Tiling a convex body into possibly similar pieces

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November 24, 2010

Abstract

Generalizing results by Valette, Zamfirescu and Laczkovich, we will here prove that the tiling of any convex body K into convex subsets contains some information on K itself, namely if many tiles are similar to K, K must be a polytope.

Consider a convex body (a compact convex set with nonempty interior) K in $\mathbb{R}^n$, which is tiled into a finite number of convex bodies. A tiling T, as a formal object, will here be the set consisting of all its tiles, and all tiles will be assumed to be convex bodies. Also, this investigation will only consider proper tilings, that is, tilings which are not trivial (meaning they consist of more than just one tile).
Let there be tiles similar to K. Does it follow that K is a polytope?
For dimension 2, M. Laczkovich ([1]) could show that if one tile is similar to K, and if the tiling is proper, K is in fact a polytope. He generalized a remark by G. Valette and T. Zamfirescu in [2].
Now the question arises: is this extendable for dimension 3? Already the original paper by Lazckovich contained a remark by Zamfirescu that this must be wrong, since a circular cone can easily be tiled in such a way that one tile is similar to the cone by just cutting near the apex. It was therefore conjectured: Consider a convex body K in 3-space which is tiled in such a way such that 2 tiles are similar to K. Then K is a polytope.
This will turn out to be true, and is a very special case of the general theorem, because in dimensions higher than 3, a condition only on the number of similar tiles will never be sufficient (see the example at the end of this paper). The additional condition will encode some information on how the tiles are located relative to the convex body itself. In consistency with our observations, the condition will degenerate in dimensions 2 and 3.
Let us state the theorem:

Theorem 1. *Let K be a n-dimensional convex body, and let T_i, $\{1, 2, 3, ..., n-1\}$ be $n-1$ proper tilings of K, each of which contains a tile L_i similar to K.*
If the convex hull of the fixed points x_{L_i} forms a nondegenerate $n-2$-dimensional simplex, then K is a polytope.

Here, the point x_L is the fixed points of the similarity f_L from K to L. If there is more than one similarity from K to L, choose one.

The simplex referred to in theorem 1 will be called the *tip simplex* of K with respect to the tilings involved.
Often, one is in the situation to have a single tiling, and several tiles are similar to K. This is a special case of above theorem, which is a bit harder to prove, since then one has to deal with interdependencies between the tiles similar to K when deforming tilings.

Corollary 2. *Let K be a n-dimensional convex body, which is properly tiled into a finite number of convex bodies, $n-1$ of which are similar to K. Denote these particular tiles by L_i.*
If the convex hull of the fixed points x_{L_i} forms a nondegenerate $(n-2)$-dimensional simplex, then K is a polytope.

In dimensions 2 and 3, the condition on the tip simplex is automatically fulfilled, giving the following theorem:

Corollary 3. *Let K be a convex body in $\mathbb{R}^n$, $n \leq 3$, which is properly tiled into a finite number of convex bodies, $n-1$ of which are similar to K. Then K is a polygon/ a polyhedron.*

Even more, in dimension 3, symmetry also plays a role in this calculation:

Corollary 4. *Let K be a convex body in $\mathbb{R}^3$, which is properly tiled into a finite number of convex bodies, 1 of which is similar to K. If the fixed point x_L is dependent on the similarity from K to L, K is a polyhedron.*

Denote by $\text{bd}(M)$, $\text{int}(M)$, $\mathscr{E}(M)$, $\text{conv}(M)$ the the boundary/ the interior/ set of extremal points/ the convex hull of a set M. Also, $B_\varepsilon(x)$ denotes the set of all points in $\mathbb{R}^n$ with distance less than ε to x.
Consider the general situation of a convex body K tiled into convex bodies P_j. Since we are considering tilings K into convex bodies, the intersection of any two tiles $P_j \cap P_{j'}$, $j \neq j'$ is a $n-1$-dimensional convex compact set. There are only finitely many of these intersections, and any element of $\mathscr{E}(P_j) \cap \text{int}(K)$ will lie in one of these. Intersecting these sets again, we can further precise the position of these extremal points, until we know that $\mathscr{E}(P_j) \cap \text{int}(K)$ is always of finite cardinality.
Let us, from now on until the end of the paper, suppose that K is not a polytope, then K has infinitely many extremal points. We will deduce a contradiction from this, which will show that K must be a polytope.
Given a tiling T, denote the tile(s) similar to K by $L_{(i)}$. $\mathscr{E}(L) \cap \text{int}(K)$ is a finite set. This in turn implies that the similarity mapping f_L from K to L preserves infinitely many extremal points in $\text{bd}(K)$, in particular $\mathscr{E}(f_L^d(K)) \cap \text{bd}(K)$ is nonempty for each natural number d. Thus,

$$x_L \in \text{bd}(K).$$

Adapting a tiling

In this section, we focus on adapting tilings until they have features that come handy in the later proof. The first feat we would like to see in our tilings is that we want the similarity from K to L to be a homothety. After that, we want to make sure we can move the fixed point, at least as long as we stay in the tip simplex with it. Along the way, we will find out about a structural property of K supposing it fulfills the conditions of theorem 1.

Before we begin, let us state our methods to deform a tiling:
Let K be a convex body, and let T be a tiling of K, containing a tile L similar to K. Let f_L be a similarity mapping from K to L. Define $f_L(T)$ to be the function f_L applied to all the tiles of T, and thus a tiling of L. The idea is that we can refine the tiling T to T' by defining $T' := f_L(T) \cup T \setminus \{L\}$. In this situation, we will also write $T' = f_L(T) + T$. The similarity mappings of newly created tiles will be defined by the composition of the similarities involved. We call this procedure *iterating a tiling*.
In a related situation, suppose we are given two tilings T_1 and T_2, we could form the tiling $T_1 * T_2 := \{P \cap Q | \ P \in T_1, \ Q \in T_2\}$. If T_1 tiled a set K_1, and T_2 tiled K_2, then $T_1 * T_2$ is a tiling of the intersection of K_1 and K_2. In particular, if $K_1 = K_2$, then $T_1 * T_2$ is a (possibly refined) tiling of K_1.
Let us apply the above methods to concrete tilings to show that simplifications are indeed possible.

Lemma 5. *Let K be a convex body as above, T a tiling which contains a tile L similar to K. Then there is a tiling T' of K which contains a homothetic copy of K whose fixed point coincides with x_L.*

Proof. Let L be given as above, and f_L the similarity with respect to K. Note that because f_L is an affine linear mapping, we can speak of eigenvectors and eigenvalues of (the linear part of) f_L.
All eigenvalues of f_L are of equal absolute value, however, some might be complex. Still, because we can iterate tilings as described above, we easily dispose of complex eigenvalues if their argument is a rational multiple of π. We can therefore assume that all eigenvalues either have argument 0 or an argument that is an irrational multiple of π.
Dirichlet's theorem on simultaneous approximation by real numbers tells us that by iterating the similarity we can get it as close to a homothety as we want: Let M_L be the matrix representing the linear part of f_L, and let λ be the absolute value of an eigenvalue of f_L. Then, we can get $\frac{M_L^d}{\lambda^d}$ as close to the identity matrix as we please by adjusting $d \in \mathbb{N}$ (with respect to some matrix norm).
Obviously, the solid tangent cones of later iterations of $f_L^d(K)$ are included in earlier:

$$TC_{x_L}(K) \supset TC_{x_L}(L) \supset TC_{x_L}(f_L^2(K)) \supset TC_{x_L}(f_L^3(K))...,$$

where $TC_{x_L}(K)$ denotes the solid tangent cone of K at x_L. But since we can get f_L as close to a homothety as we want, we can get $B_\varepsilon(x) \cap TC_{x_L}(f_L^d(K))$ as close to $B_\varepsilon(x) \cap TC_{x_L}(K)$ (with respect to, for example, the Pompeiu-Hausdorff metric and a fixed $\varepsilon > 0$) as we want by choosing $d \in \mathbb{N}$. Thus, the solid tangent cones of K and $f_L^d(K)$, $d \in \mathbb{N}$ at x_L coincide. In particular, because all tiles are compact, convex and have nonempty interior, their tangent cones are never degenerate, and it follows that L is the only tile of T which contains x_L. Then there is an $\varepsilon > 0$ such that

$$B_\varepsilon(x_L) \cap K = B_\varepsilon(x_L) \cap L = B_\varepsilon(x) \cap TC_{x_L}(K)$$

In particular, x_L is not an accumulation point of extremal points of K.
Next, choose $d \in \mathbb{N}$ large enough so that

$$f_L^d(K) \subset B_\varepsilon(x) \cap TC_{x_L}(K).$$

Then $\mathrm{bd}(f_L^d(K)) \backslash \mathrm{bd}(TC_{x_L}(K)) \subset \mathrm{int}(K)$ and can thus be written as a finite union of $n-1$-dimensional compact convex sets. We will state this as a separate Lemma:

Lemma 6. *K can be written as*

$$\mathrm{conv}(\{x_L\} \bigcup_i B_i),$$

where the $B_i \in \mathrm{bd}(K)$ are finitely many convex compact $n-1$-dimensional manifolds with boundary (which can be chosen to be disjoint from $\{x_L\}$). We will call these manifolds a base of K.

Introduce the isometry $g := \lambda^d (f_L - x_L)^{-d} + x_L$, where λ is the absolute value of an eigenvector of f_L, and define $h = g \circ f_L^d$, which is a homothety. Still,

$$\mathrm{bd}(h(K)) \setminus \mathrm{bd}(TC_{x_L}(K)) \subset \mathrm{int}(K)$$

holds. Apply g to the tiling given by

$$T' = T + \sum_{i\ \in \{1,2,3,\ldots,d\}} f_L^i(T).$$

$g(T')$ contains $h(K)$ as one of its tiles (and thus, it contains a homothetic copy of K), but it may not be a tiling of K.
Enlarging $g(T')$ by a factor α using the homothety $H_\alpha(x) = \alpha(x - x_L) + x_L$, we get a new tiling T''_α. If α is large enough, $T''_\alpha * \{K\}$ will be a proper tiling of K into a finite set of convex bodies, and $H_\alpha(h(K))$ will be a tile. Since we haven't changed the fixed point during the process, it still coincides with x_L. □

Now that the tiles are directed nicely, we want to turn to adjusting their position.

Lemma 7. *Let K be a convex body as above, T_i tilings fulfilling the conditions of theorem 1, S the tip simplex, x_0 a point in the tip simplex. Then there is a tiling T' which fulfills the conditions of theorem 1, with a tile L similar to K such that $x_L = x_0$.*

Proof. If the tip simplex is just a point, the Lemma is trivial. Suppose therefore that there are at least two tilings with T_1 and T_2 with tiles L_1 respectively L_2 which are similar to K and have distinct fixed points, and suppose the corresponding similarities are chosen to be homotheties. Then the tiling $T_1 + f_{L_1}(T_2)$ will contain a homothetic copy of K: $L_3 = f_{L_1}(L_2)$. L_3 is new to us, because the corresponding fixed point will not coincide with x_{L_1} or x_{L_2}, but will lie in their convex hull. By iterating this procedure, we see that there is a dense subset M of $\mathrm{conv}(\{x_{L_1}, x_{L_2}\})$ we can make the fixed points of similarities lie in.

Next, suppose $x_0 \in \mathrm{conv}(\{x_{L_1}, x_{L_2}\}) \setminus M$. Find a tiling T with a similar copy L of K such that $x_L \in \mathrm{conv}(\{x_{L_1}, x_0\})$, and denote the eigenvalue of the homothety from K to L by λ. (Again, we assume the similarities to be homotheties.) Define $\alpha := \frac{|x_0 - x_{L_1}|}{|\lambda(x_0 - x_L) + x_L - x_{L_1}|}$, $H(x) := \alpha(x - x_{L_1}) + x_{L_1}$ and $T' := H(T)$. $T' * \{K\}$ is a tiling of K with tile $H(L)$ which has fixed point x_0. With higher dimensional simplices, this construction works just in the same way. □

Conclusion

We will prove theorem 1 by induction. In dimension 1, it is trivial, in dimension 2, it was proven by Laczkovich.

Let us assume theorem 1 is proven in dimension $n-1$. Let K be a convex body fulfilling the conditions of the theorem 1 in Dimension n, let x be some point in the relative interior of the tip simplex S, and let H be some $n-1$-dimensional affine subspace of $\mathbb{R}^n$ containing x. $H \cap K$ is a $n-1$-dimensional convex set. It could even be of smaller dimensions, so let us just assume $H \cap K$ is not a point. Let T_i' be proper tilings of K whose similarities are homotheties and whose fixed points coincide with the extremal points of the $n-3$-dimensional simplex $S \cap H$.

$H \cap K$ inherits a tiling structure from K by means of intersection: T_i' induces the tiling $\{H\} * T_i'$ on $H \cap K$. Note that because L_i' is a homothetic copy of K whose fixed point lies in H, $H \cap L_i'$ will be an element of T_i' which is a homothetic copy of $H \cap K$. The fixed point of $H \cap L_i'$ in $H \cap K$ coincides with the fixed point of L_i' in K and is therefore an extremal point of $S \cap H$. Since x is a relative interior point of S, $S \cap H$ is a nondegenerate $n-3$-dimensional simplex spanned by the fixed points of $H \cap L_i'$, which in turn are elements of proper tilings $\{H\} * T_i'$ of $H \cap K$.

Using the induction hypothesis, we see that $H \cap K$ must be a polytope.

Proposition 8. *Let K be a convex body in $\mathbb{R}^n$, which is tiled as in the description of theorem 1. Let S be the tip simplex, x a point in the relative interior of this simplex, and let H be some $n-1$-dimensional affine subspace of $\mathbb{R}^n$ containing x. Then $H \cap K$ is a polytope.*

We are almost done, it seems, and turn to the proof of theorem 1 in Dimension n, which in turn will prove proposition 8 in Dimension $n+1$ and so on.

Proof of theorem 1. As already stated, K is the convex combination of a finite number of $n-1$ dimensional compact convex manifolds and a point x in the relative interior of the tip simplex. Since we supposed K is not a polytope, one of these manifolds has infinitely many extremal points which coincide with extremal points of K. Call this manifold B, and recall that $B \subset \mathrm{bd}(K)$. Pick another point y in the relative interior of the tip simplex of K. Choose a $n-1$-dimensional affine subspace H parallel to B and containing y. If $H \cap \mathrm{conv}(\{x\} \cup B) = \emptyset$, interchange the roles of y and x. If H contains x, tilt H just a little such that it intersects $\mathrm{conv}(\{x\} \cup B)$ but neither $\{x\}$ nor the base B. The intersection of H and K is, as we know from proposition 8, a polytope. But $H \cap B$ will have infinitely many extremal points, as it is a (possibly dilated, if we tilted H) homothetic copy of B. Thus, K and B can only share a finite number of extremal points, in contradiction with the assumption. □

Sharpness of results

We could now ask if the above results are optimal, and indeed, they are. First note that as soon as we have constructed (in any dimension) a tiling T of a convex body K which contains (at least) 2 similar copies L, L' of K, then we could create a tiling with more similar copies by regarding the tiling $f_L(T) + T$. Note however that we can never make a degenerate tip simplex nondegenerate using this method. The tip simplex is thus the real condition to make K a polytope, a feat not visible in dimensions 2 or 3.

Using induction on dimensions, we can construct a convex body which even allows us to see that the condition on the tip simplex is optimal, in particular, $n-3$-dimensional tip simplices are not enough to conclude that K is a polytope. Figuratively speaking, we take a circular cone, which shows theorem 1 to be sharp in dimension 3, and take it as a base for a 4-dimensional cone, which in turn forms a base of a five-dimensional cone etc. This example is an extension of an example Zamfirescu gave for dimension 3.

To make a concrete example with $n-3$ dimensional tip simplex in dimension $n > 2$, use the following construction: Regard the convex body K which is the set of all points

$$\sqrt{x_1^2 + x_2^2} + \sum_{i \in \{3,4,5,\dots,n\}} x_i \leq 1;\ x_i \geq 0\ \forall i \in \{3,4,5,\dots,n\}$$

where the x_i are coordinates with respect to some base $\{e_1, e_2, e_3, \dots, e_n\}$ of $\mathbb{R}^n$. We will show that the tip simplex S is $\mathrm{conv}(\bigcup_{i \in \{3,4,5,\dots,n\}} \{e_i\})$.

We regard the homotheties

$$f_i(x) = \frac{1}{2}(x - e_i) + e_i$$

for $i \in \{3, 4, 5, \dots, n\}$. Their fixed points span the said tip simplex, and the interior of the images of K does not intersect. Thus, it remains to show that the remaining tile is convex. But this is simple, since it can be written as intersection of the convex sets K and X_i, $i \in \{3, 4, 5, \dots, n\}$,

$$X_i := \{x \in \mathbb{R}^n | x_i \leq \frac{1}{2}\}$$

References

[1] M. Laczkovich; *Decomposition of convex figures into similar pieces*, Discrete and Comp. Geometry Vol. 13, 143-148, 1995

[2] G. Valette, T. Zamfirescu; *Les partages d'un polygone convexe en 4 polygones semblambes au premier*, J. Combin. Theory Ser. B, Vol 16, 1-16, 1974