NOTES ON TWO ELEMENTARY EVOLUTIONARY GAMES

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ABSTRACT. Thus spoke the wise Queen Delores, saying, "I have studied mathematics. I will question this young man, in my tent tonight, and in the morning I will report the truth as to his pretensions".

Jurgen, James Branch Cabell (1919), Ch. XXXII: Jurgen proves it by mathematics

Evolutionary game theory [3] defines something like a functor from the classical theory of games to dynamical systems, imagining biological entities whose rate of reproduction is proportional to their success at playing the game in question. It is a beautiful and accessible subject; these notes on two interesting examples grew out of JHU's 2007 BioCalc I course, and I'd like to thank the students there for their interest and forebearance. This work was also suggested by DARPA's Fundamental Problems of Biology initiative.

1. The Battle of the Sexes

" C'est magnifique, mais ce n'est pas la guerre ..."

Marshal Pierre Bosquet, Balaclava 1854

1.1 In the evolutionary version of Richard Dawkins' toy model for marriage markets as asymmetric two-player games, the rate of reproduction of a population type is proportional to its success at beating the mean expectation for the game. Following Hofbauer and Sigmund [3 §10.2], this is expressed by a system

$$\dot{x_k} = x_k \cdot [(A\mathbf{y})_k - \mathbf{x} \cdot A\mathbf{y}] \ ,$$
$$\dot{y_k} = y_k \cdot [(B\mathbf{x})_k - \mathbf{y} \cdot B\mathbf{x}] \ .$$

of 'replicator' equations, in which there are two types of players: those of type I (males) characterized by a state vector $\mathbf{x} = (x_1, \ldots, x_n)$ in a unit simplex in some space of strategies, and type II (females) defined similarly by a vector $\mathbf{y} = (y_1, \ldots, y_m)$.

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The details of the model are specified by matrices A, B: a type I player choosing mixed strategy $\mathbf{x}$ against a type II player choosing strategy $\mathbf{y}$ receives payoff $\mathbf{x} \cdot A\mathbf{y}$, while his type II 'opponent' [the game is **not** zero-sum!] receives payoff $\mathbf{y} \cdot B\mathbf{x}$. In the simplest version of the game both types of players have two strategies:

$$\mathbf{x} \;=\; (x, 1-x),\; \mathbf{y} \;=\; (y, 1-y)\;.$$

The traditional terminology is that $x(t)$ is the proportion of 'fast' males and $y(t)$ is the proportion of 'slow' females; this choice of parameters foregrounds analogies with a Lotka-Volterra system.

The payoff matrices are

$$A = \left[\begin{array}{cc} 0 & G \\ G - E - \frac{1}{2}C & G - \frac{1}{2}C \end{array} \right]$$

and

$$B = \left[\begin{array}{cc} 0 & G - E - \frac{1}{2}C \\ G - C & G - \frac{1}{2}C \end{array} \right]\;.$$

In the four possible types of interaction, a fast ('macho'?) male in an encounter with fast female receives payoff G while the female receives $G - C$; when a fast male encounters a slow ('coy'?) female both receive payoff 0. When a slow male encounters a fast female, both players receive payoff $G - \frac{1}{2}C$, and in an encounter between a slow male and a slow female both players receive payoff $G - E - \frac{1}{2}C$. The model assumes that

$$0 \;<\; E \;<\; G \;<\; C < \; 2(G - E)\;;$$

the engagement cost E (borne equally by the players) is less than the (individual) payoff G for successful reproduction, and the total cost C of reproduction, though bigger than the payoff for a single participant, is less than the total payoff to both parties, less the total cost of engagement.

1.2 The resulting system

$$\dot{x} \;=\; x(1-x)[\tfrac{1}{2}C + (E - G)y]$$

$$\dot{y} \;=\; y(1-y)[-E + (C + E - G)x]$$

of replicator equations is completely integrable, with five critical points, hyperbolic at the corners of the unit square[1] and a more interesting elliptic point at

$$X \;=\; \frac{E}{E - G + C}\;,\; Y \;=\; \frac{C}{2(G - E)}\;.$$

[1]Corresponding to classical models: the Garden of Eden, the Summer of Love, Total War, and Christian Heaven . . .

The linearization near this critical point is defined by the Hessian or Jacobian matrix of the right-hand side of this system; it has purely imaginary eigenvalues Λ satisfying

$$\Lambda^2 = -\frac{CE(2(G-E)-C)(C-G)}{4(G-E)(C+E-G)} \;.$$

Any point in the interior of the square lies on a closed orbit, whose period approaches $T = 2\pi|\Lambda|^{-1}$ at the fixed point. The desire to understand this relatively complicated 'observable' was the initial motivation for this note.

1.3 Proposition $(E, G, C) \mapsto (X, Y)$ extends to a map

$$[E : G : C] \mapsto [EC : 2E(G-E) : C(E-G+C) : 2(G-E)(E-G+C)]$$

from $\mathbb{P}_2$ to a quadric surface $\mathbb{P}_1 \times \mathbb{P}_1 \subset \mathbb{P}_3$.

Proof: If we write $[Z_0 : Z_1 : Z_2 : Z_3]$ for the coordinates in $\mathbb{P}_3$, then evidently $X = Z_0 Z_2^{-1}$, $Y = Z_1 Z_3^{-1}$; in other words, the map above is the composition of the map from Dawkins' space of economic parameters E, G, C to psychosocial parameters X, Y followed by the Segre embedding

$$(x_0, x_1) \times (y_0, y_1) \mapsto (x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1)$$

of the quadric surface $Z_0 Z_3 = Z_1 Z_2$; for example, $[1 : 1 : 0] \mapsto [1 : 0 : 0 : 0]$ and $[0 : 1 : 1] \mapsto [1 : 1 : 0 : 1]$. $\square$

The function Λ is homogeneous of weight one in the economic parameters; its value thus depends on a choice of units. This issue is familiar from physics: similar considerations in the case of the van der Waals model for liquid-gas transitions, for example, led historically to thermodynamics' law of corresponding states [2 §6.3].

Corollary

$$\Lambda^2 \;=\; (2Z_3)^{-1}(Z_0 - Z_1)$$

in units defined by the geometric mean of C and $C - G$.

[More precisely, we have

$$\Lambda^2 \;=\; \frac{Z_0 - Z_1}{2Z_3} \cdot C(C-G) \;,$$

and in 'natural' units such that $C(C-G) = 1$, ie

$$G = C - C^{-1}$$

we can omit the factor on the right.]

2. Progressives and Conservatives

"If I can't sell it gonna sit down on it,
never catch me givin it away!"

Ruth Brown, Fantasy Records (1989)

2.1 Similar techniques can be used to analyze the much simpler game defined by payoff matrices

$$A = aH, B = bH^{\mathrm{T}}$$

with

$$H = \left[\begin{array}{cc} 0 & \eta((b-a) \\ \eta(a-b) & 0 \end{array} \right] ,$$

where $\eta(x)$ is the Heaviside function ($= 1$ if $x > 0$, $= \frac{1}{2}$ if $x = 0$, and $= 0$ otherwise; I'll assume that a and b are both positive, and that $a \neq b$ to exclude trivial cases. In this example there are again two types of players, with strategy vectors $(x, 1-x)$ and $(y, 1-y)$ as above, now interpreted as the proportion of progressive (resp. conservative) participants of type I (resp II); with payoff parameters a, b for the two types.

An encounter between a progressive and a conservative yields zero, unless the conservative receives the larger payoff. An encounter between two progressives yields payoff a for the type I player and payoff b for the type II player, and an encounter between two conservatives produces nothing for either. This is therefore a rather silly game: the progressive agrees with anything that benefits anybody, while the conservative strategy amounts to pure bullying; but because the game is not zero-sum, it is not completely trivial.

2.2 The obvious symmetries of the payoff matrices result in very simple replicator equations

$$\dot{x} \; = \; ax(1-x)\, \eta(b-a)$$

$$\dot{y} \; = \; by(1-y)\, \eta(a-b) \; .$$

One or the other of these equations is thus trivial, depending on the relative sizes of a and b: the proportion of participants whose payoff is larger (let's call them fat cats) does not change with time, while the group with lower payoff (underdogs?) become more progressive, following a logistic growth pattern; the population thus evolves toward political polarization.

References

1. R. Dawkins, **The selfish gene**, OUP (1989)
2. DL Goodstein, **States of matter**, Prentice-Hall (1975)
3. J. Hofbauer, K. Sigmund, **Evolutionary games and population dynamics**, CUP (1998)

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