ON THE DUAL OF THE MOBILE CONE.

SUNG RAK CHOI

Abstract. We prove that the cone of mobile divisors and the cone of curves birationally movable in codimension 1 are dual in the $(K+B)$-negative part for a klt pair $(X/Z, B)$. We also prove the structure theorem and the contraction theorem for the expanded cone of curves birationally movable in codimension 1. The duality of the cones gives a partial answer to the problem posed by Sam Payne.

Contents

1. Introduction

Let $X \to Z$ be a projective morphism of $\mathbb{Q}$-factorial normal algebraic varieties defined over an algebraically closed field k (of characteristic 0). It is well-known that the cone of nef divisors $\mathrm{Nef}(X/Z)$ in $\mathrm{N}^1(X/Z)$ and the cone of curves $\overline{\mathrm{NE}}(X/Z)$ (often called the Mori cone) in $\mathrm{N}_1(X/Z)$ are dual to each other. The cone of pseudoeffective divisors $\overline{\mathrm{Eff}}(X/Z) \subseteq \mathrm{N}^1(X/Z)$ and the cone of movable curves $\overline{\mathrm{NM}}(X/Z) \subseteq \mathrm{N}^1(X/Z)$ are dual to each other (Theorem 4.8):

$$\begin{array}{ccc} \overline{\mathrm{Eff}}(X/Z) & \supseteq & \mathrm{Nef}(X/Z) \\ {\scriptstyle\text{dual}}\updownarrow & & {\scriptstyle\text{dual}}\updownarrow \\ \overline{\mathrm{NM}}(X/Z) & \subseteq & \overline{\mathrm{NE}}(X/Z). \end{array}$$

The next most important cone in $\mathrm{N}^1(X/Z)$ is probably the cone $\overline{\mathrm{Mob}}(X/Z)$ of mobile divisors. Mobile divisors are the divisors whose $\mathbb{R}$-base loci are of codimension ≥ 2. The mobile cone $\overline{\mathrm{Mob}}(X/Z)$ is a subcone of $\overline{\mathrm{Eff}}(X/Z)$ which contains the nef cone $\mathrm{Nef}(X/Z)$: $\mathrm{Nef}(X/Z) \subseteq \overline{\mathrm{Mob}}(X/Z) \subseteq \overline{\mathrm{Eff}}(X/Z)$. It is natural to think about the the dual in $\mathrm{N}^1(X/Z)$ of the cone $\overline{\mathrm{Mob}}(X/Z)$. In this paper, we study the duality between these two cones.

Let $f : X \dashrightarrow X'/Z$ be a small birational map between $\mathbb{Q}$-factorial normal projective varieties $X, X'/Z$. Since it is known that $\mathrm{N}^1(X/Z)$ and $\mathrm{N}^1(X'/Z)$ are isomorphic under f_* [17, 12-2-1], their dual spaces $\mathrm{N}_1(X/Z)$ and $\mathrm{N}_1(X'/Z)$, respectively, are also isomorphic: $\mathrm{N}_1(X/Z) \cong \mathrm{N}_1(X'/Z)$. Under this isomorphism, a class $\alpha = [C] \in \mathrm{N}_1(X'/Z)$ defined by a mov^1(*movable in codimension* 1)-curve C in X'/Z can be pulled back to $\mathrm{N}_1(X/Z)$ and we can simply consider α as a class in $\mathrm{N}_1(X/Z)$. The mov^1-curve C in X'/Z is called a b-mov^1(*birationally movable in codimension* 1)-curve of X/Z. (See Section 4.)

We define

$$\mathrm{NM}^1(X, X'/Z) \subseteq \mathrm{N}_1(X/Z)$$

as the image of the convex cone $\mathrm{NM}^1(X'/Z)$ in $\mathrm{N}_1(X/Z)$ under the isomorphism $\mathrm{N}_1(X/Z) \cong \mathrm{N}_1(X'/Z)$. We define

$$\overline{\mathrm{bNM}}^1(X/Z) := \sum_{X \dashrightarrow X'} \overline{\mathrm{NM}}^1(X, X'/Z)$$

where the summation is taken over all $\mathbb{Q}$-factorial X' isomorphic to X in codimension 1.

We have the following partial duality result.

Theorem 1.1. *Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial klt pair. Then*

$$\overline{\mathrm{NE}}(X/Z)_{K+B\geq 0} + \overline{\mathrm{Mob}}(X/Z)^\vee = \overline{\mathrm{NE}}(X/Z)_{K+B\geq 0} + \overline{\mathrm{bNM}}^1.$$

In other words, the dual cone $\overline{\mathrm{Mob}}(X/Z)^\vee$ is spanned by the b-mov^1-curves of X/Z in the $(K + B)$-negative part.

We also have the following cone theorem for $\overline{\mathrm{bNM}}^1(X/Z)$.

Theorem 1.2. (The Cone Theorem for $\overline{\mathrm{bNM}}^1(X/Z)$) *Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial klt pair. There exists a countable set of b-mov^1-curves $\{C_i\}_{i\in I}$ of X/Z such that*

$$\overline{\mathrm{NE}}(X/Z)_{K+B\geq 0} + \overline{\mathrm{bNM}}^1(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K+B\geq 0} + \sum_{i\in I} \mathbb{R}_{\geq 0} \cdot [C_i]$$

and for any ample H and any $\varepsilon > 0$, there exists a finite subset $J \subseteq I$ such that

$$\overline{\mathrm{NE}}(X/Z)_{K+B+\varepsilon H\geq 0} + \overline{\mathrm{bNM}}^1(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K+B+\varepsilon H\geq 0} + \sum_{j\in J} \mathbb{R}_{\geq 0} \cdot [C_j].$$

The rays $\{R_i = \mathbb{R}_{\geq 0}[C_i]\}_{i\in I}$ in the first equality can accumulate only at the hyperplanes supporting both $\overline{\mathrm{NE}}(X/Z)_{K+B\geq 0}$ and $\overline{\mathrm{bNM}}^1(X/Z)$.

Note that this is actually a structure theorem for the expanded cone $\overline{\mathrm{NE}}(X/Z) + \overline{\mathrm{bNM}}^1(X/Z)$ (cf. Figure 1 in Section 5) and we cannot replace $\overline{\mathrm{NE}}(X/Z)$ by $\overline{\mathrm{bNM}}^1(X/Z)$ to have the genuine form of the cone theorem as in the original cone theorem for $\overline{\mathrm{NE}}(X/Z)$. See Example 5.4 or [16, Section 6].

We also prove the following contraction theorem, inspired by the result in [1] and [16]. We call an extremal ray R of $\overline{\mathrm{bNM}}^1(X/Z)$ a *mov*^{1}*-extremal ray* for $(X/Z, B)$ if it is $(K+B)$-negative and it is also an extremal ray for the cone $\overline{\mathrm{NE}}(X/Z)_{K+B\geq 0} + \overline{\mathrm{bNM}}^1(X/Z)$. See Section 5.

Theorem 1.3. (Contraction Theorem for mov^1-extremal rays) *Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial klt pair. Let R be a mov^1-extremal ray of $\overline{\mathrm{bNM}}^1(X/Z)$ for $(X/Z, B)$. Then the following hold;*

(1) *there exists a small birational map $\varphi : X \dashrightarrow X'$ and a contraction $\psi : X' \to Y$ which is either a divisorial contraction or a Mori fibration such that the mov^1-extremal ray R is spanned by a mov^1-curve C on X' if and only if C is contracted by ψ, and*
(2) *the composition map $\psi \circ \varphi$ is uniquely determined by R.*

These results give a partial answer to the duality problem posed in [19]. See Section 5.

This paper is organized as follows:
In section 2, we recall some basic notions used throughout the paper. In section 3, we review the necessary results from the papers [4],[7]. In section 4, we review the definitions and properties of the non-ample locus $\mathbf{B}_+(D)$, non-nef locus $\mathbf{B}_-(D)$, and volume function $\mathrm{vol}(D)$ of divisors D. The proof of Theorem 1.1 is given in this section. In section 5, we prove Theorem 1.2 and Theorem 1.3.

I would like to thank S. Boucksom and Z. Ran for pointing out an error in the preliminary version of this paper. I also would like to thank V. Shokurov for the encouragement.

2. Preliminaries

Let X be a $\mathbb{Q}$-factorial normal projective variety and $X \to Z$ be a projective morphism to another variety Z. We simply denote this by X/Z throughout the paper. The relations $\equiv, \sim_{\mathbb{R}}$ and the properties such as ampleness, nefness, $\equiv 0$, bigness for divisors on X/Z will be considered relatively over Z.

We define the cones in the numerical space $\mathrm{N}^1(X/Z)$:

$$\begin{aligned} \mathrm{Amp}(X/Z) &= \{[D] \in \mathrm{N}^1(X/Z) \mid C \cdot D > 0 \text{ for any curve } C \text{ on } X/Z\}, \\ \mathrm{Mob}(X/Z) &= \{[D] \in \mathrm{N}^1(X/Z) \mid D \equiv D' \text{ for some } \mathbb{R}\text{-mobile } D'\}, \\ \mathrm{Eff}(X/Z) &= \{[D] \in \mathrm{N}^1(X/Z) \mid D \equiv D' \text{ for some effective } D'\}. \end{aligned}$$

The closures $\mathrm{Nef}(X/Z) = \overline{\mathrm{Amp}}(X/Z)$, $\overline{\mathrm{Mob}}(X/Z)$, and $\overline{\mathrm{Eff}}(X/Z)$ are called the *nef cone, mobile cone*, and *pseudoeffective cone*, respectively. They satisfy the inclusion: $\mathrm{Nef}(X/Z) \subseteq \overline{\mathrm{Mob}}(X/Z) \subseteq \overline{\mathrm{Eff}}(X/Z)$. For a cone $V \subseteq \mathrm{N}_1(X/Z)$, a divisor D and $\square \in \{=, <, >, \geq, \leq\}$, we define

$$V_{D\square 0} = V \cap \{C \in \mathrm{N}_1(X/Z) \mid D \cdot C \square 0\}.$$

An *extremal face* F of a closed convex cone V satisfies the two conditions; 1)if $v \in F$, then $rv \in F$ for any $r > 0$, and 2)if $v + u \in F$ for $u, v \in V$, then $u, v \in F$. A one dimensional extremal face is called an *extremal ray*.

We use the standard notions of singularities of pairs (X, B) in the log minimal model program (LMMP, for short) [13] [12]. We briefly recall the basics of LMMP.

Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial lc pair and let $\varphi : X \dashrightarrow Y$ be a birational map. The *log birational transform* of B on Y is $B_Y^{\log} := B_Y + \sum E_i$, where B_Y is a proper transform of B on Y and E_i are φ^{-1}-exceptional prime divisors on Y. For an exceptional prime divisor E over X/Z, $a(E, X/Z, B)$ denotes the log discrepancy of $(X/Z, B)$ at E.

Definition 2.1. *Let the notations be as above.*

1. *A pair $(Y/Z, B_Y^{\log})$ is called a* wlc model of $(X/Z, B)$ *if the pair $(Y/Z, B_Y^{\log})$ is lc, $K_Y + B_Y^{\log}$ is nef, and the inequality $1 - \mathrm{mult}_E B \leq a(E, Y, B_Y^{\log})$ holds for any φ-exceptional prime divisor E. If, furthermore, $K_Y + B_Y^{\log}$ is ample, then the pair $(Y/Z, B_Y^{\log})$ is called the* lc model of $(X/Z, B)$.
2. *A pair $(Y/Z, B_Y^{\log})$ equipped with a fibration $Y \to T/Z$ is called a* Mori log fibration of $(X/Z, B)$ *if $(Y, B_Y^{\log})$ is lc, $\dim T < \dim Y$, the relative Picard number $\rho(Y/T) = 1$, $-(K_Y + B_Y^{\log})$ is ample over T and the inequality $1 - \mathrm{mult}_E B \leq a(E, Y, B_Y^{\log})$ holds for any φ-exceptional prime divisor E.*
3. *A* resulting model of $(X/Z, B)$ *is either a wlc model (1) or a Mori log fibration (2).*

If a resulting model $(Y/Z, B_Y^{\log})$ in (1),(2) is a projective $\mathbb{Q}$-factorial lt pair and the strict inequalities $1 - \mathrm{mult}_E B < a(E, Y, B_Y^{\log})$ hold for φ-exceptional divisors E, then it is called a *strictly lt(slt)* resulting model. A slt wlc model is called a *log terminal model* [4]. By the LMMP, any $\mathbb{Q}$-factorial klt pair $(X/Z, B)$ is expected to have a resulting model and it cannot have both resulting models simultaneously [21, 2.4.1].

We will use the following lemma often.

Lemma 2.2. *Let (X, B) be a klt pair and H be an ample divisor on X. Then there exists an effective divisor $H' \sim_{\mathbb{R}} H$ such that $(X, B + H')$ is klt.*

Proof. See [15, Example 9.2.29]. □

If the pairs $(X, B), (X, B')$ are klt and $B \sim_{\mathbb{R}} B'$, then (X, B) and (X, B') have the same resulting models by the LMMP. Therefore by Lemma 2.2, given a klt pair $(X/Z, B)$ and an ample divisor H, we may assume that $(X/Z, B + H)$ is klt in order to run the LMMP or to study the resulting models of $(X/Z, B + H)$.

3. Geography

In this section, we review some necessary results from [7],[4]. First, we state an important result about the decomposition of the following set:

$$\mathcal{E}_A := \{B \in V |\ B \geq 0 \text{ and } K + A + B \text{ is klt and pseudo-effective } \}$$

where V is a finite dimensional subspace of real Weil divisors and A is an ample divisor on X.

Theorem 3.1. ([4, Theorem 1.1, Theorem 1.1.5]) *Let (X, B_0) be a klt pair and suppose that $B_0 \in \mathcal{E}_A$. Then for any $B \in \mathcal{E}_A$, the pair (X, B) has a log terminal model. Furthermore, there exist finitely many birational maps $\varphi_i : X \dashrightarrow X_i$ $(1 \leq i \leq p)$ and the set $\mathcal{E}_A$ is decomposed into finitely many rational polytopes;*

$$\mathcal{E}_A = \bigcup_i^p \mathcal{W}_i,$$

satisfying the following condition: if, for $B \in \mathcal{E}_A$, there exists a birational contraction $\varphi : X \dashrightarrow Y$ which is a log terminal model of (X, B), then $\varphi = \varphi_i$ for some $1 \leq i \leq p$.

In [7], the similar decomposition problem (which we call the *geography*) is studied in detail in terms of b-divisors. The polytopes $\mathcal{W}_i$ in Theorem 3.1 correspond to the $\sim_{\text{wlc}}$ classes (*countries*) in [7].

Definition 3.2. *Let $(X/Z, B)$ be an lc pair and H a fixed ample divisor such that $K + B + H$ is $\mathbb{R}$-Cartier. We define the following thresholds:*

- effective threshold

$$e_H(X/Z, B)(= e_H) := \sup\{t \geq 0 |\ t(K + B) + H \in \overline{\text{Eff}}(X/Z)\}$$

- mobile threshold *(or* mobility*)*

$$m_H(X/Z, B)(= m_H) := \sup\{t \geq 0 |\ t(K + B) + H \in \overline{\text{Mob}}(X/Z)\}$$

- nef threshold

$$n_H(X/Z, B)(= n_H) := \sup\{t \geq 0 |\ t(K + B) + H \in \text{Nef}(X/Z)\}$$

If $K + B \in \text{Nef}(X/Z)$ (resp. $\overline{\text{Mob}}(X/Z), \overline{\text{Eff}}(X/Z)$), then we define $n_H = \infty$ (resp. $m_H = \infty, e_H = \infty$). Clearly, $e_H \geq m_H \geq n_H$.

The mobile threshold and effective threshold are invariant under certain birational modifications in the LMMP with scaling.

Lemma 3.3. *Let $\varphi : X \dashrightarrow X'$ be a birational map which is either a log flip or a divisorial contraction in the LMMP on (X, B) with scaling of H. Let*

$$\begin{aligned} m &= m_H(X, B), \quad m' = m_{H'}(X', B'), \\ e &= e_H(X, B), \quad\;\; e' = e_{H'}(X', B') \end{aligned}$$

where $H' = f_*H$ and $B' = f_*B$. Then

(1) *$e = e'$, and*
(2) *$m = m'$ if φ is a log flip of (X, B).*

Proof. (1): See [1, Lemma 5.1].
(2): This immediately follows from the fact that a log flip φ induces an isomorphism of the cones $\overline{\mathrm{Mob}}(X/Z) \cong \overline{\mathrm{Mob}}(X'/Z)$ and $\mathrm{N}^1(X/Z) \cong \mathrm{N}^1(X'/Z)$. □

See Section 5 for the definitions of co-extremal rays and mov^1-extremal rays.

Theorem 3.4. *Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial klt pair. Then the following hold;*

(1) *if $K + B \notin \overline{\mathrm{Eff}}(X/Z)$, then there exists a birational map $\varphi : X \dashrightarrow X'/Z$ and a Mori fibration $\psi : X' \to Y/Z$, and*
(2) *if $K + B \in \overline{\mathrm{Eff}}(X/Z)$ but $\notin \mathrm{Int}\,\overline{\mathrm{Mob}}(X/Z)$, then there exists a small birational map $\varphi : X \dashrightarrow X'/Z$ and a map $\psi : X' \to Y/Z$ which is either a divisorial contraction or a Mori fibration.*

Proof. (1): Since $K + B \notin \overline{\mathrm{Eff}}(X/Z)$, there exists a co-extremal ray of $\overline{\mathrm{NM}}(X/Z)$ for $(X/Z, B)$ and an ample divisor H such that $K+B+A$ is co-bounding for the ray [16, Lemma 4.2] and $(X/Z, B+A)$ is klt. (Note that by our definition, co-bounding divisors define supporting planes away from $\overline{\mathrm{NE}}(X/Z)_{K+B>0}$.) Since $K+B+A \in \partial\overline{\mathrm{Eff}}(X/Z)$, it follows from [4, Corollary 1.3.2].
(2): This can be proved in a way parallel to the proof of [1, Theorem 3.9]. Since $K + B \notin \overline{\mathrm{Mob}}(X/Z)$, there exists a mov^1-extremal ray of $\overline{\mathrm{bNM}}^1(X/Z)$ for $(X/Z, B)$. By Lemma 5.5, there exists an ample divisor A such that $K+B+A$ is nef and $K+B+tA$ ($0 < t = m_A(X/Z, B)^{-1} < 1$) is mov^1-bounding for the ray and $(X/Z, B+A)$ is klt. If we first run the LMMP on the pair $(X, B+tA = \Delta)$ with scaling of $H = (1-t)A$ as in [1, 3.8] with the same notations, we obtain a log terminal model $\varphi : X \dashrightarrow X' = X_n/Z$ of $(X/Z, B+tA)$. Since $K+B+t'A$ for $t' > t$ is $\mathbb{R}$-mobile by Lemma 4.2, the map φ is small. By Lemma 3.3, the mobile threshold is unchanged: $m_{A'}(X', B') = m_A(X, B)$ where $A' = \varphi_*A$ and $B' = \varphi_*B$. Now let $X'_0 = X', B'_0 = B', H'_0 = tA'$ and run the LMMP on $(X'_0/Z, B'_0)$ with scaling of H'_0. The divisor $K_{X'_0} + B'_0$ is not nef since $K + B \notin \overline{\mathrm{Mob}}(X/Z)$ and φ_* preserves the numerical class in $\mathrm{N}^1(X/Z) \cong \mathrm{N}^1(X'/Z)$. Note that $1 = \lambda'_0 = \inf\{\lambda \in [0,1] \mid K_{X'_0} + B'_0 + \lambda H'_0 \text{ is nef }\}$ (see Lemma 4.2, Definition-Lemma 4.1). Thus there exists a $(K_{X'} + B')$-negative extremal ray R'_0 such that $R'_0 \cdot (K_{X'_0} + B'_0 + H'_0) = 0$. We may assume that the ray R'_0 is of divisorial type. Indeed, first of all, it is not of fibering type because $K + B$ is big. Suppose that the birational map associated to R'_0 is small and let $\varphi'_0 : X'_0 \dashrightarrow X'_1/Z$ be the

associated log flip. Then $K_{X'_1}+B'_1=\varphi_*(K_{X'_0}+B'_0)\notin\overline{\mathrm{Mob}}(X/Z)$ and $K_{X'_1}+B'_1$ is not nef. Inductively, we see that possibly after finitely many log flips $\varphi':X'_0 \dashrightarrow X'_i/Z$, we must have an extremal ray R'_i of divisorial type with a divisorial contraction $\psi:X'_i\to Y/Z$, otherwise $K_{X'_i}+B'_i$ does not become nef. Therefore, we have a small birational map $\varphi'\circ\varphi:X\dashrightarrow X'_i/Z$ and a divisorial contraction $\psi:X'_i\to Y/Z$. $\square$

4. Movable cone

For a $\mathbb{Z}$-divisor D, its base locus $\mathrm{Bs}(D)$ is defined as the support of the intersection of the elements in the usual $\mathbb{Z}$-linear system $|D|=\{D'\in\mathrm{Div}_{\mathbb{Z}}(X/Z)\mid D\sim D'\geq 0\}$. For a $\mathbb{Q}$-divisor D, the *stable base locus* of D is defined as $\mathbf{B}(D):=\bigcap_m \mathrm{Bs}(mD)$ where the intersection is taken over the positive integers m such that mD is integral. It is known that there exists an integer m_0 such that m_0D is integral and for all large positive integers n, $\mathbf{B}(D)=\mathrm{Bs}(nm_0D)$ [18]. For an $\mathbb{R}$-divisor D, the $\mathbb{R}$-linear system is defined as $|D|_{\mathbb{R}}:=\{D'\in\mathrm{Div}_{\mathbb{R}}(X/Z)\mid D\sim_{\mathbb{R}} D'\geq 0\}$ and its $\mathbb{R}$-*stable base locus* as $\mathbf{B}_{\mathbb{R}}(D):=(\cap|D|_{\mathbb{R}})_{\mathrm{red}}$. A big divisor D is $\mathbb{R}$-*mobile* if the set $\mathbf{B}^1_{\mathbb{R}}(D)$ of divisorial components in $\mathbf{B}_{\mathbb{R}}(D)$ is empty. Clearly, $\mathbf{B}_{\mathbb{R}}(D)\subseteq\mathbf{B}(D)$ for a $\mathbb{Q}$-divisor D. From now on, we always use $\mathbb{R}$-divisors unless otherwise stated.

For a big $\mathbb{R}$-divisor D, we define the *non-ample locus* (or *augmented base locus*) of D as

$$\mathbf{B}_+(D):=\bigcap_{\text{ample } A}\mathbf{B}(D-A)$$

where the intersection is taken over all ample divisors A such that $D-A$ are $\mathbb{Q}$-divisors. We define $\mathbf{B}_+(D):=X$ if D is not big. It is known that $\mathbf{B}_+(D)=\mathbf{B}_+(D-A)$ for any sufficiently small ample divisor A. As the name suggests, D is ample if and only if $\mathbf{B}_+(D)=\emptyset$.

For a pseudoeffective divisor D, we define the *non-nef locus* of D as

$$\mathbf{B}_-(D):=\bigcup_{\varepsilon>0}\mathbf{B}_+(D+\varepsilon A)$$

for any fixed ample divisor A. Of course, the definitions is independent of the choice of A. If D is not pseudoeffective, then we define $\mathbf{B}_-(D):=X$. Since $\mathbf{B}_+(D+\varepsilon' A)\subseteq\mathbf{B}_+(D+\varepsilon A)$ for $0<\varepsilon<\varepsilon'$, it is enough to take only small $\varepsilon>0$: for any fixed $r>0$, $\mathbf{B}_-(D)=\cup_{r>\varepsilon>0}\mathbf{B}_+(D+\varepsilon A)$. It is easy to see that D is nef if and only if $\mathbf{B}_-(D)=\emptyset$. See [5],[9],[10],[15] for more details about the non-ample loci and non-nef loci.

Let D be a big divisor in X/Z. We let $\mathbf{B}^1_+(D)$ (resp. $\mathbf{B}^1_-(D)$) be the set of divisorial components of $\mathbf{B}_+(D)$ (resp. $\mathbf{B}_-(D)$). Let $\mathrm{Mob}_+(X)$ (resp. $\mathrm{Mob}_-(X)$) be the cone in $\mathrm{N}^1(X)$ spanned by the big divisors D such that $\mathbf{B}^1_+(D)=\emptyset$ (resp. $\mathbf{B}_-(D)=\emptyset$). Since $\mathbf{B}_-(D)\subseteq\mathbf{B}_+(D)$, $\mathrm{Mob}_+(X)\subseteq\mathrm{Mob}_-(X)$. It is easy to see that $\mathbf{B}_-(D)\subseteq\mathbf{B}_{\mathbb{R}}(D)\subseteq\mathbf{B}_+(D)$. Therefore $\mathrm{Mob}_+(X/Z)\subseteq\mathrm{Mob}(X/Z)\subseteq\mathrm{Mob}_-(X/Z)$.

Definition-Lemma 4.1. *The cone* $\mathrm{Mob}_+(X/Z)$ *is open and dense in* $\mathrm{Mob}_-(X/Z)$*. In particular, the cone* $\overline{\mathrm{Mob}}_-(X/Z)$ *is closed and*

$$\overline{\mathrm{Mob}_+}(X/Z) = \overline{\mathrm{Mob}}(X/Z) = \mathrm{Mob}_-(X/Z).$$

We call $\overline{\mathrm{Mob}}(X)$ *the* mobile cone*.*

Proof. Fix a divisor $D \in \mathrm{Mob}_+(X/Z)$. To show that $\mathrm{Mob}_+(X/Z)$ is open, it is enough to show that for any sufficiently small divisor F, $D+F \in \mathrm{Mob}_+(X/Z)$. For an ample divisor A, we may assume that $A-F$ is also ample. We may also assume that $A-F$ is sufficiently small so that $\mathbf{B}_+(D) = \mathbf{B}_+(D-(A-F))$. Therefore, $\mathbf{B}_+(D+F) \subseteq \mathbf{B}_+(D+F-A) = \mathbf{B}_+(D)$. Since $\mathbf{B}_+(D) = \emptyset$, $D+F \in \mathrm{Mob}_+(X/Z)$.

Now suppose that $\overline{\mathrm{Mob}_+}(X/Z) \subsetneq \mathrm{Mob}_-(X/Z)$. Then there exists an open set $U \subset \mathrm{Mob}_-(X/Z) \setminus \overline{\mathrm{Mob}_+}(X/Z)$. Let $D \in U$. Then for a small ample divisor A such that $D+A \in U$, there exists a divisorial component $E \subseteq \mathbf{B}_+(D+A)$. However, since $\mathbf{B}_+(D+A) \subseteq \mathbf{B}_-(D)$, it is a contradiction. □

Lemma 4.2. *Let D be a big divisor such that $D \in \partial\overline{\mathrm{Mob}}(X/Z)$ and fix an ample divisor A on X/Z. Then (1) $D+A$ is $\mathbb{R}$-mobile, and (2) $D-A$ is not $\mathbb{R}$-mobile. In particular, there exists an irreducible divisorial component $E \subseteq \mathbf{B}_+(D)$.*

Proof. Since D is in the boundary $\partial\overline{\mathrm{Mob}_+}(X/Z)$, there exists an arbitrarily small divisor D' such that $\mathbf{B}^1_+(D+D') = \emptyset$. By taking a positive number r small and choosing D' sufficiently small, we may assume that $rA - D'$ is a sufficiently small ample divisor such that $\mathbf{B}_+(D+A) = \mathbf{B}_+(D+A-(rA-D'))$. Then (1) follows from the following inclusion:

$$\mathbf{B}_{\mathbb{R}}(D+A) \subseteq \mathbf{B}_+(D+A) = \mathbf{B}_+(D+(1-r)A+D') \subseteq \mathbf{B}_+(D+D').$$

Since D is in the boundary $\partial\overline{\mathrm{Mob}_+}(X/Z)$, there also exists an arbitrarily small divisor D' such that $\mathbf{B}^1_+(D+D') \neq \emptyset$. We may assume that $A+D'$ is ample. Then (2) follows from the following inclusion:

$$\mathbf{B}_{\mathbb{R}}(D-A) \supseteq \mathbf{B}_-(D-A) = \bigcup_{\varepsilon>0} \mathbf{B}_+(D-A+\varepsilon(A+D')) \supseteq \mathbf{B}_+(D+D').$$

□

Note that for a big divisor D in $\partial\overline{\mathrm{Mob}}(X/Z)$, $\mathbf{B}^1_-(D)$ may be empty. In dimension 2, the nef cones and the mobile cones coincide. Therefore, if D is a big and nef divisor which is not ample, then $\mathbf{B}^1_-(D) = \emptyset$, but $\mathbf{B}^1_+(D) \neq \emptyset$.

Proposition 4.3. *Let $f : X \dashrightarrow X'/Z$ be a small birational map between $\mathbb{Q}$-factorial projective varieties $X, X'/Z$. Suppose that for a big divisor D, there exists an irreducible divisorial component V in $\mathbf{B}_+(D)$. Then $V_{X'} \subseteq \mathbf{B}_+(D')$, where $V_{X'} = f_*V$ and $D' = f_*D$.*

Proof. Let $D' \equiv A' + E'$ be a decomposition into an ample divisor A' and an effective divisor E'. Then since f is small and the numerical classes are preserved in $\mathrm{N}^1(X/Z) \cong \mathrm{N}^1(X'/Z)$ under f_* (and f_*^{-1}), $D \equiv f_*^{-1}A' + f_*^{-1}E'$ is a decomposition into an ample divisor $f_*^{-1}A'$ and an effective divisor $f_*^{-1}E'$. Since V is an irreducible component of $\mathbf{B}_+(D)$, $V \subseteq \operatorname{Supp} f_*^{-1}E'$. Therefore, $V' \subseteq \operatorname{Supp} E'$. This implies that $V' \subseteq \mathbf{B}_+(D')$. □

Definition 4.4. *For a $\mathbb{Q}$-divisor D on X/Z of dimension d, we define the* volume *as*

$$\operatorname{vol}(D) := \limsup \frac{H^0(X, mD)}{m^d/d!}$$

where lim sup *is taken over the positive integers m such that mD is integral.*

For a nef divisor D, the volume $\operatorname{vol}(D)$ can be defined as the self intersection number (D^d) and D is big if and only if $(D^d) > 0$. Volume $\operatorname{vol}(D)$ depends only on the numerical class of D. Furthermore, it extends uniquely to a continuous function:

$$\operatorname{vol} : \mathrm{N}^1(X/Z) \longrightarrow \mathbb{R}.$$

In particular, vol defines a nonnegative function on the pseudoeffective cone $\overline{\mathrm{Eff}}(X/Z)$ and $\operatorname{vol}(D) = 0$ on the boundary $\partial\overline{\mathrm{Eff}}(X/Z)$. For more detailed properties of vol, see [14]. For an irreducible closed subvariety $E \subseteq X$ of positive dimension d' such that $E \not\subseteq \mathbf{B}_+(D)$, we can also define the *restricted volume* $\operatorname{vol}_{X|E}(D)$ of a divisor D on E [9]:

$$\operatorname{vol}_{X|E}(D) := \limsup \frac{\dim_k \operatorname{Im}\Big(H^0(X, mD) \to H^0(E, mD|_E)\Big)}{m^{d'}/d'!}$$

where lim sup is taken over the positive integers m such that mD is integral. For an ample divisor H, the three notions coincide: $\operatorname{vol}_{X|E}(H) = \operatorname{vol}(H|_E) = H^{\dim E} \cdot E$. The restricted volume extends uniquely to a continuous function on the set $\operatorname{Big}^E(X/Z)$ of $\mathbb{R}$-divisor classes ξ such that E is not properly contained in any irreducible component of $\mathbf{B}_+(\xi)$:

$$\operatorname{vol}_{X|E} : \operatorname{Big}^E(X/Z) \longrightarrow \mathbb{R}$$

having the property that $\operatorname{vol}_{X|E}(\xi) = 0$ if and only if E is an irreducible component of $\mathbf{B}_+(\xi)$ [9]. In particular, we have the following continuity property.

Theorem 4.5. *Let E be an irreducible component of $\mathbf{B}_+(D)$, then*

$$\lim_{\xi \to D} \operatorname{vol}_{X|E}(\xi) = 0,$$

where the limit is taken over $\xi \in \operatorname{Big}^E(X/Z)$ such that ξ approaches the numerical class of D.

Proof. See Theorem 5.6 and Remark 5.7 in [9]. □

We will need the following result restricted on some subvariety.

Theorem 4.6 ([14, Theorem 1.6.1]). *Let X be a variety of dimension d and D_i be nef divisors. Then*

$$D_1 \cdot D_2 \cdots D_d \geq \mathrm{vol}(D_1)^{\frac{1}{d}} \cdot \mathrm{vol}(D_2)^{\frac{1}{d}} \cdots \mathrm{vol}(D_d)^{\frac{1}{d}}.$$

We define the following three types of curves.

Definition 4.7. *Let X/Z be a $\mathbb{Q}$-factorial normal algebraic variety of dimension d.*

- *A curve C on X/Z is called a* movable curve *if it is a member of a family of curves covering X/Z.*
- *A curve C on X/Z is called a* mov^1(movable in codimension 1)-curve *if it is a member of a family of curves covering a subvariety of codimension* 1.
- *A mov^1-curve C on some $\mathbb{Q}$-factorial X'/Z which is isomorphic to X/Z in codimension* 1 *is called a* b-mov^1(birationally movable in codimension 1)-curve of *X/Z.*

Note that a b-mov^1-curve C defines a class $\alpha = [C] \in \mathrm{N}_1(X/Z)$ even though the curve C may not be defined in X/Z. Thus we may define a b-mov^1-curve C as a class in $\mathrm{N}_1(X/Z)$. We let $\mathrm{NM}(X/Z)$, $\mathrm{NM}^1(X/Z)$ be the cones in $\mathrm{N}_1(X/Z)$ that are spanned by the movable curves and mov^1-curves in X/Z, respectively. We define $\mathrm{NM}^1(X, X'/Z)$ as the image in $\mathrm{N}_1(X/Z)$ of the cone $\mathrm{NM}(X'/Z)$ under the isomorphism $\mathrm{N}_1(X'/Z) \cong \mathrm{N}_1(X/Z)$. Lastly, we define $\overline{\mathrm{bNM}}^1(X/Z)$ as the cone in $\mathrm{N}_1(X/Z)$ defined by b-mov^1-curves of X/Z. It is easy to see that

$$\overline{\mathrm{bNM}}^1(X/Z) = \sum_{X \dashrightarrow X'} \overline{\mathrm{NM}}^1(X, X'/Z)$$

where the summation is taken over all $\mathbb{Q}$-factorial X'/Z which is isomorphic to X/Z in codimension 1.

By definition, a movable curve is mov^1 and a mov^1-curves is b-mov^1. Thus

$$\overline{\mathrm{NM}}(X/Z) \subseteq \overline{\mathrm{NM}}^1(X/Z) \subseteq \overline{\mathrm{bNM}}^1(X/Z).$$

The second inclusion above is not an equality in general. See [19, Example 1] for a counterexample.

Theorem 4.8. *The following hold:*

(1) $\mathrm{Nef}(X/Z) = \overline{\mathrm{NE}}(X/Z)^\vee$.
(2) $\overline{\mathrm{Eff}}(X/Z) = \overline{\mathrm{NM}}(X/Z)^\vee$.

Proof. (1) It is a well known result in algebraic geometry. See [14, Proposition 1.4.28].
(2) It is the main result of [6]. □

According to Theorem 1.1, the cones $\overline{\mathrm{Mob}}(X/Z)$ and $\overline{\mathrm{bNM}}^1(X/Z)$ are dual to each other at least in some part of the cones. In order to prove Theorem 1.1, we prove the following equivalent dual statement:

the cones $\overline{\mathrm{Mob}}(X/Z)$ *and* $\overline{\mathrm{bNM}}^1(X/Z)^\vee$ *coincide inside the convex cone* $P = \mathrm{Nef}(X/Z) + \mathbb{R}_{\geq 0} \cdot [K+B]$.

We start with an easy observation.

Lemma 4.9. *We have the following nonnegative intersection pairing:*

$$(\alpha, \beta) \in \overline{\mathrm{Mob}}(X/Z) \times \overline{\mathrm{bNM}}^1(X/Z) \longmapsto \alpha \cdot \beta \geq 0.$$

Proof. Let D be an $\mathbb{R}$-mobile divisor and C be a b-mov^1-curve in X/Z. Since the numerical classes in $\mathrm{N}_1(X/Z)$ are preserved under a small birational map, we may assume that C is a mov^1-curve in X/Z. Then since C moves in a family of curves covering a subvariety of codimension 1, we may assume that C is disjoint from the base locus of D which is of codimension≥ 2. Thus $C \cdot D \geq 0$. The classes α and β are the limits of the classes of such curve C and divisor D. Therefore $\alpha \cdot \beta \geq 0$ by continuity. □

Proof. of Theorem 1.1
(Step 1. Dualizing) As we stated above, we prove its dual statement. We use the argument used in [6, Theorem 2.2]. By Lemma 4.9, we have $\overline{\mathrm{Mob}}(X/Z) \subseteq \overline{\mathrm{bNM}}^1(X/Z)^\vee$. This in particular implies

$$\overline{\mathrm{Mob}}(X/Z) \cap P \subseteq \overline{\mathrm{bNM}}^1(X/Z)^\vee \cap P,$$

where $P = \overline{\mathrm{Nef}}(X/Z) + \mathbb{R}_{\geq 0} \cdot [K+B]$. Suppose that the strict inclusion $\subsetneq$ holds. Note that since $\overline{\mathrm{bNM}}^1(X/Z) \supseteq \overline{\mathrm{NM}}(X/Z)$, $\overline{\mathrm{bNM}}^1(X/Z)^\vee \subseteq \overline{\mathrm{Eff}}(X/Z)$ holds by (2) of Theorem 4.8. Note also that $\overline{\mathrm{bNM}}^1(X/Z)^\vee \subseteq \bigcap \overline{\mathrm{NM}}^1(X, X'/Z)^\vee$, where the intersection is taken over all $\mathbb{Q}$-factorial X'/Z which are isomorphic to X/Z in codimension 1. Therefore, if the strict inclusion holds, then there exists a big divisor $D \in \partial\overline{\mathrm{Mob}}(X/Z) \cap \mathrm{Int}\, P$ and a small open neighborhood of D contained in $\overline{\mathrm{NM}}^1(X, X'/Z)$ for any $\mathbb{Q}$-factorial X'/Z which are isomorphic to X/Z in codimension 1.

(Step 2. MMP) There exists an ample divisor H such that $rD \equiv K + B + H$ for some $r > 0$. By rescaling, we may assume that $D \equiv K + B + H$. By Lemma 2.2, we may assume that $(X/Z, B+H)$ is klt. By running the LMMP on $(X/Z, B+H)$ with scaling of $r'H$ for some sufficiently large $r' > 0$, we obtain a log terminal model $f : X \dashrightarrow Y/Z$ of $(X/Z, B)$. Note that since $D \in \partial\overline{\mathrm{Mob}}(X/Z)$, $K + B + H + aH$ is $\mathbb{R}$-mobile for any $a > 0$. Therefore, the modification f is small.

(Step 3. Approximation by restricted volume) Since small modifications preserve the numerical classes of $\mathrm{N}^1(X/Z) \cong \mathrm{N}^1(Y/Z)$, $D_Y - \varepsilon A \in \overline{\mathrm{NM}}^1(Y/Z)^\vee$ for a fixed ample divisor A and for all sufficiently small $\varepsilon > 0$. In particular, this implies that $(D_Y - \varepsilon A) \cdot C \geq 0$ for any mov^1-curve C on Y/Z, i.e.,

$$(*) \qquad \frac{D_Y \cdot C}{A \cdot C} \geq \varepsilon.$$

Now since $D \equiv K+B+H \in \partial\overline{\mathrm{Mob}}(X/Z)$ and D is big, there exists an irreducible divisorial component $E \subseteq \mathbf{B}_+(D)$ by Lemma 4.2. Since f is small, E_Y is also an irreducible divisorial component $E_Y \subseteq \mathbf{B}_+(D_Y)$ by Proposition 4.3. Furthermore, since D_Y is nef, $D_Y + \lambda A$ is ample for any $\lambda > 0$. We may assume that $D + \lambda A$ is a $\mathbb{Q}$-divisor and let M_λ be a general member of the linear system defined by an integral divisor $m(D_Y + \lambda A)$ for some integer $m > 0$. Let $C = E_Y \cap M_\lambda^{d-2}$ be a mov^1-curve on Y/Z covering E_Y. Then

$$\begin{aligned} mD_Y \cdot C &= mD_Y \cdot M_\lambda^{d-2} \cdot E_Y \\ &\leq m(D_Y + \lambda A) \cdot M_\lambda^{d-2} \cdot E_Y \\ &= M_\lambda \cdot M_\lambda^{d-2} \cdot E_Y \\ &= M_\lambda^{d-1} \cdot E_Y \\ &= \mathrm{vol}_{Y|E_Y}(M_\lambda). \end{aligned}$$

By the inequality of Theorem 4.6 restricted on E_Y, we also have

$$A \cdot C = A \cdot M_\lambda^{d-2} \cdot E_Y \geq \mathrm{vol}_{Y|E_Y}(A)^{\frac{1}{d-1}} \, \mathrm{vol}_{Y|E_Y}(M_\lambda)^{\frac{d-2}{d-1}}.$$

Combining the above two inequalities, we obtain

$$(**) \qquad \frac{D_Y \cdot C}{A \cdot C} \leq \frac{\frac{1}{m}\,\mathrm{vol}_{Y|E_Y}(M_\lambda)}{\mathrm{vol}_{Y|E_Y}(A)^{\frac{1}{d-1}}\,\mathrm{vol}_{Y|E_Y}(M_\lambda)^{\frac{d-2}{d-1}}} \leq \frac{\mathrm{vol}_{Y|E_Y}(M_\lambda)^{\frac{1}{d-1}}}{\mathrm{vol}_{Y|E_Y}(A)^{\frac{1}{d-1}}}.$$

Note that $E_Y \not\subseteq \mathbf{B}_+(D_Y + \lambda A)(=\emptyset)$ because $D_Y + \lambda A$ is ample. In particular, $D_Y + \lambda A \in \mathrm{Big}^{E_Y}(Y/Z)$. Since E_Y is an irreducible component of $\mathbf{B}_+(D_Y)$, we have

$$\mathrm{vol}_{Y|E_Y}(M_\lambda) = \mathrm{vol}_{Y|E_Y}(D_Y + \lambda A) \to 0 \ \text{ as } \ \lambda \to 0$$

by Theorem 4.5. Thus we can make $\frac{D_Y \cdot C}{A \cdot C}$ in $(**)$ arbitrarily small, but it is a contradiction to $(*)$. □

5. Cone theorems

We now study the structure of the cone $\overline{\mathrm{bNM}}^1(X/Z)$. We prove Theorem 1.2 and Theorem 1.3 in this section.

Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial klt pair. In the numerical space $\mathrm{N}^1(X/Z)$, we consider the following two convex cones:

$$\begin{aligned} V(X/Z,B)(=V) &:= \overline{\mathrm{NE}}(X/Z)_{K+B\geq 0} + \overline{\mathrm{bNM}}^1(X/Z) \\ V'(X/Z,B)(=V') &:= \overline{\mathrm{NE}}(X/Z)_{K+B\geq 0} + \overline{\mathrm{NM}}(X/Z) \end{aligned}$$

An extremal face F of $\overline{\mathrm{bNM}}^1(X/Z)$ is called a *mov*^{1}*-extremal face* for the pair $(X/Z, B)$ if F is a $(K+B)$-negative extremal face of V. A divisor D which is positive on $\overline{\mathrm{NE}}(X/Z)_{K+B\geq 0} \setminus \{0\}$ and such that the plane $\{\alpha \in \mathrm{N}_1(X/Z) | \alpha \cdot D = 0\}$ supports the cone V exactly at F is called a *mov*^{1}*-bounding divisor* of F. An extremal face F' of $\overline{\mathrm{NM}}(X/Z)$ is called a *co-extremal face* for the pair $(X/Z, B)$ if F' is a $(K+B)$-negative extremal face of V'. A divisor D which is positive on $\overline{\mathrm{NE}}(X/Z)_{K+B\geq 0} \setminus \{0\}$

and such that the plane $\{\alpha \in \mathrm{N}_1(X/Z) | \alpha \cdot D = 0\}$ supports the cone V' exactly at F' is called a *co-bounding divisor* of F'.

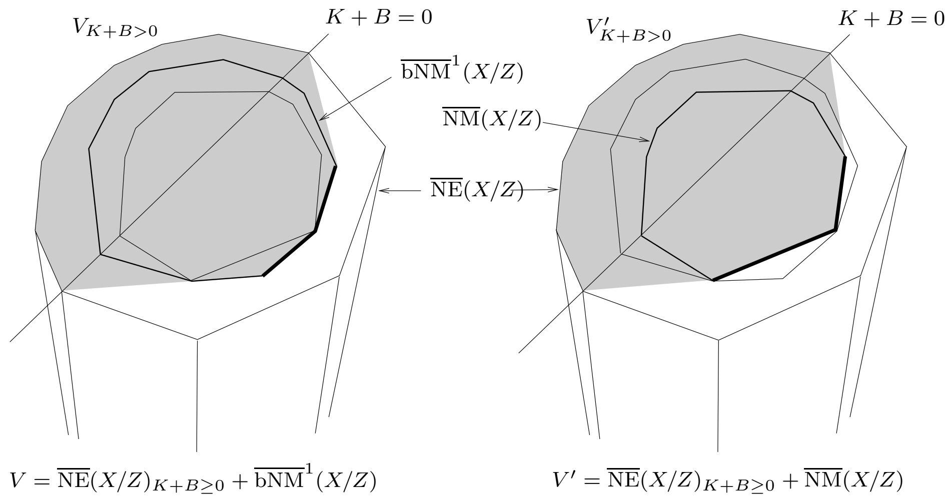

FIGURE 1

As illustrated in Figure 1, an extremal face of $\overline{\mathrm{bNM}}^1(X/Z)$ (resp. $\overline{\mathrm{NM}}(X/Z)$) in $\overline{\mathrm{NE}}_{K+B<0}(X/Z)$ is not necessarily a mov^1-extremal face (resp. a co-extremal face) because it may not be exposed in the cone V (resp. V'). Note also that a co-extremal face of $\overline{\mathrm{NM}}(X/Z)$ can coincide with a mov^1-extremal face of $\overline{\mathrm{bNM}}^1(X/Z)$.

It is easy to see that the cone $\overline{\mathrm{bNM}}^1(X/Z)$ has a mov^1-extremal ray if and only if $K+B \notin \overline{\mathrm{Mob}}(X/Z)$ by Theorem 1.1 and the cone $\overline{\mathrm{NM}}(X/Z)$ has a co-extremal ray if and only if $K+B \notin \overline{\mathrm{Eff}}(X/Z)$ by (2) of Theorem 4.8.

We have the following cone theorem for $\overline{\mathrm{NM}}(X/Z)$ and the contraction theorem for co-extremal rays.

Theorem 5.1 (Cone Theorem for $\overline{\mathrm{NM}}(X/Z)$)**.** [1, Theorem 1.1], [16, Theorem 1.1] *Let $(X/Z, B)$ be a dlt pair. There are countably many $(K+B)$-negative movable curves $\{C_i\}_{i\in I}$ such that*

$$\overline{\mathrm{NE}}(X/Z)_{K+B\geq 0} + \overline{\mathrm{NM}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K+B\geq 0} + \overline{\sum_{i\in I} \mathbb{R}_{\geq 0} \cdot [C_i]}.$$

The rays $\mathbb{R}_{\geq 0} \cdot [C_i]$ can accumulate only along the hyperplanes supporting both $\overline{\mathrm{NM}}(X/Z)$ and $\overline{\mathrm{NE}}(X/Z)_{K+B\geq 0}$.

Theorem 5.2 (Contraction theorem for co-extremal faces)**.** [16, Theorem 1.3] *Let $(X/Z, B)$ be a dlt pair. Suppose that F' is a co-extremal face of $\overline{\mathrm{NM}}(X/Z)$ and D be* *a co-bounding divisor of F'. Then there exists a birational morphism $\varphi : W \to X$ and a contraction $h : W \to Z$ such that*

(1) *Every movable curve C on W with $[\varphi_* C] \in F'$ is contracted by h.*
(2) *For a general pair of points in a general fiber of h, there is a movable curve C through the two points with $[\varphi_*(C)] \in F'$.*

These properties determine the pair (W, h), up to a birational equivalence. In fact, the map we construct satisfies a stronger property:

(3) *There is an open set $U \subset W$ such that the complement of U has codimension 2 in a general fiber of h and a complete curve C in U is contracted by h if and only if $[\varphi_* C] \in F'$.*

Remark 5.3. *If $K+B \in \partial\overline{\mathrm{Eff}}(X/Z)$, then $\overline{\mathrm{NM}}(X/Z) \subseteq \overline{\mathrm{NE}}(X/Z)_{K+B\geq 0}$ by Theorem 4.8. Thus there are no co-extremal faces for the pair $(X/Z, B)$. However, there exists an extremal face F' of $\overline{\mathrm{NM}}(X/Z)$ in $\overline{\mathrm{NE}}(X/Z)_{K+B=0}$. If B is big, then $K+B-\varepsilon B$ is not pseudoeffective for any $\varepsilon > 0$ because F' is $(K+(1-\varepsilon)B)$-negative. Thus there exists a co-extremal ray of $\overline{\mathrm{NM}}(X/Z)$ for the pair $(X/Z, (1-\varepsilon)B)$ and since $(X/Z, (1-\varepsilon)B)$ is klt for small $\varepsilon > 0$, the above theorems can be applied to this pair. In particular, some extremal rays of F' are contractible on some birational model of X/Z.*

We prove the analogous results for the cone $\overline{\mathrm{bNM}}^1(X/Z)$. Before we prove Theorem 1.2, we note the following.

Example 5.4. *As explained in* [16, Example 6.2]*, a Cutkosky's example* [8] *shows that the genuine form of the cone theorem does not hold for $\overline{\mathrm{NM}}(X/Z)$, that is, we cannot replace $\overline{\mathrm{NE}}(X/Z)$ in Theorem 5.1 by $\overline{\mathrm{NM}}(X/Z)$. The same is true for the cone theorem for $\overline{\mathrm{NM}}^1(X/Z)$. For readers' convenience, we construct the example following* [16, Example 6.2]*. Let Y be an abelian surface with Picard number $\rho \geq 3$. Then the three cones coincide: $\overline{\mathrm{Eff}}(Y) = \overline{\mathrm{Mob}}(Y) = \mathrm{Nef}(Y)$ and the cone is circular in $\mathrm{N}^1(Y)$. Let L be a divisor on Y such that $-L$ is ample and let $X = \mathbb{P}_Y(\mathcal{O} \oplus \mathcal{O}(L))$ be the $\mathbb{P}^1$-bundle over Y with $\pi : X \to Y$. If S is the section of π such that $S|_S \sim L$, then any divisor on X has the form $aS + \pi^* D$ and $K = -2S + \pi^* L$. It is shown in* [16] *that the cone $\overline{\mathrm{Eff}}(X)$ is spanned by S and $\pi^*\overline{\mathrm{Eff}}(Y)$, which implies that $-K$ is big. It is also easy to see that $\overline{\mathrm{Eff}}(X)$ coincides with the nef cone $\mathrm{Nef}(X)$. Therefore $-K$ is also mobile. Since $\overline{\mathrm{bNM}}^1(X)$ is K-negative and it has circular part in $\partial\overline{\mathrm{bNM}}^1(Y)$, Theorem 1.2 fails when $\overline{\mathrm{NE}}(X)$ is replaced by $\overline{\mathrm{bNM}}^1(X)$. Similarly, Theorem 5.1 also fails when $\overline{\mathrm{NE}}(X)$ is replaced by $\overline{\mathrm{NM}}(X)$ as explained in* [16, Example 6.2].

To prove Theorem 1.2, we start with a lemma.

Lemma 5.5. *Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial klt pair. Let $K+B \notin \overline{\mathrm{Mob}}(X/Z)$ and D be a mov^1-bounding divisor for a mov^1-extremal face F of $\overline{\mathrm{bNM}}^1(X/Z)$ for $(X/Z, B)$.*

Then there exists an ample divisor H such that $K+B+H$ is ample and $\alpha D \equiv K+B+cH$ for some $\alpha > 0$ and $0 < c < 1$.

Proof. Let G be the 2-dimensional closed cone in $\mathrm{N}^1(X/Z)$ spanned by D and $-(K+B)$. It is enough to prove that the $\emptyset \neq G \cap \mathrm{Amp}(X/Z) \setminus \{0\}$. Indeed, a sufficiently large ample divisor $H \in G \cap \mathrm{Amp}(X/Z)$ satisfies the conditions with $\frac{1}{t} = m_H(X,B)$.

Suppose that $G \cap \mathrm{Amp}(X/Z) = \{0\}$. Then there exists a curve class C which separates the two cones: $L \cdot D' < 0$ for all $D' \in G \setminus \{0\}$ and $L \cdot D'' > 0$ for all $D'' \in \mathrm{Amp}(X/Z) \setminus \{0\}$. By the second inequality, $L \in \overline{\mathrm{NE}}(X/Z)$. The first inequality with $D' = -(K+B)$ gives $L \in \overline{\mathrm{NE}}(X/Z)_{K+B>0}$. However the first inequality with $D' = D$ also gives $L \cdot D < 0$, contradicting the fact that D is positive on $\overline{\mathrm{NE}}(X/Z)_{K+B>0}$. □

Conversely, it is easy to see that the divisors of the form $D \equiv K+B+H$ for an ample divisor H which are in $\partial\overline{\mathrm{Mob}}(X/Z)$ are mov^1-bounding divisors of some mov^1-extremal face.

Proposition 5.6. *Let $(X/Z,B)$ be a $\mathbb{Q}$-factorial klt pair. Consider the cone*

$$\overline{\mathrm{NE}}(X/Z)_{K+B+H\geq 0} + \overline{\mathrm{bNM}}^1(X/Z)$$

for some ample divisor H such that $(X/Z, B+H)$ is klt. There exists a finite set $\{C_i\}$ of b-mov^1-curves of X/Z such that for any mov^1-bounding divisor D for a mov^1-extremal face of the cone $\overline{\mathrm{bNM}}^1(X/Z)$ for $(X/Z, B+H)$, $[C_i]\cdot[D] = 0$ for some C_i.

Proof. Let D be a mov^1-bounding divisor as in the statement. Then by Lemma 5.5, there exists an ample divisor A such that $K+B+H+A$ is ample and $\alpha D \equiv K+B+H+cA$ for $\alpha > 0$ and $0 < c < 1$. By Lemma 2.2, we may assume that the pair $(X/Z, B+H+A)$ is klt. By Theorem 3.4, running the LMMP on $(X/Z, B+H+cA)$ with scaling of $(1-c)A$, we obtain a birational map $\varphi : X \dashrightarrow X'/Z$ which is an isomorphism in codimension 1 and a contraction $\psi : X' \to Y$ which is either a divisorial contraction or a Mori fibration. If $K+B+H+cA$ is big, then ψ is divisorial and there exists a mov^1-curve C' on X'/Z contracted by ψ. If $K+B+H+cA$ is in the boundary $\partial\overline{\mathrm{Eff}}(X/Z)$, then the curve C' is movable. Thus, in either case, we obtain a b-mov^1-curve C' of X/Z. By the finiteness of geography (Theorem 3.1), as we vary D in $\partial\overline{\mathrm{Mob}}(X/Z) \cap \mathcal{E}_A$, we obtain only finitely many maps $\psi \circ \varphi$ and consequently finitely many b-mov^1-curves. □

Proof. of Theorem 1.2 We may assume that $(X/Z, B+H)$ is klt. Let $\{\varepsilon_j\}$ be a strictly decreasing positive sequence converging to 0. Let $\{C_{ji}\}_{i\in I_j}$ be the finite set of all b-mov^1-curves obtained by running the LMMP on $(X/Z, B+\varepsilon_j H)$ as in Proposition 5.6. Then clearly,

$$\overline{\mathrm{NE}}(X/Z)_{K+B+\varepsilon_j H\geq 0} + \overline{\mathrm{bNM}}^1(X/Z) \supseteq \overline{\mathrm{NE}}(X/Z)_{K+B+\varepsilon_j H\geq 0} + \overline{\sum_{i\in I_j} \mathbb{R}_{\geq 0}\cdot [C_{ji}]}.$$

Suppose that the strict inclusion $\supsetneq$ holds. Then there exists a mov^1-extremal ray R such that $R \setminus \{0\}$ is disjoint from $\overline{\mathrm{NE}}(X/Z)_{K+B+\varepsilon_j H \geq 0} + \overline{\sum_{i \in I_j} \mathbb{R}_{\geq 0} \cdot [C_{ji}]}$. If D is a mov^1-bounding divisor of R for $(X/Z, B + \varepsilon_j H)$, then by Lemma 5.5, there exists an ample divisor A such that $K + B + \varepsilon_j H + A$ is ample and $\alpha D \equiv K + B + \varepsilon_j H + cA$ for $\alpha > 0$ and $0 < c < 1$. Since we may assume that $(X/Z, B + \varepsilon_j H + A)$ is klt, by running the LMMP on $(X/Z, B + \varepsilon_j H + cA)$ with scaling of $(1 - c)A$, we obtain a b-mov^1-curve C of X/Z (see the proof of Proposition 5.6) such that $R = \mathbb{R}_{\geq 0} \cdot [C]$. Since $C \notin \{C_i\}$, it is a contradiction.

Suppose that the set $\cup_{j \in \mathbb{N}} I_j$ is infinite. By removing ε_{j+1} such that $I_j = I_{j+1}$ from the sequence $\{\varepsilon_j\}$, we may assume that $I_j \subsetneq I_{j+1}$ for all j. By taking the limit $\lim_{j \to \infty}$, we obtain the second equality of the cones and the last statement. □

Proof. of Theorem 1.3 If $K + B \in \overline{\mathrm{Mob}}(X/Z)$, then $\overline{\mathrm{NE}}(X/Z)_{K+B \geq 0} \supseteq \overline{\mathrm{bNM}}^1(X/Z)$ and there are no mov^1-extremal rays. Thus statements are vacuous. Suppose that $K + B \notin \overline{\mathrm{Mob}}(X/Z)$. By Lemma 5.5, there exists an ample divisor H such that $K + B + H$ is ample and $D = K + B + cH$ $(0 < c < 1)$ is a mov^1-bounding divisor for R. We may also assume that $(X/Z, B + H)$ is klt by Lemma 2.2. By Theorem 3.4, we can run the LMMP on $(X/Z, B + cH)$ with scaling of $(1 - c)H$ to obtain a resulting log terminal model $\varphi : X \dashrightarrow X'/Z$ of $(X/Z, B + cH)$. Since $D \equiv K + B + cH \in \partial \overline{\mathrm{Mob}}(X/Z)$, the divisors $K + B + cH + rH$ for $0 < r \leq 1$ are $\mathbb{R}$-mobile. Thus the birational map $\varphi : X \dashrightarrow X'$ is small.

If $D \in \partial \overline{\mathrm{Eff}}(X/Z)$, then the ray R is a co-extremal ray of $\overline{\mathrm{NM}}(X/Z)$ for $(X/Z, B)$. By Theorem 3.4, there exists a Mori fibration $X' \to Y$ and the statements follow from Theorem 5.2.

Assume that $D \in \mathrm{Int}\, \overline{\mathrm{Eff}}(X/Z)$. Then the ray R is spanned by a mov^1-curve C' in X'/Z (which is a b-mov^1-curve of X/Z). Its associated contraction $\psi : X' \to Y$ is divisorial by Theorem 3.4. Therefore, by applying the usual Contraction Theorem ([13, Theorem 3.6]), we obtain the statements (1) and (2). □

Remark 5.7. *As illustrated in the Figure 1, there may be an extremal ray R of $\overline{\mathrm{bNM}}^1(X/Z)$ which is not mov^1-extremal, but co-extremal. This ray is not necessary in the expression $\overline{\mathrm{NE}}_{K+B \geq 0}(X/Z) + \overline{\sum_{i \in I} \mathbb{R}_{\geq 0} \cdot [C_i]}$ because it is not exposed in this cone. However, the statements in Theorem 1.3 also hold for this ray by Theorem 5.2.*

In the statements of Theorem 1.3, if $K + B$ is not big, then $\psi : X' \to Y$ is a Mori fibration and this is a resulting model of the given pair $(X/Z, B)$. Note that if $K + B \in \partial \overline{\mathrm{Eff}}(X/Z)$, then $K_{X'} + B_{X'}$ is ψ-trivial and Y is the lc Iitaka model of $(X/Z, B)$. If $K + B$ is big and $K + B \in \partial \overline{\mathrm{Mob}}(X/Z)$, then $(X'/Z, B_{X'})$ is a resulting model which is a log terminal model of $(X/Z, B)$ and the contraction $\psi : X' \to Y$ is the lc contraction to the lc model $Y = X_{\mathrm{lcm}}$ of $(X/Z, B)$. For all other cases, namely, when $K+B$ is big but not in $\overline{\mathrm{Mob}}(X/Z)$, the divisorial contraction ψ is only one of the intermediate modifications of the LMMP.

Remark 5.8. *If $K+B \in \overline{\mathrm{Mob}}(X/Z)$, then the cone $\overline{\mathrm{bNM}}^1(X/Z)$ does not have any mov^1-extremal faces. However, if $K+B \in \partial\overline{\mathrm{Mob}}(X/Z)$, then $\overline{\mathrm{bNM}}^1(X/Z)$ has extremal faces in $\overline{\mathrm{NE}}(X/Z)_{K+B=0}$. If $K+B$ is big or $B \in \mathrm{Int}\,\overline{\mathrm{Mob}}(X/Z)$, then some of such faces F are mov^1-extremal for some pair and Theorem 1.3 holds for these rays too. Indeed, suppose $\overline{\mathrm{Mob}}(X/Z) \subseteq \overline{\mathrm{NE}}(X/Z)_{K+B\geq 0}$ and let F be an extremal face of $\overline{\mathrm{bNM}}^1(X/Z)$ in $\overline{\mathrm{NE}}(X/Z)_{K+B=0}$. If $K+B$ is big, then $K+B \equiv H+E$ for some ample H and effective E. For small $\varepsilon > 0$, $K+B+\varepsilon E$ is big and $(X/Z, B+\varepsilon E)$ is still klt. However, $K+B+\varepsilon E \notin \overline{\mathrm{Mob}}(X/Z)$ since we can easily check that F is $(K+B+\varepsilon E)$-negative and $\overline{\mathrm{NE}}(X/Z)_{K+B+\varepsilon E=0}$ does not intersect with the supporting plane $\{[C] \in \mathrm{N}_1(X/Z) \mid C\cdot(K+B)=0\}$. Therefore, F is a mov^1-extremal face of $\overline{\mathrm{bNM}}^1(X/Z)$ for the pair $(X/Z, B+\varepsilon E)$ and $K+B$ is a mov^1-bounding divisor for F. Since the extremal rays of F are mov^1-extremal rays, Theorem 1.2 and Theorem 1.3 can be applied to this case. The similar argument works for the case when $B \in \mathrm{Int}\,\overline{\mathrm{Mob}}(X/Z)$ (cf. Remark 5.3).*

Lastly, we give a partial answer to the problem posed in [19].

A $\mathbb{Q}$-factorial variety X/Z is said to be *Fano type(FT)* if there exists a boundary divisor B on X/Z such that $(X/Z,B)$ is klt and $K+B \sim_{\mathbb{R}} 0/Z$ (see [20, Lemma-Definition 2.8] for equivalent definitions).

Corollary 5.9. *For an FT(Fano type) variety X/Z, the following duality holds:*

$$\overline{\mathrm{Mob}}(X/Z)^\vee = \overline{\mathrm{bNM}}^1(X/Z).$$

Furthermore, the cones $\overline{\mathrm{Mob}}(X/Z)$ and $\overline{\mathrm{bNM}}^1(X/Z)$ are closed convex and rational polyhedral.

Proof. There exists a boundary divisor B such that $(X/Z,B)$ is klt, $K+B \sim_{\mathbb{R}} 0$ and the components of B generate $\mathrm{N}^1(X/Z)$ (cf. [20, Lemma-Definition 2.8]). There exists an ample divisor A such that $\mathrm{Supp}\,A = \mathrm{Supp}\,B$. The pair $(X/Z, B-\varepsilon A)$ is klt for sufficiently small $\varepsilon > 0$ and $-(K+B-\varepsilon A)$ is ample. Therefore, the cone $\overline{\mathrm{NE}}(X/Z)$ is $(K+B-\varepsilon A)$-negative and it follows immediately from Theorem 1.1 and Theorem 1.2. □

The second part of Corollary 5.9 also follows from the rational polyhedral property and the finiteness of the Geography. See [7, Corollary 3.5].

Remark 5.10. *Corollary 5.9 gives an affirmative answer to the problem posed in* [19] *for the Fano type varieties X with the case $k = \dim X - 1$. It is also easy to see from the proof of Theorem 1.1 that the same result holds for Mori dream spaces*

[11]. *Indeed, the duality holds in the portion of the cone* $\overline{\mathrm{Mob}}(X/Z)$ *where we can run the MMP. In* [19]*, it is shown that for complete* $\mathbb{Q}$*-factorial toric varieties* X *and* $0 \leq k \leq \dim X$*, the duality holds between the closed cone in* $\mathrm{N}^1(X/Z)$ *spanned by divisors that are ample in codimension* k *and the closed cone in* $\mathrm{N}_1(X/Z)$ *spanned by the curves that are birationally movable in codimension* k*. (see* [19, Theorem 2]*). It is also explained that considering only the* mov^1*-curves in Theorem 1.1 is not enough (see* [19, Example 1]*).*

Question 5.11. *In* [3]*, Batyrev conjectured that the extremal rays* $R_i = \mathbb{R}_{\geq 0} \cdot [C_i]$ *in Theorem 5.1 do not accumulate away from* $\overline{\mathrm{NE}}(X/Z)_{K+B=0}$*. Similarly, we can ask whether the* mov^1*-extremal rays in* $R_i = \mathbb{R}_{\geq 0} \cdot [C_i]$ *in Theorem 1.2 can accumulate away from* $\overline{\mathrm{NE}}(X/Z)_{K+B=0}$*. For the results related to the conjecture of Batyrev or the cone* $\overline{\mathrm{NM}}(X/Z)$*, see* [1],[2],[3],[22].

References

[1] C. Araujo, The cone of pseudo-effective divisors of log varieties after Batyrev, Math. Z. 264, no. 1, 179-193, (2010)

[2] S. Barkowski, The cone of moving curves of a smooth Fano three- or fourfold, manuscripta mathematica. Vol 131, Numbers 3-4, 305-322, (2010)

[3] V.V. Batyrev, The cone of effective divisors of threefolds, Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989), Contemp. Math., 131, Part 3, AMS, 337–352, (1992)

[4] C. Birkar, P. Cascini, C. Hacon, J. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. 23, no. 2, 405-468, (2010)

[5] S. Boucksom, A. Broustet, G. Pacienza, Uniruledness of stable base loci of adjoint linear systems with and without Mori Theory, arXiv:0902.1142v2 [math.AG], (2009, Preprint)

[6] S. Boucksom, J.-P. Demailly, M. Paun, T. Peternell, The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension, arXiv:math/0405285v1 [math.AG], (2004, Preprint)

[7] S. Choi, V.V. Shokurov, Geography of models: theory and applications, arXiv:0909.0288v2 [math.AG], (2009, Preprint)

[8] S. Cutkosky, Zariski decomposition of divisors on algebraic varieties, Duke Math. J. 53, no. 1, 149-156, (1986)

[9] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, Mihnea Popa, Restricted volumes and base loci of linear series, Amer. J. Math. 131, no. 3, 607-651, (2009)

[10] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, Mihnea Popa, Asymptotic invariants of base loci, Ann. Inst. Fourier, 56 no.6, 1701-1734, (2006)

[11] Y. Hu, S. Keel, Mori dream spaces and GIT, Michigan Math. J. 48, 331-348, (2000)

[12] V.A. Iskovskikh and V.V. Shokurov, Birational models and flips, Russian Mathematical Surveys, 60, 27-94, (2005)

[13] J. Kollár and S. Mori, Birational geometry of algebraic varieties, Cambridge tracts in mathematics, vol. 134, Cambridge University Press, Cambridge UK, (1998)

[14] R. Lazarsfeld, Positivity in algebraic geometry I, 48. xviii+385 pp. Springer-Verlag, Berlin, (2004)

[15] R. Lazarsfeld, Positivity in algebraic geometry II, 49. xviii+387 pp. Springer-Verlag, Berlin, (2004)

[16] B. Lehmann, A cone theorem for nef curves, arXiv:0807.2294v3 [math.AG], (2008, Preprint)
[17] K. Matsuki, Introduction to the Mori program, Universitext. Springer-Verlag, New York, (2002)
[18] N. Nakayama, Zariski-decomposition and abundance, MSJ Memoirs, 14. Mathematical Society of Japan, Tokyo, (2004)
[19] S. Payne, Stable base loci, movable curves, and small modifications, for toric varieties, Math. Z. 253, no. 2, 421-431, (2006)
[20] Yu.G. Prokhorov and V.V. Shokurov, Towards the second main theorem on complements, J. Algebraic Geom. 18, no. 1, 151-199, (2009)
[21] V.V. Shokurov, 3-fold log models, Algebraic Geometry, 4, J.Math. Sci. 81, Consultants Bureau, New York, 2667-2699, (1996)
[22] Q. Xie, The Nef Curve Cone Theorem Revisited, arXiv:math/0501193v2 [math.AG], (2005, Preprint)

Department of Mathematics
University of California, Riverside
900 University Ave
Riverside, CA 92521, USA
E-mail address: schoi@math.ucr.edu