A Functorial Map from Knots in Thickened Surfaces to Classical Knots and Generalisations of Parity

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Abstract

We construct a functorial map from knots in thickened surfaces to classical knots. This method uses a natural generalisation of the notion of parity: we localise the information about non-triviality of knots in surfaces (virtual knots) at crossings which allows to construct a functorial. In particular, this allows to lift all classical knot invariants to knots in thickened sufraces.

The aim of the present paper is to consturct a map (projection) from the set of knots in thickened surfaces onto the set of classical knots. This is obtained by using the *universal parity* which is a generalisation of *parity*; and can be further generalised to the *universal non-commutative parity*.

This map can be used in order to "lift" all invariants of classical knots to the realm of knots in thickened surfaces. Partially, this can be done for virtual knots, however, the map itself does not agree with stabilisation for virtual knots.

Many invariants of classical knots do not admit any evident generalisation for the case of virtual knots: in some cases, e.g. for Khovanov homology, [7], one has to revisit completely the original definition in the classical case, whence some other invariants rely on geometry and topology of the 3-space. The presence of a "right" projection usually allows not only to lift many invariants, but also to refine them in many different ways [2, 1]. For further applications of parity in other prob-

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lems of topology, see [3, 4]. The parity idea goes as follows: all classical crossings of a diagram can be naturally split into "even ones" and "odd ones" so that this way of splitting is well-behaved under Reidemeister moves. The parity can be used for construction of many new invariants, refinements of old invariants and construction of functorial maps. Here odd crossings are "responsible" for non-triviality. In the present paper, we strengthen the notion of parity to get the *universal parity* and the *the universal non-commutative parity*; they are valued in certain groups (which has its own interest) rather than in the group $\mathbb{Z}_2$, in such a way that in the case of a non-classical diagram at least one crossing turns out to be odd. The latter allows to construct a projection. Note that the proof of the main result of the present paper, the universal (commutative) parity suffices, however, the non-commutative parity construction is of its own interest. Certainly, these refinements of parity can be used for further improvements of virtual knot invariants.

The *universal commutative parity* group is nothing but the $\mathbb{Z}_2$-homology group of the underlying surface. The *non-commutative parity group* is related to the fundamental group of the underlying surface, however, the explicit presentations of these groups given in the present paper, are related with the crossings: they *localise the non-trivial information about a knot in a crossing* .

Note that in some previous works (see [2]) we have constructed a well-defined mapping from virtual knots to "virtual knots with orientable atoms" (see [2]).

Virtual knots were invented by Kauffman [5]; as a natural way of stabilising knots in thickened surfaces studied by F.Jaeger, L.Kauffman, and H.Saleur [JKS].

The reader is assumed to be familiar with the basics of virtual knots, see, e.g., [5, 6] and the parity theory (see., e.g., [2]).

Remark 1. In the sequel, all virtual diagrams are considered only up to the detour move. We call a virtual diagram *classical* if it becomes classical after an application of detour moves.

Remark 2. We shall construct a map from knots in surfaces (not links) having orientable atoms, to classical knots. Later on, we assume all atoms to be orientable, i.e. admitting a *source-sink structure*. Recall, following [8], that a four-valent graph with a formal relation of half-edges at vertices to be opposite, is said to admit a source-sink structure if its edges can be oriented in such a way that in every vertex some two opposite edges are incoming and the other two edges (which are also opposite) are emanating. Note that for a connected graph with a structure of opposite half-edges specified, if a source-sink structure exists then it is unique up to the orientation reversal for all edges.

Let K be a virtual knot diagram given by its planar diagram D. Consider the surface $S(D)$ corresponding to D. This surface is constructed in the following way. Starting with a virtual diagram D we construct a surface with boundary as follows. At every classical crossing we put a "cross" (upper Fig. 1), and at each virtual crossing we put a pair of skew lines (lower Fig. 1). Connecting these crosses and bands by non-intersecting and non-twisted bands going along arcs of the knot diagram, we get an oriented 2-manifold with boundary to be denoted by $S'(D)$.

In a natural way the diagram D can be drawn in $S'(D)$ in such a way that the arcs of the diagram (which are allowed to pass through virtual crossings) are mapped to the middle lines of bands, and classical (planar) crossings correspond to intersections of middle lines of crosses. Thus we obtain a collection of curves $\delta \subset S'(D)$. Pasting the boundary

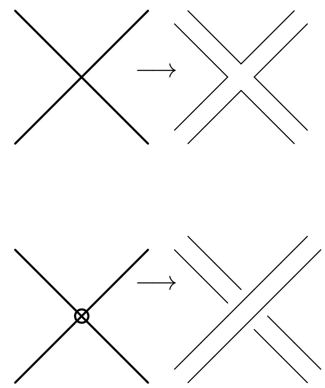


Figure 1: Local Structure of M'

components of the manifold $S'(D)$ by discs, we obtain an oriented manifold $S = S(D)$ without boundary with a collection of immersed curves δ. Note that every boundary component of $S'(D)$ corresponds to some "rotating cycle" on the graph (projection of the knot), i.e. a cycle on a four-valent graph (with opposite edge structure) where every two adjacent edges correspond to two *non-opposite* half-edges at their common vertex. Note that vertices of the knot projection correspond *to classical crossings only*, and the relation of half-edges to be opposite comes from the surface $S(D)$.

Note that the detour move does not change the surface $S(D)$ at all; neither it changes the curve inside it. The first classical Reidemeister move does not change the surface $S(D)$; neither does the third classical Reidemeister move. As for the second classical Reidemeister move, there are two cases: the *local* one which does not change the surface $S(D)$, and the (de)stabilising one where a new handle is attached in such a way that the two newborn crossings appear inside the new handle.

This is exactly the place where virtual knot theory differs from theory of knots in thickened surfaces. A band-pass presentation of a virtual knot given above can be treated as a presentation for a knot in a thickened surface; the surface is minimal by construction, and the only Reidemeister move which can change the genus of the surface is the (de)stabilising version of the second Reidemeister move.

So, we shall deal with two equivalences: the usual one (with (de)stabilisation) and the equivalence without (de)stabilisation which preserves the genus of the underlying surface.

Now, let us construct the *unversal parity group* $G(D)$. We shall use the additive notation for this group. For generators of $G(D)$ we take crossings of the diagram D, and the relations will be $2a_i = 0$ for every crossing and there will also be relations correspond to *pasted cycles.* Namely, a pasted cycle is just a rotated cycle on the 4-valent graph (shadow of the knot): the sum of crossings corresponding to any pasting cycle is zero.

Analogously, one defines the *two non-commutative parity groups*, $\{NG^1(D), NG^2(D)\}$. So far we consider these pair of groups as unordered.

The generators will coorespond to the crossings (as well as those of $G(D)$; we do not impose the relation that the square of every generator is equal to the neutral element of the group and we shall write the relations of the group multiplicatively.

The source-sink structure on the projection graph generates an orientation for *every rotating cycle.* This yields a cyclic ordering on every pasted cycle: $a_1, \dots, a_n$. For every such cycle we write down the relation $a_1 \cdots a_n = 1$.

Since we have two source-sink structure, we shall have two different sets of relations which will lead us to presentations of the groups to be denoted by NG^1 and NG^2.

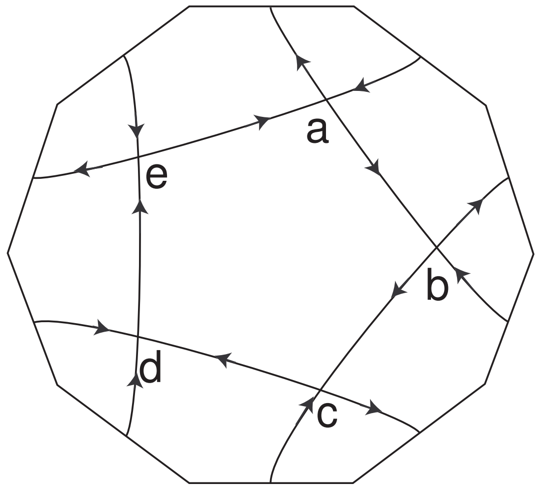


Figure 2: A knot in a surface of genus two

It is obvious that for a classical knot diagram D any of the groups $G(D), NG^1(D), NG^2(D)$ is trivial.

Denote the element of the group G corresponding to a crossing X of the knot diagram, by $g(X)$.

Example 1. *Consider the curve in the surface of genus two given in Fig. 2. The surface is represented as a decagon with opposite edges identified.*

The corresponding group of parity (say, $\{NG^1\}$) has five generators and the following relations

$abcde = 1, abdebceacd = 1, acebd = 1.$

The other group (say, $\{NG^2\}$) has the same set of generators and the following relations: $edcba = 1, dcaecbedba = 1, dbeca = 1.$

Remark 3. Every time when we prove the invariance under a certain Reidemeister or the fact that some map is functorial, we shall consider the "coordinated" source-sink structures (i.e., those coinciding outside the domain of the Reidemeister move) for two diagrams related by a Reidemeister move, and the corresponding groups.

The two groups NG^1 and NG^2 evidently have the same ablianisation

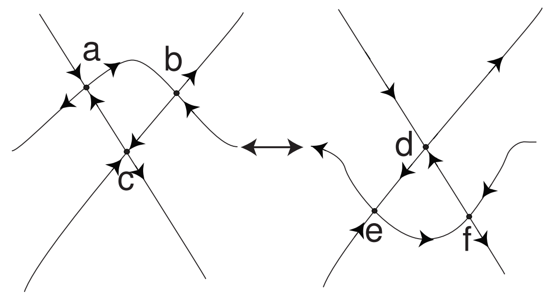


Figure 3: The third Reidemeister move

which is isomorphic to G.

Theorem 1. *For two non-stably equivalent virtual diagrams D, D' the groups $G(D), G(D')$ are isomorphic. Moreover, after an coordinated choice of the source-sink structure we have $NG_1(D) = NG_1(D'), NG_2(D) = NG_2(D')$.*

Proof. We have to prove the invariance under classical Reidemeister moves except the second (de)stabilising Reidemeister move; the presentation itself does not change under the detour move. We shall fix the source-sink orientation, and prove the claim of the the theorem for one of the groups NG_1 or NG_2; the proof for the other group is the same; the claim for the abelianisation follows.

When applying the first Reidemeister move, we add one crossing corresponding to a new generator a_N; we add the relation $a_N = 1$. The old relations either do not change (in the case if they correspond to pasted cycles not passing through the edge, the new crossing belongs to), or the letter a_N is added to these relations. The third Reidemeister move looks as follows, see Fig. 3.

The presentation corresponding to the left picture, has $n+3$ generators $a_1, \dots, a_n, a, b, c$, some relations $r_1, \dots, r_k$, and the relation $abc = 1$ (the generators $a_1, \dots a_n$ correspond to vertices outside the domain of the

Reidemeister move).

In the right picture we get a persentation with $n+3$ generators $a_1,\dots,a_n,d,e,f$, relations $r'_1,\dots,r'_k$, and also $fed=1$, where r'_j are obtained from r_i by the following replacements: $a\to fe, b\to df, c\to ed, ab\to f, bc\to d, ca\to e$.

It is easy to see that such presentations yield isomorphic groups where the isomorphism is obtained from $a\mapsto fe, b\mapsto df, c\mapsto ed, a_j\mapsto a_j, j=1,\dots,n$, and the inverse map is given by $d\mapsto bc, e\mapsto ca, f\mapsto ab$.

The second Reidemeister move (which does not lead to a (de)stabilisation) goes as follows, see Fig. 4. It takes place inside some polygon $a_1\dots a_k$ and is applied to edges connecting a_j,a_{j+1} and a_l,a_{l+1} (we add 1 modulo k). Then in the presentation we add two generators a_N and a_{N+1} and the relation $a_Na_{N+1}=1$. Besides that, for those edges the Reidemeister moves are applied two, the relations are changed as follows: we get $a_1\cdots a_ja_N\cdot a_{l+1}\cdots a_n=1$ and $a_{N+1}a_{j+1}\cdots a_l=1$; for all crossings related to edges participating in the Reidemeister moves, the relations acquire a_Na_{N+1}.

It is evident here that the presentation leads to the same group, indeed, by conjugating the relations $a_1\cdots a_ja_N\cdot a_{l+1}\cdots a_N=1$ and $a_{N+1}a_{j+1}\cdots a_l=1$ and multiplying them, we get some relation $a_1\cdots a_n=1$, and the relation $a_1\cdots a_ja_N\cdot a_{l+1}\cdots a_n=1$ itself can be treated as the defining relation for the generator a_N to be excluded.

Note that some of the letters $a_1\dots a_n$ may coincide.

$\square$

So, the above theorem provides a powerful tool for studying curves on 2-surfaces and knots in thickened surfaces. The groups NG_1 and NG_2 are invariants of the pair (surface S of genus g, conjugacy class in

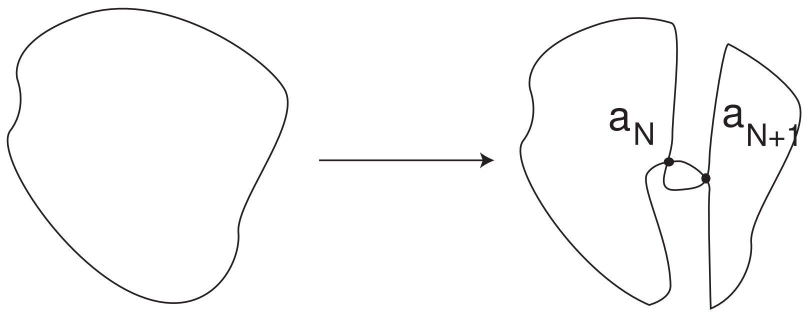


Figure 4: The second Reidemeister move

$\pi_1(S_g))$.

Theorem 2. *Let h be a homomorphism from the group $G(D)$ to the group $\mathbb{Z}_2$. Then the map $X \to h(g(X))$ generates a parity of crossings for knots in a given thickened surface. Moreover, every parity for this concrete surface factors through the group $G(D)$.*

The proof follows directly from the comparison of the relations in the group and the parity axioms.

Let us consider now the map f from virtual diagrams to virtual diagrams that deletes (makes virtual) those crossings X of the knot diagram for which $g(X) \neq 1$ in $G(K)$.

Theorem 3. *For a virtual knot diagram D we have $f(D) = D$ if and only if K is classical.*

Proof. Let D be a virtual knot diagram. By definition of the surface $S(D)$, the four-valent graph corresponding to the diagram D of K splits this surface into cells that can be coloured in a checkerboard manner.

Let $i \in H^1(S(D), \mathbb{Z}_2)$ be the non-trivial homology class of the surface $S(D)$. Consider the parity p_i for knots in $S(D)$ corresponding to this class. Namely, for every crossing v of the diagram D consider the "halves" the diagram is split into in this crossing, $D_{v,1}$ and $D_{v,2}$, and set

$p(v) = i(D_{v,1}) = i(D_{v,2})$.

Let γ be a cycle on $S(D)$ such that $i(\gamma) = 1$. Consider the cycle γ' homologous to γ and lying on the frame of D.

Let $a_1, \dots, a_l$ be the set of crossings where the cycle γ' rotates. We shall use the additive notation for the group $\mathbb{Z}_2$: the element 0 is neutral in the group, and 1 is non-trivial. By definition of the homological parity we have $\sum_{i=1}^{l} p(a_i) = 1 (mod\ 2)$, which yields that the parity p is nontrivial for at least one crossing a_j amongst $a_1, \dots, a_l$.

Consider the homomorphism $G(K) \to \mathbb{Z}_2$ corresponding to the parity p. It is clear that the element $g(a_j)$ is non-trivial in $G(K)$.

Thus, the crossing a_j is virtualised by the map f. □

Denote the k-th iteration of the map f by f^k. Let us now construct the map pr from knots in thickened surfaces (with orientable atoms) to classical knots. Let D be a virtual diagram. Since the number of classical crossings of D is finite (say, is equal to n), we see that $f^{n+1}(D) = f^n(D)$. By definition, set $pr = f^{(n)}$. Obviously, $pr(D)$ is classical.

This leads to the following

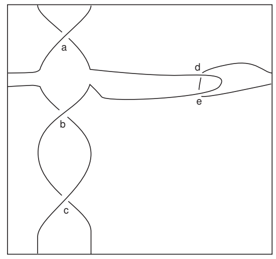
Theorem 4. *The map pr is a well-defined map from knots in thickened surfaces to classical knots.*

Remark 1. *Unfortunately, the map f is not well behaved under stabilisations, so, the map pr does not lead to a well defined map from virtual knots to the classical knots.*

Consider the following example. Let T be the simplest diagram of the classical right trefoil knot.

By definition, $f(T) = T$, since all crossings are trivial.

Now, represent T as the closure of the $(2, 3)$-braid in the cylinder. Close this cylinder to get the torus. On this torus we get a diagram T'



which is obtained from T by a stabilisation (note that the source-sink structure is satisfied).

Now, if we look at the group, we see that none of the generators corresponding to vertices is trivial; so, $f(T)$ gives the unknot.

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