v := e^{-t(\Delta_D + V)}1$, which is zero on ∂K_3 for $t > 0$, we get:

$$\int_{K_3} p_i(t, x, y)dy \leq 1,\ \forall x \in K_3,$$

and therefore $||e^{-tH_i}||_{\infty,\infty} \leq 1$. By duality, it is true also on L^1, and by the subordination identity we have:

$$||e^{-t\sqrt{H_i}}||_{1,1} + ||e^{-t\sqrt{H_i}}||_{\infty,\infty} \leq C.$$

Interpolating this with the L^2 bound, we get that

$$||e^{-t\sqrt{H_i}}||_{p,p} \leq Ce^{-ct},$$

for $1 < p < \infty$, where the constants C and c depend on p. Then we write for $t \geq 1$:

$$||S^i(t)u||_\infty \leq ||[\Delta, \phi_1]e^{-\frac{1}{2}\sqrt{H_i}}||_{L^p \to L^\infty} ||e^{-(t-1/2)\sqrt{H_i}}\rho_1 u||_{L^p} \leq Ce^{-ct}||u||_p.$$

Here we have used that the heat kernels $p_i^D(\frac{1}{2}, ., .)$ are C^∞. Thus we have proven:

Lemma 5.3

$$||S_t^i(u)||_1 + ||S_t^i(u)||_p \leq \frac{C}{(1+t)^{n/p}}||u||_p, \, \forall t > 0.$$

The error term, when we approximate $e^{-t\sqrt{L_i}}$ by the above parametrix is $G_i(S_t^i(u))$. We cannot control it directly, but the main argument of [10] shows that when we integrate the error term, we can control it well: more precisely, given the result of Lemma (5.3), we have the following Lemma that sums up Carron's result:

Lemma 5.4 *Assume $n > 3$.*
Let $(g_i(u))(x) := \int_0^\infty (G_i(S_t^i(u)))(t, x)dt$.
Then for any $\frac{n}{n-1} < p < n$, there is a constant C such that for all $u \in L^p$,

$$||L_i(g_i(u))||_p + ||g_i(u)||_p \leq C||u||_p.$$

Applying Lemma (5.4), we deduce that

$$||d(g_i(u))||_p \leq C||u||_p.$$

We can now finish the proof of Theorem (5.2). We use the formula

$$L_i^{-1/2} = c\int_0^\infty e^{-t\sqrt{L_i}}dt,$$

to get:

$$L_i^{-1/2}u = \phi_1 H_i^{-1/2}\rho_1 u + \phi_0 L_1^{-1/2}\rho_0 u - cg_i(u).$$

Therefore:

$$dL_1^{-1/2}u - dL_0^{-1/2}u = \big(d(\phi_1 H_1^{-1/2}\rho_1 u) - d(\phi_1 H_0^{-1/2}\rho_1 u)\big) + c\big(dg_0(u) - dg_1(u)\big).$$

(here is where we use the fact that we have taken for parametrices $e^{-t\sqrt{L_1}}$ for both operators outside a compact set). Write $d(\phi_1 H_i^{-1/2}\rho_1 u) = (d\phi_1)H_i^{-1/2}\rho_1 u + \phi_1 dH_i^{-1/2}\rho_1 u$. $(d\phi_1)H_i^{-1/2}\rho_1$ has a smooth kernel with compact support, therefore is bounded on L^p. Applying Proposition (5.1), we get that $\phi_1 dH_i^{-1/2}\rho_1$ is bounded on L^p, hence we have the result.

□

5.2 Boundedness of $d(\Delta + V)^{-1/2}$

We now show:

Theorem 5.3 *Assume $n > 3$.*
Let (M^m, g) be an m-dimensional complete Riemannian manifold which satisfies the n-Sobolev inequality, and whose negative part of the Ricci tensor is in $L^{\frac{n}{2}\pm\epsilon}$ for an $\epsilon > 0$. We also assume that the Ricci curvature is bounded from below, and that the volume growth of M is compatible with the Sobolev dimension. Let $V \in C_0^\infty$ be non-negative, such that $\vec{\Delta} + V$ is strongly positive.
Then the Riesz transform $d(\Delta + V)^{-1/2}$ is bounded on L^p for every $1 < p < n$.

Remark 5.2 *This is an analog of the result on local Riesz transforms (Theorem (4.3)).*

We first show a preliminary result:

Lemma 5.5 $(\vec{\Delta}+V)^{-1/2}d$ *is bounded on* L^p *for every* $2 \leq p < \infty$.

Proof of Lemma (5.5):
Since $\vec{\Delta}+V$ is strongly positive, we have a Gaussian upper estimate on $e^{-t(\vec{\Delta}+V)}$ by Corollary (3.5). Thus we are in position to apply Theorem 5 in [38], to get that $d^*(\vec{\Delta}+V)^{-1/2}$ is bounded on L^q for all $1 < q \leq 2$. By taking duals, $(\vec{\Delta}+V)^{-1/2}d$ is bounded on L^p for any $2 \leq p < \infty$.

□

Proof of Theorem (5.3):
First, let us note that we can restrict ourselves to the case $\frac{n}{n-1} < p < n$. Indeed, for $1 < p < 2$, since the hypotheses that we have made imply the Faber-Krahn inequality, and given the domination $e^{-t(\Delta+V)} \leq e^{-t\Delta}$, we have a Gaussian upper bound for $e^{-t(\Delta+V)}$. Thus the result of [14] shows that $d(\Delta+V)^{-1/2}$ is bounded on L^p for every $1 < p \leq 2$. So let $p \in (\frac{n}{n-1}, n)$.
The problem to get from Lemma (5.5) the boundedness of the Riesz transform $d(\Delta+V)^{-1/2}$ is that it is not true that $d(\Delta+V)^{-1/2} = (\vec{\Delta}+V)^{-1/2}d$ anymore. To circumvent this difficutly, we use again the method of [10]. We will use the following:

Lemma 5.6 *For* $1 \leq r \leq s \leq \infty$, *we have the existence of a constant* C *such that:*

$$||e^{-t(\vec{\Delta}+V)}||_{L^r \to L^s} \leq \frac{C}{t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{s})}}.$$

We postpone the proof of Lemma (5.6) until the end of this section.
Let E be the vector bundle of basis $M \times \mathbb{R}_+$, whose fiber in (t,p) is $\Lambda^1 T_p^* M$. Let G be the operator (the "Green operator") acting on sections of E, whose kernel is given by

$$G(\sigma,s,x,y) = \int_0^\infty \left[\frac{e^{-\frac{(\sigma-s)^2}{4t}} - e^{-\frac{(\sigma+s)^2}{4t}}}{\sqrt{4\pi t}} \right] \vec{p_t}^V(x,y)dt,$$

where $\vec{p_t}^V$ is the kernel of $e^{-t(\vec{\Delta}+V)}$. We can see that G satisfies:

$$\left(-\frac{\partial^2}{\partial\sigma^2} + (\vec{\Delta}_x + V)\right) G = I,$$

and that $G(\sigma,s,x,y)$ is finite if $x \neq y$ and $\sigma \neq s$ (here we use the estimate $|p_t^V(x,y)| \leq \frac{C}{t^{n/2}}$, given by Theorem (3.2) and Corollary (3.3)). We want to write, as in the proof of Theorem (5.2), for $u \in C_0^\infty(M)$:

$$e^{-t\sqrt{\vec{\Delta}+V}}du = de^{-t\sqrt{\Delta+V}}u - G\left(\left(-\frac{\partial^2}{\partial t^2} + (\vec{\Delta}+V)\right) de^{-t\sqrt{\Delta+V}}u\right). \tag{2}$$

Now, we justify formula (2) and in passing we show some estimates that will be used later. We compute:

$$\begin{aligned}\left(-\tfrac{\partial^2}{\partial t^2} + (\vec{\Delta}+V)\right) de^{-t\sqrt{\Delta+V}}u &= -d(\Delta+V)e^{-t\sqrt{\Delta+V}}u + (\vec{\Delta}+V)de^{-t\sqrt{\Delta+V}}u \\ &= -\big(e^{-t\sqrt{\Delta+V}}u\big)(dV).\end{aligned}$$

We have:

$$||e^{-t\sqrt{\Delta+V}}||_{L^p \to L^\infty} \leq \frac{C}{t^{n/2p}}, \ \forall t > 0,$$

and

$$||e^{-t\sqrt{\Delta+V}}||_{L^p\to L^p} \leq 1, \forall t>0.$$

(this comes from the domination $e^{-t(\Delta+V)} \leq e^{-t\Delta}$).
Thus if we denote $f := \left(-\frac{\partial^2}{\partial t^2} + (\vec{\Delta}+V)\right) de^{-t\sqrt{\Delta+V}}u$, we have:

Lemma 5.7

$$||f(t,.)||_1 + ||f(t,.)||_p \leq \frac{C}{(1+t)^{n/p}}||u||_p.$$

Now we show:

Lemma 5.8 $||G(f)(t,.)||_2$ *is bounded uniformly with respect to* $t>0$*, and*

$$\lim_{t\to 0} ||G(f)(t,.)||_2 = 0.$$

Proof of Lemma (5.8):
Denote $K_s(t,\sigma) := \frac{e^{-\frac{(\sigma-t)^2}{4s}} - e^{-\frac{(\sigma+t)^2}{4s}}}{\sqrt{4\pi s}}$, and $H_t(x,y)$ the kernel of e^{-tL}.

$$\begin{array}{rcl} G(f)(t,x) & = & \int G(\sigma,t,x,y)f(\sigma,y)d\sigma dy \\ & = & \int_M \int_0^\infty \int_0^\infty K_s(t,\sigma)\, H_s(x,y) f(\sigma,y)\, ds d\sigma dy \\ & = & \int_0^\infty \int_0^\infty K_s(t,\sigma)\, \left(\int_M H_s(x,y) f(\sigma,y)\, dy\right)\, ds d\sigma \\ & = & \int_0^\infty \int_0^\infty K_s(t,\sigma)\, e^{-s\sqrt{L}}(x)\, ds d\sigma \end{array}$$

Consequently,

$$||G(f)(t,.)||_2 \leq \int_0^\infty \int_0^\infty K_s(t,\sigma)\, ||e^{-s\sqrt{L}} f(\sigma,.)||_2\, ds d\sigma.$$

But we have

$$\begin{array}{rcl} ||e^{-s\sqrt{L}} f(\sigma,.)||_2 & \leq & \min\left(\frac{1}{s^{n/4}}||f(\sigma,.)||_1, ||f(\sigma,.)||_2\right) \\ & \leq & C||u||_2 \min\left(\frac{1}{s^{n/4}} \frac{1}{(1+\sigma)^{n/2}}, \frac{1}{(1+\sigma)^{n/2}}\right) \end{array}$$

Therefore,

$$||G(f)(t,.)||_2 \leq C||u||_2 \int_0^\infty \frac{1}{(1+\sigma)^{n/2}} \left(\int_0^1 \frac{e^{-\frac{(\sigma-t)^2}{4s}} - e^{-\frac{(\sigma+t)^2}{4s}}}{\sqrt{s}}\, ds + \int_1^\infty \frac{e^{-\frac{(\sigma-t)^2}{4s}} - e^{-\frac{(\sigma+t)^2}{4s}}}{s^{\frac{n}{4}+\frac{1}{2}}}\, ds\right) d\sigma$$

Since $n\geq 3$, the three integrals $\int_0^\infty \frac{d\sigma}{(1+\sigma)^{n/2}}$, $\int_0^1 \frac{ds}{\sqrt{s}}$ and $\int_1^\infty \frac{ds}{s^{\frac{n}{4}+\frac{1}{2}}}$ converge, and this yields immediately the fact that $||G(f)(t,.)||_2$ is bounded uniformly with respect to $t>0$. Furthermore, we can apply the Dominated Convergence Theorem to conclude that $\lim_{t\to 0} ||G(f)(t,.)||_2 = 0$.

□

Therefore, letting

$$\phi(t,.) := e^{-t\sqrt{\vec{\Delta}+V}}du - de^{-t\sqrt{\Delta+V}}u + G\left(\left(-\frac{\partial^2}{\partial t^2} + (\vec{\Delta}+V)\right) de^{-t\sqrt{\Delta+V}}u\right),$$

$\phi(t,.)$ satisfies:

$$\left(-\frac{\partial^2}{\partial t^2} + (\vec{\Delta}+V)\right)\phi = 0,$$

and

$$L^2 - \lim_{t\to 0} \phi(t,.) = 0.$$

This last assertion uses that $L^2 - \lim_{t\to 0} e^{-t\sqrt{\vec{\Delta}+V}} du = L^2 - \lim_{t\to 0} de^{-t\sqrt{\Delta+V}} u = du$. To justify $L^2 - \lim_{t\to 0} de^{-t\sqrt{\Delta+V}} u = du$, we can say that by the Spectral Theorem ((c) in Theorem VIII.5 in [33]), $\sqrt{\Delta+V} e^{-t\sqrt{\Delta+V}} u$ converges in L^2 for $u \in C_0^\infty(M)$; since $V \geq 0$, the Riesz transform $d(\Delta+V)^{-1/2}$ is bounded on L^2, we deduce that $de^{-t\sqrt{\Delta+V}} u$ converges in L^2, and the limit is necessarily du. Together with the fact that $\phi(t,.)$ is bounded in L^2 uniformly with respect to $t > 0$, we deduce that $\phi \equiv 0$. This proves the formula (2).

Letting $(g(u))(x) := \int_0^\infty (G(f))(t,x)dt$, we have by integration of formula (2):

$$(\vec{\Delta}+V)^{-1/2} du = d(\Delta+V)^{-1/2} u - cg.$$

By Lemma (5.7) and Lemma (5.6), we have as in [10]:

$$||g||_p \leq C||u||_p.$$

Applying Lemma (5.5), we conclude that $d(\Delta+V)^{-1/2}$ is bounded on L^p.

□

Proof of Lemma (5.6):

Let us denote $L := \vec{\Delta} + V$. If we can prove that $||e^{-tL}||_{\infty,\infty} \leq C$, $||e^{-tL}||_{1,1} \leq C$ and $||e^{-tL}||_{1,\infty} \leq \frac{C}{t^{n/2}}$, then by standard interpolation arguments we are done. The fact that $||e^{-tL}||_{\infty,\infty} \leq C$ comes from the Gaussian estimate we have on e^{-tL} (we have this by Corollary (3.5)), plus the fact that $\frac{1}{V(x,\sqrt{t})} \int_M e^{-c\frac{d^2(x,y)}{t}} dy$ is bounded uniformly in $x \in M$ and $t > 0$. Then by duality $||e^{-tL}||_{1,1} \leq C$. Moreover, by Theorem (3.2) we also have the estimate:

$$|e^{-tL}||_{2,\infty} \leq \frac{C}{t^{n/4}}, \forall t > 0.$$

By duality, we deduce

$$|e^{-tL}||_{1,2} \leq \frac{C}{t^{n/4}}, \forall t > 0,$$

and by composition

$$|e^{-tL}||_{1,\infty} \leq ||e^{-tL}||_{1,2}||e^{-tL}||_{2,\infty} \leq \frac{C^2}{t^{n/2}}, \forall t > 0.$$

□

References

[1] Akutagawa, K. Yamabe metrics of positive scalar curvature and conformally flat manifolds. *Differential Geom. Appl.*, 4:239–258, 1994. no. 3.

[2] Alexopoulos, G. An application of homogeneisation theory to harmonic analysis : Harnack inequalities and Riesz transforms on Lie groups of polynomial growth. *Canad. J. Math*, 44:691–727, 1992.

[3] Auscher, P. On necessary and sufficent conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates.

[4] Auscher, P.; Coulhon, T.; Duong, X.T.; Hofmann, S. Riesz transforms on manifolds and heat kernel regularity. *Ann. Scient. ENS Paris*, 37:911–995, 2004. no. 6.

[5] Bakry, D. Etude des Transformations de Riesz dans les variétés Riemanniennes à courbure de Ricci minorée. *Séminaire de probabilités XIX*, 351. Lecture Notes in Mathematics, 1123.

[6] Bérard, P.H. *Spectral Geometry : Direct and Inverse Problems*. Springer-Verlag, 1986. no. 1207.

[7] Carron, G. Inégalités de Faber-Krahn et conséquences. *Actes de la table ronde de géométrie différentielle en l'honneur de M. Berger*, 1:205–232, 1994.

[8] Carron, G. Une suite exacte en L^2-cohomologie. *Duke Mathematical Journal*, 95:343–371, 1998. no. 2.

[9] Carron, G. L^2-Cohomologie et inégalités de Sobolev. *Mathematische Annalen*, 314:613–639, 1999.

[10] Carron, G. Riesz Transforms on Connected Sums. *Annales de l'Institut Fourier*, 57:2329–2343, 2007.

[11] Carron, G.; Coulhon, T.; Hassell, A. Riesz transform and L^p-cohomology for manifolds with Euclidean ends. *Duke Mathematical Journal*, 133:59–93, 2006.

[12] Coulhon, T. Inégalités de Gagliardo-Nirenberg pour les semi-groupes d'opérateurs et applications. *Potential Anal.*, 1. no. 4.

[13] Coulhon, T.; Dungey, N. Riesz transform and perturbation. *J. Geometric Anal.*, 17. no.2.

[14] Coulhon, T.; Duong, X.T. Riesz Transforms for $1 \leq p \leq 2$. *Trans. Amer. Math. Soc.*, 351. no.3.

[15] Coulhon, T.; Duong, X.T. Riesz Transform and Related Inequalities on Noncompact Riemannian Manifolds. *Communications on Pure and Applied Mathematics*, LVI:0001–0024, 2003.

[16] Coulhon, T.; Li, H.Q. Estimations inférieures du noyau de la chaleur sur les variétés coniques et transformée de riesz. *Archiv der Mathematik*, 82.

[17] Coulhon, T.; Russ, E.; Tardivel-Nachef, V. Sobolev Algebras on Lie groups and Riemannian manifolds. *Amer. J. Math.*, 123:283–342, 2001.

[18] Coulhon, T.; Saloff-Coste, L.; Varopoulos, N. *Analysis and geometry on groups*. Cambridge University Press, 1993.

[19] Coulhon, T.; Zhang, Q. Large Time behavior of Heat Kernel on Forms. *J. Diff. Geometry*, 77. no.3.

[20] Davies, E.B.; Simon, B. L^p norm of noncritical Scrödinger semigroups. *J. Funct. Anal.*, 102:95–115, 1991. no. 1.

[21] Devyver, B. Finiteness of the Morse Index for Schrödinger Operators. Preprint.

[22] Dodziuk, J. Maximum Principle for Parabolic Inequalities and the Heat Flow on Open Manifolds. *Indiana Univ. Math. J.*, 32:703–716, 1983. no. 5.

[23] Grigor'yan, A. The heat equation on non-compact Riemannian manifolds. *Matem. Sbornik*, 182:55–87, 1991. Engl. Transl. Math. USSR Sb., 72, 1992, 47-77.

[24] Grigor'yan, A. Heat Kernel Upper Bounds On A Complete Non-compact Manifold. *Revista Mathematicà Iberoamericana*, 10:395–452, 1994. no. 2.

[25] Grigor'yan, A.; Saloff-Coste, L. Stability Results for Harnack Inequalities. *Ann. Institut Fourier*, 55:825–890, 2005. no. 3.

[26] Guillarmou, C.; Hassell, A. The resolvent at low energy and Riesz transform for Schrodinger operators on asymptotically conic manifolds, Part I. *Math. Annalen*, 341:859–896, 2008. no. 4.

[27] Guillarmou, C.; Hassell, A. The resolvent at low energy and Riesz transform for Schrodinger operators on asymptotically conic manifolds, Part II. *Ann. Institut Fourier*, 2008.

[28] Hess H.; Schrader R.; Uhlenbrock, D.A. Domination of semigroups and generalisation of Kato's inequality. *Duke Math. J.*, 44:893–904, 1977. no. 4.

[29] Li, H.Q. Estimations du noyau de la chaleur sur les variétés coniques et applications. *Bull. Sci. math.*, 124:365–384, 2000. no. 5.

[30] Lohoué, N. Comparaison des champs de vecteurs et des puissances du Laplacien sur une variété Riemannienne à courbure non positive. *J. Funct. Anal.*, 61.

[31] Pigola, S.; Rigoli, M.; Setti, A. *Vanishing and Finiteness Results in Geometric Analysis.* Birkhäuser, 2008.

[32] Ray, D.B.; Singer, I.M. R-Torsion and the Laplacian on Riemannian Manifolds. *Advances in Math.*, 7:145–210, 1971.

[33] Reed, M.; Simon, B. *Methods of Modern Mathematical Physics I.* Academic Press, 1978.

[34] Reed, M.; Simon, B. *Methods of Modern Mathematical Physics II.* Academic Press, 1978.

[35] Saloff-Coste, L. A note on Poincaré, Sobolev and Harnack inequalities. *Duke Math. J.*, 65:27–38, 1992. no. 2.

[36] Saloff-Coste, L. Uniformly elliptic operators on Riemannian manifolds. *J. Diff. Geometry*, 36:417–450, 1992. no. 2.

[37] Saloff-Coste, L. *Aspects of Sobolev-Type Inequalities.* Cambridge University Press, 2002.

[38] Sikora, A. Riesz transform, Gaussian bounds and the method of wave equation. *Mathematische Zeitschrift*, 247:643–662, 2004.

[39] Simon, B. Brownian Motion, L^p Properties of Schrödinger Operators and the Localization of Binding. *Journal of Functional Analysis*, 35:215–229, 1980.

[40] Stein, E.M. *Topics in Harmonic Analysis related to the Littlewood-Paley Theory.* Princeton University Press. Annals of Mathematics Studies, 63.

[41] Strichartz, R. Analysis of the Laplacian on complete Riemannian manifolds. *J. Funct. Anal.*, 52:48–79, 1983. no.1.

[42] Varopoulos, N. Hardy-Littlewood Theory for Semigroups. *Journal of Functional Analysis*, 63:240–260, 1985. no.2.