# STRICHARTZ ESTIMATES ON ASYMPTOTICALLY DE SITTER SPACES

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Abstract. In this article we obtain two families of weighted Strichartz estimates with derivative loss for the Klein-Gordon equation on asymptotically de Sitter spaces, one for nonnegative $\lambda$ and a stronger one for $\lambda > \frac{n^2}{4}$. We provide an application of these estimates to establish small-data global and almost-global existence results for a class of semilinear equations on these spaces.

## 1. Introduction

In this paper we prove a family of weighted Strichartz estimates with derivative losses for the Klein-Gordon equation on asymptotically de Sitter spaces. The estimates improve significantly in the setting corresponding to positive mass. As an application, we establish small-data global and almost-global existence results for a class of semilinear equations on these spaces.

Strichartz estimates are mixed $L^p$ (in time) and $L^q$ (in space) estimates that provide a measure of dispersion for the wave equation. These estimates first appeared in the works of Mockenhaupt-Seeger-Sogge [MSS93] and Kapitanskiĭ [Kap91] and have been useful for proving the well-posedness of semilinear wave and Schrödinger equations. In the context of general relativity, Marzuola-Metcalfe-Tataru-Tohaneanu established Strichartz estimates for the Schwarzschild black hole background [MMTT10]. If $u$ satisfies the wave equation

$$\Box u = 0,$$
$$(u, \partial_t u)|_{t=0} = (\phi, \psi),$$

on Minkowski space $\mathbb{R} \times \mathbb{R}^n$, then for allowable exponents $(p, q, s)$, $u$ satisfies the following estimate.

$$\left( \int_{\mathbb{R}} \| u(t, \cdot) \|^p_{W^{1-s,q}} \right)^{1/p} + \left( \int_{\mathbb{R}} \| \partial_t u(t, \cdot) \|^p_{W^{-s,q}} \right)^{1/p} \lesssim \| \phi \|_{H^1} + \| \psi \|_{L^2}$$

The allowable exponents $(p, q, s)$ must satisfy two conditions: the *admissibility* condition, and the *scaling* condition.

$$\text{(admissibility)} \qquad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}$$

$$\text{(scaling)} \qquad \frac{1}{p} + \frac{n}{q} = \frac{n}{2} - s$$

If $s$ is larger than the value indicated by the scaling condition, the Strichartz estimates are said to have a loss of derivatives.

The main result of this paper is a weighted Strichartz-type estimate on asymptotically de Sitter spaces. We now describe these spaces.

De Sitter space is the constant curvature spherically symmetric solution of the vacuum Einstein equations with a positive cosmological constant. It can be realized as the one-sheeted hyperboloid $\{-X_0^2+X_1^2+\ldots+X_{n+1}^2=1\}$ in Minkowski space and so is diffeomorphic to $\mathbb{R}\times\mathbb{S}^n$. In coordinates $(\tau,\theta)$ given by

$$X_0 = \sinh\tau,$$
$$X_i = \theta_i\cosh\tau,$$

where $\theta\in\mathbb{S}^n$, its metric is

$$g_{dS} = -\,d\tau^2+\cosh^2\tau\,d\theta^2.$$

If we restrict our attention to large $\tau$, then de Sitter space provides a model of a closed but expanding universe. If we let $T=e^{-\tau}$ near $\tau=+\infty$, then $\tau=+\infty$ is given by $T=0$ and the metric has the form

$$g_{dS} = \frac{-\,dT^2+\frac{1}{4}\left(1+T^2\right)\,d\theta^2}{T^2}, \tag{1}$$

i.e., the metric is *conformally compact* with a spacelike boundary at infinity.

The class of asymptotically de Sitter spaces considered in the current paper is the same as the class studied by Vasy [Vas09]. In particular, we demand that asymptotically de Sitter spaces be conformally compact in the sense of equation (1). In other words, if $X$ is a compact manifold with boundary and boundary defining function $x$, then $g$ is an asymptotically de Sitter metric on $X$ if it is Lorentzian on the interior of $X$ and, in a collar neighborhood of $\partial X$, it has the form

$$g=\frac{-\,dx^2+h}{x^2},$$

where $h$ is a smoothly varying (in $x$) family of symmetric $(0,2)$-tensors on $X$, $h|_{\partial X}$ is a section of $T^*\partial X\otimes T^*\partial X$ (rather than $T^*_Y\partial X\otimes T^*_Y\partial X$), and is a Riemannian metric on $\partial X$. After appealing to Proposition 2.1 of the paper of Joshi and Sá Barreto [JB00], we may assume that in fact $h=h(x,y,\,dy)$ (and not merely its restriction) is a section of $T^*\partial X\otimes T^*\partial X$. We also impose two global assumptions on $(X,g)$ to ensure that the metric is globally hyperbolic. We state these assumptions precisely in Section 2.

The main result of this paper is the following Strichartz-type estimate.

**Theorem 1.** *Suppose that $(X,g)$ is an asymptotically de Sitter space, $\lambda\geq 0$, and $u$ satisfies the following "odd" Klein-Gordon equation on $X$.*

$$(\Box_g+\lambda)\,u=0$$
$$(u,\partial_t u)|_{t=t_0}=(0,\psi)$$

*Suppose further that $\epsilon,\delta>0$ and that $(p,q,s)$ satisfy the following admissibility and scaling relationships.*

$$\frac{2}{p}+\frac{n-1}{q}\leq\frac{n-1}{2}$$
$$\frac{1}{p}+\frac{n}{q}=\frac{n}{2}-s+\epsilon$$

*The function $u$ then satisfies a weighted Strichartz-type estimate.*

$$\|\partial_t u\|_{e^{(n+\delta)(t-t_0)/q}L^p([t_0,\infty);W^{-s,q}(\,dk_t))}+\|u\|_{e^{(n+\delta)(t-t_0)/q}L^p([t_0,\infty);W^{1-s,q}(\,dk_t))}\lesssim\|\psi\|_{L^2(\,dk_{t_0})}$$

*Furthermore, if $\lambda > \frac{n^2}{4}$ and $p, q, s, \epsilon, \delta$ are as above, then $u$ satisfies a stronger Strichartz-type estimate.*

$$\|\partial_t u\|_{e^{\delta(t-t_0)}L^p([t_0,\infty);W^{-s,q}(\,dk_t))} + \|u\|_{e^{\delta(t-t_0)}L^p([t_0,\infty);W^{1-s,q}(\,dk_t))} \lesssim \|\psi\|_{L^2(\,dk_{t_0})}$$

As a consequence of the Strichartz estimates, we obtain a (forward in time) small-data global well-posedness result for the following semilinear wave equation on asymptotically de Sitter spaces.

$$\left(\Box_g + \lambda\right) u = e^{-nt} F_k(u)$$
$$(u, \partial_t u)|_{t=t_0} = (0, \psi)$$

Here we assume that $F_k(u)$ is similar to $|u|^{k-1}u$ in the following manner.

$$|F_k(u)| \lesssim |u|^k$$
$$|u| \cdot |F_k'(u)| \sim |F_k(u)|$$

For general $\lambda \geq 0$, we require the exponential damping on the nonlinear term as a means of compensating for the exponential growth seen in the Strichartz and energy estimates. Yagdjian studied this equation without the $e^{-nt}$ factor but on the static model of de Sitter space [Yag09]. He established small-data global well-posedness for values of $k$ below a threshold value. The transformation he applies to solutions introduces an exponentially decreasing factor similar to the one we use, but we are not able to recover his results for these values of $\lambda$. For large $\lambda$ ($\lambda > \frac{n^2}{4}$), we further obtain a small-data almost global existence result for the standard semilinear equation.

For both the Strichartz estimates and the semilinear equation, we restrict our attention only to the future. This is done both for convenience and for physical reasons, as the past of de Sitter space (and, indeed, asymptotically de Sitter spaces) are poor models for our universe.

We believe that the loss of derivatives in Theorem 1 is an artifact of our method, which requires regularizing a Fourier integral operator. Because we lack Littlewood-Paley theory in this geometric setting, we are unsure how to obtain the sharper estimates. The exponential weights seen in Theorem 1 and in Theorems 30, 33, and 35 are due to two sources. The main contribution is from the expanding nature of the spacetime, which prevents energy conservation (or a global energy bound). The energy estimate we use contains an exponentially growing bound, which carries through the proof and can be eliminated when $\lambda$ is large by a conjugation argument. The other term is the slight loss of decay. This is an artifact of the non-decay of the fundamental solution along the light cone shown by the author [Bas10a]. We also state uniform local Strichartz estimates. In [Bas10b], the author established Strichartz estimates without loss and with better decay for the conformal value of the Klein-Gordon mass. With this parameter, the Strichartz estimates are conformally equivalent to local in time estimates for the wave equation on a compact Lorentzian cylinder. In this current manuscript we do not recover those stronger estimates.

The proof of the Strichartz estimates relies on energy estimates and a dispersive estimate obtained by analyzing the fundamental solution given by the author [Bas10a, Bas10c]. The main ingredient in the proof of the existence result for the semilinear equation is a contraction mapping argument using the Strichartz estimate. We do not prove existence for a wider range of powers because we do not have an inhomogeneous $L^1L^2 \to L^pL^q$ Strichartz estimate. This is also due to the non-static nature of the spacetime, as the propagator no longer forms a semigroup.

1.1. **Notation.** Throughout this paper, we write $D$ for $\frac{1}{i}\partial$ and $P(\lambda) = \Box_g + \lambda$, where $\lambda$ is a real parameter.

Throughout this manuscript, we study the inhomogeneous Cauchy problem:

$$\begin{aligned} P(\lambda)u &= f, \\ (u, \partial_t u)|_{t=t_0} &= (\phi, \psi). \end{aligned} \tag{2}$$

Here $P(\lambda) = \Box_g + \lambda$, where $\lambda$ is a real parameter. We use the term "odd" to refer to the problem when $f = 0, \phi = 0$, "even" to refer to when $f = 0, \psi = 0$, and the inhomogeneous problem to refer when $\phi = \psi = 0$.

We require the use of Sobolev-type spaces as well. Throughout this paper, $H_E(t)$ refers to an energy space at time $t$, $H_{E,r}(t)$ denotes an energy space measuring $r$ additional derivatives at time $t$. In addition, ${}^0W^{s,p}(\,dk_t)$ refers to the $L^p$-based Sobolev space of order $s$ with respect to the measure $dk_t$. Finally, for a Banach space $Z$, we denote by $e^{at}L^p([t_0, T]; Z)$ the space of $Z$-valued functions $u$ on $[t_0, T]$ that can be written as $u = e^{at}v$, where

$$\left( \int_{t_0}^{T} \| v(t, \cdot) \|_Z^p \, dt \right)^{1/p} < \infty.$$

We give more precise characterizations of these spaces later.

## 2. Asymptotically de Sitter spaces

Suppose that $X$ is a compact manifold with boundary with boundary defining function $x$.

**Definition 2.** $(X, g)$ is an asymptotically de Sitter space if $g$ is a Lorentzian metric on the interior $X^\circ$ of $X$, and, in a collar neighborhood $[0, \epsilon)_x \times \partial X$, $g$ has the form

$$g = \frac{-\,dx^2 + h(x, y, \,dy)}{x^2},$$

where $h(x, y, \,dy)$ is a family of Riemannian metrics on $\partial X$.

We further require two global assumptions:

(A1) The boundary can be written as a disjoint union $\partial X = Y_+ \cup Y_-$, where $Y_\pm$ are unions of connected components of $\partial X$.
(A2) Each nullbicharacteristic (or light ray) $\gamma(t)$ of $g$ tends to $Y_\pm$ as $t \to \pm\infty$, or vice versa.

Assumptions (A1) and (A2) imply that $(X, g)$ is globally hyperbolic and that the interior of $X$ is diffeomorphic to $\mathbb{R} \times Y_\pm$. We let $Y = Y_+$. In particular, the global assumptions imply the existence of a global time foliation of the manifold. We may take this foliation so that $t = \log x$ near $Y_-$ and $t = -\log x$ near $Y_+$. We denote by $Y_t$ the leaves of this foliation, i.e., $Y_{t_0} = \{t = t_0\}$. We denote by $k_t$ the restriction of the metric $g$ to the slice $Y_t$. In particular, near $Y_+$, $k_t = e^{2t}h$. We set $dk_t$ to be the measure associated to this metric, which is equal to $e^{nt}\,dh_t$ near $Y_+$.

*Remark* 3. The boundary of an asymptotically de Sitter space is space-like and so one should consider the coordinate $x$ as $e^{-|t|}$.

If we set
$$D = \frac{1}{i}\partial,$$
then, in local coordinates $(x, y)$ near $\partial X$, we may write
$$\Box_g = -(xD_x)^2 - nixD_x - \frac{xD_x\sqrt{h}}{\sqrt{h}}xD_x + x^2\Delta_h.$$
We set $P(\lambda) = \Box_g + \lambda$. This convention is chosen so that $\lambda > \frac{n^2}{4}$ corresponds to positive mass.

Near $Y_+$, $x = e^{-t}$ and so in this region we have, in $(t, y)$ coordinates,

$$P(\lambda) = -D_t^2 + niD_t - \frac{D_t\sqrt{h}}{\sqrt{h}}D_t + e^{-2t}\Delta_{h_t} + \lambda, \tag{3}$$

where $\Delta_{h_t}$ is the positive Laplacian for the metric $h(t)$.

## 3. Energy estimates

In this section we prove a family of energy estimates for the equation (2).

We start by defining a norm for an energy space on a spacelike slice $Y_t$.

**Definition 4.** For $I = (\phi, \psi) \in C^\infty(Y_t) \times C^\infty(Y_t)$, we define its energy norm by

$$\|I\|^2_{H_E(t)} = \frac{1}{2}\int_{Y_t}\left(\left|(\Delta'_{k_t})^{1/2}\phi\right|^2 + |\psi|^2 + \Re\lambda\,|\phi|^2\right)\,dk_t. \tag{4}$$

For $\Re\lambda \geq 0$, this is a positive definite form, and we define the energy space $H_E(t)$ as the completion of $C^\infty(Y) \times C^\infty(Y)$ with respect to this norm.

We further define the shifted energy norm by
$$\|I\|^2_{H_{E,r}(t)} = \|A_{r,t}I\|^2_{H_E(t)},$$
and let the shifted energy space $H_{E,r}(t)$ be the completion of $C^\infty \times C^\infty$ with respect to this norm.

We now prove an energy estimate for the shifted energy spaces $H_{E,r}(t)$.

**Proposition 5.** *Suppose that $(X, g)$ is asymptotically de Sitter, $P(\lambda) = \Box_g + \lambda$, and $u$ is a smooth function on the interior of $X$. If $I = (u, \partial_t u)(t_0)$, then $u$ satisfies the following energy estimate:*

(5)
$$\|(u, \partial_t u)(t)\|_{H_{E,r}(t)} \lesssim e^{(n-2r)(t-t_0)/2}\|I\|_{H_{E,r}(t)} + e^{(n-2r)t/2}\int_{t_0}^t \|A_{r,s}P(\lambda)u\|_{L^2(\,dk_s)}\,e^{-(n-2r)s/2}\,ds.$$

*In particular if $P(\lambda)u = 0$, then $u$ satisfies:*

$$\|(u, \partial_t u)(t)\|_{H_{E,r}(t)} \lesssim e^{(n-2r)(t-t_0)/2}\|I\|_{H_{E,r}(t)}. \tag{6}$$

In particular, we obtain the following classical energy estimate by setting $r = 0$.

**Corollary 6.** *If $(X, g)$, $u$, and $I$ are as in Proposition 5, then $u$ satisfies the following energy estimate:*

$$\|(u, \partial_t u)\|_{H_E(t)} \lesssim e^{n(t-t_0)/2}\|I\|_{H_E(t_0)} + e^{nt/2}\int_{t_0}^t \|P(\lambda)u\|_{L^2(\,dk_s)}\,e^{-ns/2}\,ds. \tag{7}$$

*In particular, if $P(\lambda)u = 0$, then $u$ satisfies:*

$$\left\| (u(t), \partial_t u(t)) \right\|_{H_E(t)} \lesssim e^{n(t-t_0)/2} \left\| I \right\|_{H_E(t_0)} . \tag{8}$$

*Proof.* We know from the local theory of hyperbolic equations that the bound holds away from infinity (see, for example, the book of Taylor[Tay96]). We must thus only show the bound near infinity for initial data on a slice near infinity.

We may compute using the form (3) (for simplicity of notation, let us assume that $\lambda$ and $u$ are real):

$$\begin{aligned}
\partial_t \left\| (u, \partial_t u)(t) \right\|^2_{H_{E,r}(t)} &= \int_Y \Big( (\partial_t A_{r+\frac{1}{2},t} u)(A_{r+\frac{1}{2},t} u) + (\partial_t A_{r,t} \partial_t u)(A_{r,t}\partial_t u) \\
&\qquad + \lambda (\partial_t A_{r,t} u)(A_{r,t} u) \Big) e^{nt} \sqrt{h}\, dy \\
&\quad + \frac{1}{2} \int_Y \left( |A_{r+\frac{1}{2},t} u|^2 + |A_{r,t}\partial_t u|^2 + \lambda |u|^2 \right) \left( n + \frac{\partial_t \sqrt{h}}{\sqrt{h}} \right) e^{nt} \sqrt{h}\, dy \\
&= (n + O(e^{-t})) \left\| (u, \partial_t u)(t) \right\|^2_{H_{E,r}(t)} + \int_Y (A_{r,t} P(\lambda) u)(A_{r,t}\partial_t u) e^{nt}\sqrt{h}\, dy \\
&\quad - \int_Y |A_{r,t}\partial_t u|^2 (n + O(e^{-t})) e^{nt} \sqrt{h}\, dy \\
&\quad + \int_Y \Big( ([\partial_t, A_{r+\frac{1}{2},t}] u)(A_{r+\frac{1}{2},t} u) + ([\partial_t, A_{r,t}]\partial_t u)(A_{r,t}\partial_t u) \\
&\quad + \lambda ([\partial_t, A_{r,t}] u)(A_{r,t} u) \Big) e^{nt} \sqrt{h}\, dy.
\end{aligned}$$

We now use the calculation in Lemma 15, the positivity of one of the above terms, and the fact that $x^{-r} Q_r$ is controlled by $A_{r,t}$ to conclude that

$$\begin{aligned}
\partial_t \left\| (u, \partial_t u) \right\|^2_{H_{E,r}(t)} &\leq (n - 2r + O(e^{-t})) \left\| (u, \partial_t u) \right\|^2_{H_{E,r}(t)} \\
&\quad + \left\| A_{r,t} P(\lambda) u \right\|_{L^2(Y; dk_t)} \left\| (u, \partial_t u) \right\|_{H_{E,r}(t)} ,
\end{aligned}$$

so that

$$\partial_t \left\| (u, \partial_t u) \right\|_{H_{E,r}(t)} \leq \left( \frac{n}{2} - r + O(e^{-t}) \right) \left\| (u, \partial_t u) \right\|_{H_{E,r}(t)} + \frac{1}{2} \left\| A_{r,t} P(\lambda) u \right\|_{L^2(Y; dk_t)} .$$

An application of Gronwall's inequality finishes the proof. ∎

When $\lambda$ is large, we also obtain a stronger energy estimate by estimating the energy of $e^{\frac{n}{2}t} u$.

**Proposition 7.** *Suppose that $\lambda > \frac{n^2}{4}$ and $(X, g)$, $u$, and $I$ are as in Proposition 5, and $P(\lambda) u = 0$, then $u$ satisfies the following energy estimate.*

$$\left\| (u, \partial_t u) \right\|_{H_E(t)} \lesssim \left\| I \right\|_{H_E(t_0)} \tag{9}$$

*Proof.* Set $v = e^{\frac{n}{2}t} u$. The same argument as in the proof of Proposition 5 shows that the following energy norm of $v$ satisfies the same exponential bound.

$$\int_{Y_t} \left( |\partial_t v|^2 + \left| \left(\Delta'_{k_t}\right)^{1/2} v \right|^2 + \left( \Re\lambda - \frac{n^2}{4} \right) |v|^2 \right) dk_t \leq C e^{n(t-t_0)} e^{nt_0} \left\| I \right\|^2_{H_E(t_0)}$$

In particular, for $\lambda > \frac{n^2}{4}$ this norm is positive definite and controls $e^{nt} \left\| (u, \partial_t u) \right\|^2_{H_E(t)}$. □

## 4. The solution operator

In this section we recall the description of the solution operator for the Cauchy problem (2). This is a Lagrangian distribution on a compactification $\widetilde{X}_0^2$ of $X \times X$. We briefly describe the result below and refer the reader to the first author's previous work for more details [Bas10a, Bas10c]. Because we are interested only in obtaining Strichartz estimates, we limit our discussion to the behavior near infinity.

The full double space $\widetilde{X}_0^2$ is a manifold with corners obtained from $X \times X$ in two steps. At the first step, we obtain the 0-double space originally due to Mazzeo and Melrose [MM87]. In the second step we blow up the intersection of the flowout of the light cone with the side faces.

This space, denoted $\widetilde{X}_0^2$, is obtained by blowing up the boundary of the diagonal in $X \times X$, yielding a manifold with corners that has a new boundary hypersurface, which we call the front face ff, and on which the lift of the diagonal and the flowout of the light cone from the diagonal intersect all boundary hypersurfaces transversely. Figure 1 illustrates this blow-up.

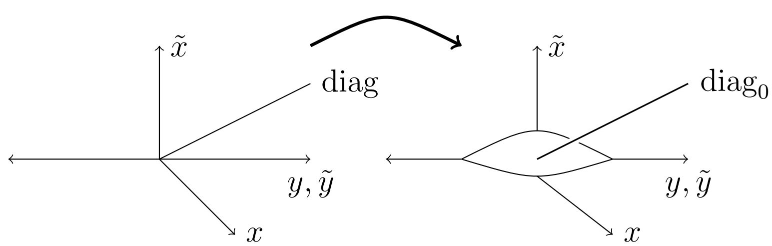


Figure 1. Passing from $X \times X$ to the 0-double space $X_0^2 = [X^2, \partial \operatorname{diag}]$.

The 0-double space has three boundary hypersurfaces: lf, the lift of the left face of $X \times X$ (given by $x = 0$ in $X \times X$); rf, the lift of the right face of $X \times X$ (given by $\tilde{x} = 0$ in $X \times X$); and ff, the front face introduced by the blow-up. Near the front face in a single coordinate chart for $Y$, the polar coordinates

$$r_{\mathrm{ff}} = \left(x^2 + \tilde{x}^2 + |y - \tilde{y}|^2\right)^{1/2}, \quad \theta = (x, y - \tilde{y}, \tilde{x})/r_{\mathrm{ff}} \in \mathbb{S}^n$$

are smooth functions. It is often more convenient to work in projective coordinates near the front face away from rf. These are given by $(s, z, \tilde{x}, \tilde{y})$, where

$$s = x/\tilde{x}, \quad z = \frac{y - \tilde{y}}{\tilde{x}}.$$

The flowout by the Hamilton vector field of $\sigma(P)$ of the characteristic set of $P$ intersected with the lifted diagonal is a smooth submanifold of $T^*\widetilde{X}_0^2$ that intersects all boundary hypersurfaces transversely. This is a Lagrangian submanifold of $T^*\widetilde{X}_0^2$ that we denote $\Lambda_1$. Near the front face but away from the diagonal, $\Lambda_1$ is the conormal bundle of a submanifold that we call the light cone LC. Because we are only interested in the region near the front face, we may assume without loss of generality that LC is an embedded submanifold away from the diagonal. The full double space is obtained by blowing up the intersection of LC with the side faces lf and rf.

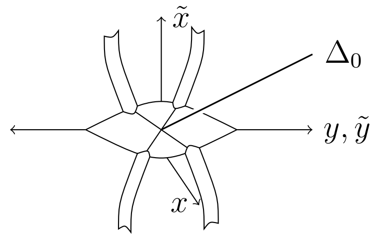


FIGURE 2. The double space $\widetilde{X}_0^2$ near $\mathrm{ff}_+$.

In the following three subsections, we summarize the results of [Bas10a]. We denote by $I_0^m\left(\widetilde{X}_0^2; \Lambda_1\right)$ the space of Lagrangian distributions of order $m$ associated to the flowout Lagrangian $\Lambda_1$ and supported near the diagonal in $\widetilde{X}_0^2$. We denote by $\mathcal{A}^{\mathcal{F}}_{\mathrm{phg}} I^m\left(\widetilde{X}_0^2; \mathrm{LC}\right)$ the space of distributions conormal to LC whose symbols have polyhomogeneous expansions with index family $\mathcal{F}$ at the side faces of $\widetilde{X}_0^2$. Finally, we denote by $\mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(\widetilde{X}_0^2)$ the space of polyhomogeneous conormal distributions with index family $\mathcal{F}$ on $\widetilde{X}_0^2$.

4.1. **The "odd" problem.** We now describe the solution operator for the "odd" initial value problem

$$\begin{aligned} P(\lambda)u &= 0 \\ (u, \partial_t u)|_{t=t_0} &= (0, \psi), \end{aligned} \tag{10}$$

where $\psi$ is a smooth function. Let us denote by $U_v(t, t_0)$ the solution operator for this problem at time $t$, i.e., if $u$ solves equation (10), then $u(t) = U_v(t, t_0)\psi$. The "v" in this notation stands for initial velocity.

**Proposition 8.** *The kernel of the operator $U_v(t, s)$, regarded as a distributional half-density on $X \times X$, can be written as $K_1 + K_2 + K_3$, where*

$$\begin{aligned} K_1 &\in I_0^{-3/2}\left(\widetilde{X}_0^2; \Lambda_1\right), \\ K_2 &\in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}} I^{-3/2}\left(\widetilde{X}_0^2; \mathrm{LC}\right), \\ K_3 &\in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(\widetilde{X}_0^2), \end{aligned}$$

*where $\Lambda_1$ is the flowout light cone, and the relevant sets in the index family $\mathcal{F}$ are given by*

$$\begin{aligned} F_{\mathrm{lcf}_1} &= \left\{(j, l) : l \leq j, j \in \mathbb{N}_0\right\}, \\ F_{\mathrm{lf}_+} &= \left\{(s_\pm(\lambda) + m, 0 : m \in \mathbb{N}_0\right\}, \\ F_{\mathrm{ff}_+} &= \left\{(m, 0) : m \in \mathbb{N}_0\right\}. \end{aligned}$$

*Remark* 9. We list only the index sets above because we are only interested in the solution operator $U(t, t_0)$ for $t_0$ and $t$ very large.

*Remark* 10. In Section 7, we need estimates for the compositions $U(t,t_0)U(s,t_0)^*$. These are roughly of the same form, but with a symmetric index set, i.e., $\mathcal{F}$ is given by

$$F_{\mathrm{lcf_l}} = F_{\mathrm{lcf_r}} = \{(j,l) : l \le j, j \in \mathbb{N}_0\}, \tag{11}$$

$$F_{\mathrm{lf}_+} = F_{\mathrm{rf}_+} = \{(s_\pm(\lambda)+m, 0 : m \in \mathbb{N}_0\}, \tag{12}$$

$$F_{\mathrm{ff}_+} = \{(m,0) : m \in \mathbb{N}_0\}. \tag{13}$$

4.2. **The "even" problem.** A similar proposition holds for the solution operator $U_p(t,t_0)$ of the "even" problem

$$\begin{aligned} P(\lambda)u &= 0 \\ (u, \partial_t u)|_{t=t_0} &= (\phi, 0). \end{aligned} \tag{14}$$

Here the subscript "p" indicates initial position.

**Proposition 11.** *The kernel of the operator $U_p(t,s)$, regarded as a distributional half-density on $X \times X$, can be written as $K_1 + K_2 + K_3$, where*

$$\begin{aligned} K_1 &\in I_0^{-1/2}\left(\widetilde{X}_0^2; \Lambda_1\right), \\ K_2 &\in \mathcal{A}_{\mathrm{phg}}^{\mathcal{F}'} I^{-1/2}\left(\widetilde{X}_0^2; \mathrm{LC}\right), \\ K_3 &\in \mathcal{A}_{\mathrm{phg}}^{\mathcal{F}'}(\widetilde{X}_0^2), \end{aligned}$$

*where $\Lambda_1$ is the flowout light cone, and the relevant sets in the index family $\mathcal{F}'$ are given by*

$$\begin{aligned} F'_{\mathrm{lcf_l}} &= \{(j-1,l) : l \le j, j \in \mathbb{N}_0\}, \\ F'_{\mathrm{lf}_+} &= \{(s_\pm(\lambda)+m, 0 : m \in \mathbb{N}_0\}, \\ F'_{\mathrm{ff}_+} &= \{(m,0) : m \in \mathbb{N}_0\}. \end{aligned}$$

Note that this index set differs from $\mathcal{F}$ only at the light cone face $\mathrm{lcf_l}$, and $F'_{\mathrm{lcf_l}} = F_{\mathrm{lcf_l}} - 1$. This difference stems from the difference in the order of the Lagrangian distributions for two solution operators.

*Remark* 12. Below we also use a symmetrized version of $\mathcal{F}'$, which differs from $\mathcal{F}$ only at $\mathrm{lcf_l}$ and $\mathrm{lcf_r}$. This difference is given by

$$F'_{\mathrm{lcf_l}} = F'_{\mathrm{lcf_r}} = F_{\mathrm{lcf_l}} - 1. \tag{15}$$

4.3. **The inhomogeneous problem.** The solution operator $E$ for the inhomogeneous problem

$$P(\lambda)u = f, \tag{16}$$

$$(u, \partial_t u)|_{t=t_0} = (0,0), \tag{17}$$

can be obtained from the solution operator for the "odd" problem. Indeed, the two operators are related by

$$(Ef)(t) = \int_s^t U_v(t,s) f(s)\, ds.$$

## 5. Regularization

We rely on the notion of semiclassical pseudodifferential operators on the slices $Y_t$. Here the variable $x$ acts as the semiclassical parameter. In particular, we consider pseudodifferential operators with Schwartz kernels given by

$$\int_{\mathbb{R}^n} e^{i(y-\tilde{y})\cdot\eta/\tilde{x}} a\left(\frac{x}{\tilde{x}}, \frac{y-\tilde{y}}{\tilde{x}}, \tilde{x}, \tilde{y}, \eta\right)\, d\eta, \tag{18}$$

where $a$ is a symbol in $\eta$.

We primarily use that powers of $\Delta_{k_t}$ are of this form.

We also use the boundedness of these operators on $L^p$ spaces. This is a standard result in semiclassical analysis.

**Lemma 13.** *Suppose that $A_x$ is a family of pseudodifferential operators of order $-\epsilon$ on $Y$ of the form* (18)*. If $1 < p < \infty$, then*

$$A_x : L^p(Y;\, dk_x) \to L^p(Y;\, dk_x),$$

*with bound independent of $x$.*

*Proof.* We start by proving the same claim for $L^p(Y;\, dh_x)$.

Using a partition of unity, we write the symbol $a = a_0 + a_\infty$, where $a_0$ is supported near the zero section and $a_\infty$ is supported away from 0. $a_0$ is a Schwartz function in $\eta$, and so we may use standard semiclassical results (see, e.g., [KTZ07]) to conclude that the bound holds for $a_0$.

For $a_\infty$, we appeal to Schur's test. Indeed, we must bound the integral over the left and right factors of $Y$, uniformly in $x$. Because $a_\infty$ is supported away from $\eta = 0$, we may use the principle of non-stationary phase to bound the integral

$$\int_Y \int_{\mathbb{R}^n} e^{i(y-y')\cdot\eta/x} a_\infty(y, y', \eta)\, d\eta \frac{dy}{x^n}.$$

Indeed, integrating by parts $n$ times gives a bound of $O(x^n)$, which cancels the factor of $x^{-n}$ in the measure. A similar bound applies to the integral in the other factor, proving the claim.

This proves that $A_x$ is bounded $L^p(Y;\, dh_x) \to L^p(Y;\, dh_x)$. To prove that it is bounded $L^p(Y;\, dk_x) \to L^p(Y;\, dk_x)$, we note only that $A_x$ commutes with multiplication by $x$. ☐

**Definition 14.** We denote by $A_{r,x}$ (or $A_{r,t}$) the operator $(\Delta'_{k_x})^{r/2}$.

We require the commutator of $A_{r,t}$ with $\partial_t$ (i.e., the commutator of $A_{r,x}$ with $-x\partial_x$).

**Lemma 15.** *For sufficiently large $t$, the commutator of $A_{r,t}$ with $\partial_t$ is given by*

$$[\partial_t, A_{r,t}] = -rA_{r,t} + x^{r+1}Q_{r,t}, \tag{19}$$

*where $Q_r$ is a family of pseudodifferential operators on $Y$ varying smoothly in $x$.*

*Proof.* We note that $\left(\Delta'_{h_x}\right)^{r/2}$ is a family of pseudodifferential operators of order $r$ on $Y$, varying smoothly in $x$ down to $x = 0$. In particular, the commutator

$$[\partial_x, \left(\Delta'_{h_x}\right)^{r/2}] = -Q_r$$

is a pseudodifferential operator of order $r$ on $Y$ also varying smoothly in $x$ down to $x = 0$.

By interpolation, $A_{r,x} = \left(\Delta'_{k_x}\right)^{r/2}$ can be written as $A_{r,x} = x^r \left(\Delta'_{h_x}\right)^{r/2}$. We may then write

$$[\partial_t, A_{r,t}] = -[x\partial_x, x^s \left(\Delta'_{h_x}\right)^{r/2}] = rA_{r,t} + x^{r+1}Q_r,$$

where $x = e^{-t}$ near $t = \infty$. □

We now prove a lemma that allows us to regularize the distributions in Section 4.

**Lemma 16.** *Suppose $K \in I_0^m\left(\widetilde{X}_0^2; \Lambda_1\right)$ is a Lagrangian distribution of order $m$ associated to $\Lambda_1$, and $A_x \in \Psi^k(Y)$ is a family of semiclassical pseudodifferential operators on $Y$ of order $k$ whose Schwartz kernels are given as $x \to 0$ by*

$$x^{-n} \int_{\mathbb{R}^n} e^{i(y-\tilde{y})\cdot\eta/x} a(x, y, \tilde{y}, \eta)\, d\eta, \tag{20}$$

*where $a$ is a symbol of order $k$. The composition $AK$ is also a Lagrangian distribution, i.e.,*

$$AK \in I_0^{m+k}\left(\widetilde{X}_0^2; \Lambda_1\right).$$

*Here we may think of $A$ as acting on the left or the right factor.*

*Remark* 17. Because $k_x \sim x^{-2}h_0$ near $x = 0$, we may put $(\Delta_{k_x})^{-k/2}$ in this form. The $x^{-n}$ in front of the oscillatory integral substitutes for the 0-half-density factor.

*Proof.* Away from the front face, we may appeal to standard composition results. Although $A$ is not pseudodifferential on $X$, the additional wavefront set of the kernel of $A$ is disjoint from the operator wavefront set of $K$. Indeed, in local coordinates near $\partial X$, it is contained in the set

$$\left\{(x, y, x, \tilde{y}, \xi, 0, -\xi, 0) : (y, \tilde{y}) \in \operatorname{supp} A_x\right\}.$$

The composition of this set with $\Lambda_1$ is empty, so we may microlocalize $A$ to be a pseudodifferential operator on $X$ without changing the singular structure of $AK$.

Near the front face, we use a different argument. Suppose that $A_x$ is a family of pseudodifferential operators on $Y$ with Schwartz kernels as in equation (20). Given a symbol $b(x, \tilde{x}, y, \tilde{y}, \xi)$ of order $m + \frac{1}{2}$, supported away from the side faces of $\widetilde{X}_0^2$, we observe that

$$A_x \left(b(x, \tilde{x}, y, \tilde{y}, \xi)e^{i(y-\tilde{y})\cdot\xi/x}\right) = c(x, \tilde{x}, y, \tilde{y}, \xi)e^{i(y-\tilde{y})\cdot\xi/x},$$

where $c$ is a symbol of order $k + m + \frac{1}{2}$. This can be seen (as in the book of Grigis and Sjöstrand [GS94]) via a careful application of stationary phase to the integral

$$\int_{\mathbb{R}^n}\int_{Y_x} e^{i(y-y')\cdot(\xi-\eta)/x} a(y, y', \eta) b(x, \tilde{x}, y', \tilde{y}, \xi) \frac{dy'}{x^n}\, d\eta.$$

□

By a similar argument, we may regularize conormal distributions with asymptotic expansions.

**Lemma 18.** *Suppose that $K \in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}} I^m\left(\widetilde{X}_0^2; \Lambda_1\right)$ is a conormal distribution and that $A_x \in \Psi^k(Y)$ is as above. Then*

$$A_x K \in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}} I^{m+k}\left(\widetilde{X}_0^2; \Lambda_1\right),$$

*where we may again think of $A$ acting on the left or on the right.*

We also require a notion of $L^p$-based Sobolev spaces, which interpolate with the $L^2$-based Sobolev spaces in the standard way.

**Definition 19.** The ${}^0W^{s,p}(\,dk_t)$ norm of a function $\phi \in C^\infty(Y)$ is given by

$$\|\phi\|^p_{{}^0W^{s,p}(\,dk_t)} = \int_{Y_t} \left|(\Delta'_{k_t})^{s/2}\right|^p \, dk_t,$$

where $\Delta'_{k_t}$ is the Laplacian of the metric $k_t$ precomposed with a projection off of its zero mode.

## 6. Dispersive estimates for Lagrangian distributions

In this section we demonstrate a family of uniform dispersive estimates for distributions in the same class as the wave evolution operator. We start by proving that 0-Lagrangian distributions associated to $\Lambda_1$ and supported near the diagonal obey a dispersive estimate.

**Lemma 20.** *Suppose that $K \in r^p_{\mathrm{ff}} I^m_0\left(\widetilde{X}^2_0; \Lambda_1\right)$ for $m = -\frac{n}{2} - 1 - \epsilon$ and $\epsilon > 0$. Suppose further that $K$ is supported near the diagonal in a small neighborhood of the front face. If we write $K = \kappa\mu$, where $\mu$ is the lift of the 0-half-density $\upsilon$ to $\widetilde{X}^2_0$, then we may bound $\kappa$ pointwise by*

$$|\kappa| \lesssim r^p_{\mathrm{ff}} \left|\log s\right|^{-(n-1)/2+\epsilon}.$$

*Moreover, because $|\log s| \leq C$ on the support of $K$, we may bound*

$$|\kappa| \lesssim r^p_{\mathrm{ff}} \left|\log s\right|^{-(n-1)/2}.$$

*Proof.* We start by showing the dispersive estimate in the case where the Lagrangian $\Lambda_1$ is parametrized by the phase function

$$\phi_0 = z \cdot \zeta \pm (1-s)|\zeta|.$$

The argument is identical for either value of plus or minus, so we fix it to be plus. We assume that the distribution $K$ is supported near the front face. We may write $\kappa$ as an oscillatory integral of the form

$$\kappa = \int_{\mathbb{R}^n} e^{i\phi_0} a(s, z, \tilde{x}, \tilde{y}, \zeta)\, d\zeta,$$

where $a$ is a symbol of order $m + \frac{1}{2}$. By using polar coordinates $\zeta = |\zeta|\hat{\zeta}$ and writing $\phi_0 = (1-s)|\zeta|\left(\frac{z}{1-s}\cdot\hat{\zeta} + 1\right)$, we may apply stationary phase to conclude that

$$\begin{aligned} |\kappa| \lesssim \int_{(1-s)^{-1}}^\infty & \left|a\left(s, z, \tilde{x}, \tilde{y}, \pm|\zeta|\hat{z}\right)\right| \cdot \\ & \left(|\zeta|^{(n-1)/2}(1-s)^{-(n-1)/2} + O(|\zeta|^{(n-2)/2}(1-s)^{-(n-2)/2})\right)\, d|\zeta| \\ + \int_0^{(1-s)^{-1}} & \int_{\mathbb{S}^{n-1}} \left|a(s, z, \tilde{x}, \tilde{y}, |\zeta|\hat{\zeta})\right| |\zeta|^{n-1}\, d\hat{\zeta}\, d|\zeta|. \end{aligned}$$

The first term is bounded by $C(1-s)^{-(n-1)/2+\epsilon}$ when $m = -\frac{n}{2} - 1 - \epsilon$. The second term within the parentheses is similarly bounded. We may bound the third term by

$$C + C\int_1^{(1-s)^{-1}} |\zeta|^{n+\frac{1}{2}+m}\, d|\zeta|,$$

which is then bounded by $C(1+(1-s)^{-(n-1)/2+\epsilon})$ when $m = -\frac{n}{2} - 1 - \epsilon$. Because this piece is supported near $s = 1$, we may bound it by $(1-s)^{-(n-1)/2+\epsilon}$.

We now use a perturbation argument to show that this estimate holds in a neighborhood of the front face for the Lagrangian $\Lambda_1$. Indeed, near the front face, we may parametrize $\Lambda_1$ by

$$\phi = \phi_0 + |\zeta| r(\tilde{x}, x, z, \hat{\zeta}),$$

where $r = \tilde{x} b$ and $b$ is a smooth function of its arguments. For small enough $\tilde{x}$, $\phi$ is still a phase function and its critical points (in $\hat{\zeta}$) are close to those of $\phi_0$. A similar argument to the one above thus shows that, in a small neighborhood of the front face the same bound holds.

We are assuming that $\kappa$ is supported near the diagonal, so $(1-s) \sim \log s$ on the support of $\kappa$, proving the claim. □

We now prove a similar estimate for the larger class of distributions used in the construction.

**Lemma 21.** *Suppose that $K \in r_{\mathrm{ff}}^{p} \mathcal{A}_{\mathrm{phg}}^{\mathcal{F}} I^{m}\left(\widetilde{X}_0^2; \Lambda_1\right)$ with $m = -\frac{n}{2} - 1 - \epsilon$, $\epsilon > 0$, and $F_{\mathrm{lcf}} \geq 0$. Suppose further that $K$ is supported near the light cone* LC *and away from the diagonal. If $K = \kappa\nu$, where $\nu$ is the lift of the 0-half-density $\upsilon$ to $\widetilde{X}_0^2$, then $\kappa$ satisfies the following bound:*

$$|\kappa(x, \tilde{x})| \lesssim r_{\mathrm{ff}}^{p} \max\left(1, |t-s|^{-\frac{n-1}{2}+\epsilon}\right),$$

*where both $x$ and $\tilde{x}$ are close to 0.*

*In fact, if $x_0$ is small and fixed so that $x, \tilde{x} \leq x_0$, $F_{\mathrm{lcf_l}} \geq \alpha$ and $F_{\mathrm{lcf_r}} \geq \beta$, then*

$$|\kappa(x, \tilde{x})| \lesssim r_{\mathrm{lcf_l}}(x)^{\alpha} r_{\mathrm{lcf_r}}(\tilde{x})^{\beta} r_{\mathrm{ff}}^{p} \max\left(1, \left|\log\left(\frac{x}{\tilde{x}}\right)\right|^{-\frac{n-1}{2}+\epsilon}\right),$$

*where $r_{\mathrm{lcf_l}}(x) = \left(x^2 + (x_0 - x - |y - \tilde{y}|)^2\right)^{1/2}$, with a similar expression for $r_{\mathrm{lcf_r}}(\tilde{x})$.*

*Proof.* Near the diagonal, the proof of this lemma is identical to the proof of Lemma 20. Away from the diagonal, $\Lambda_1$ is the conormal bundle of an embedded submanifold and the phase function for this distribution is a perturbation of

$$\phi_1(s, z, \tilde{x}, \tilde{y}, \eta) = \frac{1}{s}\left((1-s) - |z|\right)\eta,$$

where $s = x/\tilde{x}$ and $z = \frac{y-\tilde{y}}{\tilde{x}}$.

The order $m$ is sufficiently negative that the symbol of the conormal distribution is integrable. For the first statement, we then apply the symbol bound given from the order of polyhomogeneity. The proof of the second statement is identical. □

## 7. Estimating the propagator

We now seek dispersive estimates for the propagator. For convenience, we first show the estimates for the "odd" initial value problem and then indicate how to modify the proof for the "even" and inhomogeneous problems.

7.1. **The "odd" initial value problem.** We start by fixing $t_0$ large enough so that $e^{-t_0}$ is close to 0. Let $U(t,t_0) = \partial_t U_v(t,t_0)$. We seek a dispersive estimate for the product $U(t,t_0)U(s,t_0)^*$, where the adjoint is taken with respect to the energy space defined above, i.e., $U(t,t_0)^*$ is the $L^2$ transpose of $U(t,t_0)$. Let us use the decomposition given in Section 4 to write $U(t,t_0) = \sum_{i=1}^{3} U_i(t,t_0)$.

Let us denote by $\tilde{U}_i$ the regularization of $U_i$ by order $r = \frac{n+1}{4} + \frac{\epsilon}{2}$, i.e., $\tilde{U}_i = A_r U_i$.

In order to estimate $\tilde{U}_1\tilde{U}_1^*$, we use the following lemma:

**Lemma 22.** *Suppose that $s,t > t_0$ and that $e^{-t_0}$ is close to zero. Then the composition*

$$\tilde{U}_1(t,t_0)\tilde{U}_1(s,t_0)^*$$

*is an element of $I_0^{-\frac{n}{2}-1-\epsilon}\left(\widetilde{X}_0^2;\Lambda_1\right)$.*

*Proof.* The Lagrangian submanifolds corresponding to $U_1$ and $U_1^*$ intersect transversely in $Y \times Y \times Y$, so we may follow the proof of Hörmander [Hör71] to see that $\tilde{U}_1\tilde{U}_1^*$ is a Fourier integral operator. The distributions are smooth down to the front face, so there are no extra powers of $r_{\mathrm{ff}}$.

The compositions have the stated order because restricting to $t' = t_0$ shifts the order by $\frac{1}{4}$. □

Combining the previous lemma with the results of Section 6 proves the following corollary.

**Corollary 23.** *The composition $\tilde{U}_1(t,t_0)\tilde{U}_1(s,t_0)^*$ is a bounded operator $L^1(\,dk_s) \to L^\infty(\,dk_t)$ with bound*

$$C|t-s|^{-\frac{n-1}{2}+\epsilon}.$$

*Proof.* An $L^1(\,dk_s) \to L^\infty(\,dk_t)$ estimate is equivalent to a pointwise bound on $\kappa$, where $\kappa\mu$ is the kernel of the operator and $\mu$ is the lift of the 0-half-density $\nu$ to $\widetilde{X}_0^2$. The claim then follows in light of Lemma 20 and Lemma 22. Indeed, the bound from Lemma 20 implies that

$$\|K\phi\|_{L^\infty(\,dk_x)} \lesssim \left|\log\frac{x}{\tilde{x}}\right|^{-(n-1)/2+\epsilon} \|\phi\|_{L^1(\,dk_{\tilde{x}})}\,.$$

Changing coordinates from $(x,y)$ to $(t,y)$ finishes the proof. □

We now estimate the pieces containing $\tilde{U}_2$. As noted above, the phase function for this distribution is a perturbation of

$$\phi_1(s,z,\tilde{x},\tilde{y},\eta) = \frac{1}{s}\left((1-s)-|z|\right)\eta,$$

where $s = x/\tilde{x}$ and $z = \frac{y-\tilde{y}}{\tilde{x}}$.

The argument proceeds in a nearly identical manner. We start by observing the following lemma, whose proof is identical to the proof of Lemma 22.

**Lemma 24.** *Suppose that $t,s > t_0$ and $e^{-t_0}$ is close to zero. Then for $i,j = 1,2$, $(i,j) \neq (1,1)$, the composition*

$$\tilde{U}_i(t,t_0)\tilde{U}_j(s,t_0)^*$$

*is an element of $\mathcal{A}^{\mathcal{F}}_{\mathrm{phg}} I^{\frac{-n}{2}-1-\epsilon}\left(\widetilde{X}_0^2;\Lambda_1\right)$, where $\mathcal{F}$ is as given in Remark 10.*

In light of Lemma 21, we immediately obtain the following corollary.

**Corollary 25.** *Suppose that $t, s > t_0$ and $e^{-t_0}$ is near 0. Then for $(i,j) = (1,2)$, $(2,1)$, or $(2,2)$, $\tilde{U}_i(t,t_0)\tilde{U}_j(s,t_0)^*$ is bounded $L^1(\,dk_s) \to L^\infty(\,dk_t)$ with bound $C$.*

*Proof.* We apply Lemma 21 to the distribution obtained in Lemma 24. For $(i,j) = (1,2)$, $(2,1)$, or $(2,2)$, $|t-s|^{-1} \lesssim 1$ in the support of $\tilde{U}_i(t,t_0)\tilde{U}_j(s,t_0)^*$, finishing the proof. □

We must finally bound the polyhomogeneous terms, i.e., the terms containing $\tilde{U}_3$. The main observation here is that the polyhomogeneous pieces "absorb" the singularities of the other pieces, and so the pointwise bounds on the polyhomogeneous distributions are enough to prove the desired estimate without regularization. In particular, we have the following lemma.

**Lemma 26.** *Suppose that $i = 1, 2, 3$ and that $\lambda \geq 0$. Then the operator $U_3(t,t_0)U_i(s,t_0)^*$ is bounded $L^1(\,dk_s) \to L^\infty(\,dk_t)$ by a constant $C$ independent of $s$ and $t$. An identical estimate holds for $U_i(t,t_0)U_3(s,t_0)^*$.*

*Proof.* The kernel of the operator is a smooth function on the interior of $X \times X$ and is polyhomogeneous on the full double space. When $\lambda \geq 0$, the values $s_\pm(\lambda) \geq 0$. In particular, if we write the Schwartz kernel of this operator as $\kappa\nu$, where $\nu$ is the lift of the 0-half-density $\upsilon$ to $\widetilde{X}_0^2$, then $\kappa$ is bounded. □

We now interpolate to obtain a family of dispersive estimates. We start by summarizing the estimates of Corollaries 23 and 25 and Lemma 26 in the following proposition.

**Proposition 27.** *Suppose that $t, s > t_0$, $e^{-t_0}$ is close to 0, and $\lambda \geq 0$. Then the operator $\tilde{U}(t,t_0)\tilde{U}(s,t_0)^*$ is bounded $L^1(\,dk_s) \to L^\infty(\,dk_t)$ with bound*

$$\max\left(1, |t-s|^{-\frac{n-1}{2}}\right).$$

By interpolating with the energy estimates in Section 3, we obtain the following family of dispersive estimates.

**Theorem 28.** *For $q \in (2,\infty)$ and $\epsilon, \gamma > 0$, $U(t,t_0)U(s,t_0)^*$ is a bounded operator*

$$^0W^{\left(\frac{1}{q'}-\frac{1}{q}\right)\left(\frac{n+1}{4}+\frac{\epsilon}{2}\right)+\gamma, q'}(Y_s,\, dk_s) \to {}^0W^{-\left(\frac{1}{q'}-\frac{1}{q}\right)\left(\frac{n+1}{4}+\frac{\epsilon}{2}\right)-\gamma, q}(Y_t,\, dk_t),$$

*with bound*

$$C_{q,\epsilon,\gamma} e^{n(t-t_0)/q} e^{n(s-t_0)/q} \max\left(1, |t-s|^{\left(\frac{1}{q'}-\frac{1}{q}\right)\left(-\frac{n-1}{2}+\epsilon\right)}\right).$$

*Here $q'$ denotes the conjugate exponent of $q$ and $^0W^{r,q}$ denotes the $L^q$-based Sobolev space of order $r$.*

*If $\lambda > \frac{n^2}{4}$, then the same estimate holds, but with bound*

$$C_{q,\epsilon,\gamma} \max\left(1, |t-s|^{\left(\frac{1}{q'}-\frac{1}{q}\right)\left(-\frac{n-1}{2}+\epsilon\right)}\right).$$

*Proof.* We may assume that $t, s > t_0$. Proposition 5 and the boundedness of pseudodifferential operators on $L^2$-based Sobolev spaces imply that $\tilde{U}(t,t_0) \to \tilde{U}(s,t_0)^*$ is a bounded operator

$$^0W^{-\frac{n+1}{4}-\frac{\epsilon}{2},2}(Y_s,\, dk_s) \to {}^0W^{\frac{n+1}{4}+\frac{\epsilon}{2},2}(Y_t,\, dk_t),$$

with bound $C_\epsilon e^{n(t-t_0)/2} e^{n(s-t_0)/2}$.

We now interpolate with the bounds in Proposition 27 to see that for $q \in (2, \infty)$, the composition $\tilde{U}(t, t_0)\tilde{U}(s, t_0)^*$ is a bounded operator

$$ {}^0W^{-\frac{2}{q}\left(\frac{n+1}{4}+\frac{\epsilon}{2}\right),q'}(Y_s,\, dk_s) \to {}^0W^{\frac{2}{q}\left(\frac{n+1}{4}+\frac{\epsilon}{2}\right),q}(Y_t,\, dk_t), $$

with bound

$$ C_{q,\epsilon} e^{n(t-t_0)/q} e^{n(s-t_0)/q} \max\left(1, |t-s|^{\left(\frac{1}{q'}-\frac{1}{q}\right)\left(-\frac{n-1}{2}+\epsilon\right)}\right). $$

We finally remove the regularization with Lemma 13 to finish the proof. This also accounts for the extra $\gamma$ in the regularity exponent, as Lemma 13 requires that the pseudodifferential operator have negative order.

The second statement follows by using the improved energy estimates in Proposition 7. □

Because powers of $e^{-(t-t_0)}$ and $e^{-(s-t_0)}$ are bounded by $|t-s|^{-1}$ for $t, s > t_0$, we obtain the following global bound.

**Corollary 29.** *For $q \in (2, \infty)$ and $\delta, \epsilon, \gamma > 0$,*

$$ e^{-\delta(t-t_0)} e^{-n(t-t_0)/q} U(t, t_0) U(s, t_0)^* e^{-n(s-t_0)/q} e^{-\delta(s-t_0)} $$

*is a bounded operator*

$$ {}^0W^{\left(\frac{1}{q'}-\frac{1}{q}\right)\left(\frac{n+1}{4}+\frac{\epsilon}{2}\right)+\gamma,q'}\left(Y_s,\, dk_s\right) \to {}^0W^{-\left(\frac{1}{q'}-\frac{1}{q}\right)\left(\frac{n+1}{4}+\frac{\epsilon}{2}\right)-\gamma,q}\left(Y_t,\, dk_t\right), $$

*with bound*

$$ C_{q,\epsilon,\gamma,\delta} |t-s|^{-\left(\frac{1}{q'}-\frac{1}{q}\right)\left(\frac{n-1}{2}-\epsilon\right)}. $$

*We may also bound this operator by*

$$ C_{q,\epsilon,\gamma,\delta} |t-s|^{-\left(\frac{1}{q'}-\frac{1}{q}\right)\frac{n-1}{2}}. $$

*If $\lambda > \frac{n^2}{4}$, then the operator*

$$ e^{-\delta(t-t_0)} U(t, t_0) U(s, t_0)^* e^{-\delta(s-t_0)} $$

*is bounded as an operator in the same spaces with the same bounds.*

*Proof.* The second statement follows because $|t-s|$ is bounded by 1 near $t = s$ and bounded by $e^{\delta(t-s)}$ away from the diagonal. □

7.2. **The "even" initial value problem.** For the "even" initial value problem, we must use the weaker index sets in Section 4. Let $U(t, t_0)$ be the operator given by multiplying $U_p(t, t_0)$ by $(\Delta_{k_t})^{1/2}$ on either side, i.e.,

$$ U(t, t_0) = \left(\Delta_{k_t}\right)^{1/2} U_p(t, t_0) \left(\Delta_{k_t}\right)^{1/2}. $$

By the energy estimates, this operator is bounded on $L^2$ with exponential bound. At this point, the only difference with the proof in Section 7.1 is the contribution from the index set. This contributes a factor of $e^{t-t_0} e^{s-t_0}$ to the $L^1 \to L^\infty$ dispersive estimates. Interpolating with the energy estimates shows that the operator $U(t, t_0)U(s, t_0)^*$ is bounded

$$ {}^0W^{\left(\frac{1}{q'}-\frac{1}{q}\right)\left(\frac{n+1}{4}+\frac{\epsilon}{2}\right)+\gamma,q'}(Y_s,\, dk_s) \to {}^0W^{\left(\frac{1}{q'}-\frac{1}{q}\right)\left(\frac{n+1}{4}+\frac{\epsilon}{2}\right)-\gamma,q}(Y_t,\, dk_t), $$

with bound

$$ C_{q,\epsilon,\gamma} e^{\left(1+\frac{n-2}{q}\right)(t-t_0)} e^{\left(1+\frac{n-2}{q}\right)(s-t_0)} \max\left(1, |t-s|^{\left(\frac{1}{q'}-\frac{1}{q}\right)\left(-\frac{n-1}{2}+\epsilon\right)}\right). $$

By using the improved energy estimates in Proposition 7, we obtain analogous bounds with fewer exponential factors when $\lambda > \frac{n^2}{4}$.

### 7.3. **The inhomogeneous problem.**

To prove dispersive estimates in this setting, we use that the solution operator $E$ for the inhomogeneous problem

$$P(\lambda)u = f,$$
$$(u, \partial_t u)|_{t=t_0} = (0,0),$$

is given by $(Ef)(t) = \int_{t_0}^t U_v(t,s) f(s)\, ds$.

In this section we prove a dispersive estimate for $U_v(t,s)$ that allows us to later prove Strichartz estimates for the inhomogeneous problem. Let $\tilde{U}(t,s) = U_v(t,s)$ regularized by order $\frac{n-1}{2}+\epsilon$. Lemmas 20 and 21 show that $\tilde{U}(t,s)$ is a bounded operator $L^1(\,dk_s) \to L^\infty(\,dk_t)$ with bound $C_\epsilon e^{\delta(t+s)}|t-s|^{-\frac{n-1}{2}+\epsilon}$. Moreover, the energy estimates in Section 3 show that $\tilde{U}(t,s)$ is a bounded operator

$$L^2(\,dk_s) \to {}^0W^{1+\frac{n-1}{2}+\epsilon,2}(\,dk_t),$$

with bound

$$C_\epsilon e^{(n(t-s)/2)(t-s)}.$$

Interpolating these two bounds and then deregularizing yields that $U(t,s)$ is a bounded operator

$$L^{q'}(\,dk_s) \to {}^0W^{\frac{2}{q}-\gamma+\left(\frac{2}{q}-1\right)\left(\frac{n-1}{2}+\epsilon,q\right)}(\,dk_t),$$

with bound

$$C_{q,\epsilon,\gamma,\delta} e^{n(t-s)/q} e^{\delta(t+s)/q} \max\left(1, |t-s|^{\left(\frac{1}{q'}-\frac{1}{q}\right)\left(-\frac{n-1}{2}+\epsilon\right)}\right)$$

where $q'$ is the dual exponent to $q$. Note that the regularity exponent on the right hand side is nonnegative when

$$q \le 2 + \frac{4-4\gamma}{n-1+2\epsilon+2\gamma}.$$

In particular, by choosing $\gamma$ and $\epsilon$ small enough, we may ensure that the regularity exponent is nonnegative as long as

$$q < 2 + \frac{4}{n-1}.$$

Note that we may eliminate the first exponential factor by using the improved energy estimates of Proposition 7 when $\lambda > \frac{n^2}{4}$.

## 8. Strichartz estimates

We first establish Strichartz estimates for the "odd" initial value problem and then indicate how the proof must be modified for the "even" and inhomogeneous problems.

8.1. **The "odd" problem.** In this section we prove the following theorem:

**Theorem 30.** *Suppose that $\lambda \geq 0$, $\epsilon, \delta > 0$ and that $u(t)$ solves the homogeneous "odd" initial value problem:*

$$P(\lambda)u = 0,$$
$$(u, \partial_t u)|_{t=t_0} = (0, \psi).$$

*Then $u$ satisfies*

$$\|\partial_t u\|_{e^{(n+\delta)(t-t_0)/q}L^p([t_0,\infty);W^{-s,q}(\,dk_t))} + \|u\|_{e^{(n+\delta)(t-t_0)/q}L^p([t_0,\infty);W^{1-s,q}(\,dk_t))} \lesssim \|\psi\|_{L^2(\,dk_{t_0})},$$

*where*

$$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2},$$
$$s = \frac{n}{2} - \frac{1}{p} - \frac{n}{q} + \epsilon,$$

*$p, q \geq 2$, and $q \neq \infty$.*

*The function $u(t)$ also satisfies uniform local Strichartz estimates*

$$\|\partial_t u\|_{e^{n(t-t_0)/q}L^p([t_0,T];W^{-s,q}(\,dk_t))} + \|u\|_{e^{n(t-t_0)/q}L^p([t_0,T];W^{1-s,q}(\,dk_t))} \lesssim |T-t_0|^{\frac{n-1}{2}} \|\psi\|_{L^2(\,dk_{t_0})},$$

*where $p, q, s$ are as above and the constant is independent of $T$.*

*Remark* 31. We could instead move the weights associated to the spaces on the left hand side into the measures. For example, $e^{n(t-t_0)/q}L^p\left([t_0,\infty);W^{-s,q}(\,dk_t)\right)$ would become

$$L^p\left([t_0,\infty);W^{-s,q}(\,dk_t);e^{-np(t-t_0)/q}\,dt\right).$$

*Proof.* We start by defining, for $\phi \in L^2(\,dk_{t_0})$, the operator

$$T_q\phi = e^{-\delta(t-t_0)}e^{-n(t-t_0)/q}U(t,t_0)\phi.$$

Consider also the formal adjoint of $T$, considered as an operator $L^2(\,dk_{t_0}) \to L^\infty_t L^2(\,dk_t)$. This is given, for $F \in L^1_{[t_0,\infty)}L^2(\,dk_t)$, by

$$T_q^* F = \int_{t_0}^\infty e^{-\delta(t-t_0)}e^{-n(t-t_0)/q}U(t,t_0)^*F(t)\,dt.$$

In particular, the operator $TT^*$ is given by

$$(TT^*F)(t) = \int_{t_0}^\infty e^{-\delta(t+s-2t_0)}e^{-n(t+s-2t_0)/q}U(t,t_0)U(s,t_0)^*F(s)\,ds$$
$$= \int_{t_0}^\infty V(t,s)F(s)\,ds.$$

Corollary 29 implies that $V(t,s)$ is bounded as an operator

$${}^0W^{\left(\frac{1}{q'}-\frac{1}{q}\right)\left(\frac{n+1}{4}+\frac{\epsilon}{2}\right)+\gamma,q'}(Y_s,\,dk_s) \to {}^0W^{-\left(\frac{1}{q'}-\frac{1}{q}\right)\left(\frac{n+1}{4}+\frac{\epsilon}{2}\right)-\gamma,q}(Y_t,\,dk_t),$$

with bound

$$C_{q,\epsilon,\gamma,\delta}|t-s|^{-\left(\frac{1}{q'}-\frac{1}{q}\right)\frac{n-1}{2}}.$$

We now apply the method of Hardy-Littlewood-Sobolev fractional integration (see, for example, the book of Stein[Ste70], Section V.1.2) to obtain that $TT*$ is bounded

$$L^{p'}_{[t_0,\infty)}\left({}^0W_y^{s,q'};\, dt\right) \to L^p_{[t_0,\infty)}\left({}^0W_y^{-s,q};\, dt\right),$$

where $p$ is given by

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2},$$

and $s$ is given by

$$s = \left(\frac{1}{q'} - \frac{1}{q}\right)\left(\frac{n+1}{4} + \frac{\epsilon}{2}\right) + \gamma.$$

To prove the remaining estimates, we note that we may regularize $U_i$ by order $\frac{n}{2} + \epsilon$ and then the compositions have integrable symbols. This proves a bound of the form

$$\left\| \tilde{U}(t,t_0)\tilde{U}(s,t_0) \right\|_{L^1 \to L^\infty} \leq C,$$

where $\tilde{U}$ is the regularization of $U$ by order $\frac{n}{2} + \frac{\epsilon}{2}$. Interpolating this bound with the energy estimate and then removing the regularization shows that, for all $2 < q < \infty$, $U(t,t_0)U(s,t_0)^*$ is bounded ${}^0W^{s,q'} \to {}^0W^{-s,q}$ with bound $Ce^{n(t+s-t_0)/q}$, where

$$s = \frac{n+\epsilon}{2} - \frac{n+\epsilon}{q} + \gamma = \left(\frac{1}{q'} - \frac{1}{q}\right)\left(\frac{n}{2} + \epsilon\right) + \gamma.$$

In particular, $T_qT_q^*$ is bounded $L^1W^{s,q'} \to L^\infty W^{-s,q}$. Interpolating these bounds with the bounds for $\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}$ proves part of the theorem.

To finish the proof of the estimate, note that we may replace $\partial_t U_v(t,t_0)$ by $(\Delta_{k_t})^{1/2}U_v(t,t_0)$ without changing the proof. The local estimates follow by replacing the constant in the dispersive estimate with $C|T-t_0|^{\frac{n-1}{2}}$ to obtain a bound in terms of $|t-s|^{-\frac{n-1}{2}}$. □

For large $\lambda$, we also have Strichartz estimates with smaller weights, which are obtained by using the stronger estimates in Section 7.1.

**Theorem 32.** *Suppose that $\lambda > \frac{n^2}{4}$, $\epsilon, \delta > 0$ and that $u(t)$ solves the homogeneous "odd" initial value problem:*

$$P(\lambda)u = 0,$$
$$(u, \partial_t u)|_{t=t_0} = (0, \psi).$$

*Then $u$ satisfies*

$$\|\partial_t u\|_{e^{\delta(t-t_0)}L^p([t_0,\infty);W^{-s,q}(\,dk_t))} + \|u\|_{e^{\delta(t-t_0)}L^p([t_0,\infty);W^{1-s,q}(\,dk_t))} \lesssim \|\psi\|_{L^2(\,dk_{t_0})},$$

*where*

$$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2},$$
$$s = \frac{n}{2} - \frac{1}{p} - \frac{n}{q} + \epsilon,$$

*$p, q \geq 2$, and $q \neq \infty$.*

*The function $u(t)$ also satisfies uniform local Strichartz estimates*

$$\|\partial_t u\|_{e^{(t-t_0)}L^p([t_0,T];W^{-s,q}(\,dk_t))} + \|u\|_{e^{(t-t_0)}L^p([t_0,T];W^{1-s,q}(\,dk_t))} \lesssim |T-t_0|^{\frac{n-1}{2}} \|\psi\|_{L^2(\,dk_{t_0})},$$

*where $p, q, s$ are as above and the constant is independent of $T$.*

8.2. **The "even" problem.** Strichartz estimates for the "even" problem are obtained in an identical fashion as for the "odd" problem. The only difference is the dispersive estimate used. This stems from the difference in the index sets in Section 4. In particular, we obtain the following theorem.

**Theorem 33.** *Suppose that $\lambda \geq 0$, $\epsilon, \delta > 0$ and that $u(t)$ solves the homogeneous "even" initial value problem:*

$$P(\lambda)u = 0,$$
$$(u, \partial_t u)|_{t=t_0} = (\phi, \psi).$$

*Then $u$ satisfies*

$$\|\partial_t u\|_{e^{\left(1+\frac{n-2}{q}+\delta\right)(t-t_0)} L^p([t_0,\infty); W^{-s,q}(\,dk_t))} + \|u\|_{e^{\left(1+\frac{n-2}{q}+\delta\right)(t-t_0)/q} L^p([t_0,\infty); W^{1-s,q}(\,dk_t))} \lesssim \|\phi\|_{{}^0W^{1,2}(\,dk_{t_0})},$$

*where*

$$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2},$$
$$s = \frac{n}{2} - \frac{1}{p} - \frac{n}{q} + \epsilon,$$

*$p, q \geq 2$, and $q \neq \infty$.*

*The function $u(t)$ also satisfies uniform local Strichartz estimates*

$$\|\partial_t u\|_{e^{\left(1+\frac{n-2}{q}\right)(t-t_0)} L^p([t_0,T]; W^{-s,q}(\,dk_t))} + \|u\|_{e^{\left(1+\frac{n-2}{q}\right)(t-t_0)/q} L^p([t_0,T]; W^{1-s,q}(\,dk_t))} \lesssim |T - t_0|^{\frac{n-1}{2}} \|\phi\|_{{}^0W^{1,2}(\,dk_{t_0})},$$

*where $p, q, s$ are as above and the constant is independent of $T$.*

We also obtain its analogue for $\lambda > \frac{n^2}{4}$.

**Theorem 34.** *Suppose that $\lambda > \frac{n^2}{4}$, $\epsilon, \delta > 0$ and that $u(t)$ solves the homogeneous "even" initial value problem:*

$$P(\lambda)u = 0,$$
$$(u, \partial_t u)|_{t=t_0} = (\phi, \psi).$$

*Then $u$ satisfies*

$$\|\partial_t u\|_{e^{\left(1-\frac{2}{q}+\delta\right)(t-t_0)} L^p([t_0,\infty); W^{-s,q}(\,dk_t))} + \|u\|_{e^{\left(1-\frac{2}{q}+\delta\right)(t-t_0)/q} L^p([t_0,\infty); W^{1-s,q}(\,dk_t))} \lesssim \|\phi\|_{{}^0W^{1,2}(\,dk_{t_0})},$$

*where*

$$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2},$$
$$s = \frac{n}{2} - \frac{1}{p} - \frac{n}{q} + \epsilon,$$

*$p, q \geq 2$, and $q \neq \infty$.*

*The function $u(t)$ also satisfies uniform local Strichartz estimates*

$$\|\partial_t u\|_{e^{\left(1-\frac{2}{q}\right)(t-t_0)}L^p([t_0,T];W^{-s,q}(\,dk_t))} + \|u\|_{e^{\left(1-\frac{2}{q}\right)(t-t_0)/q}L^p([t_0,T];W^{1-s,q}(\,dk_t))} \lesssim |T-t_0|^{\frac{n-1}{2}} \|\phi\|_{{}^0W^{1,2}(\,dk_{t_0})},$$

*where $p, q, s$ are as above and the constant is independent of $T$.*

8.3. **The inhomogeneous problem.** The inhomogeneous problem is also similar. We use the dispersive estimates in Section 7.3 to estimate the solution operator directly, yielding the following theorem.

**Theorem 35.** *Suppose that $\lambda \geq 0$, $\epsilon, \delta > 0$ and that $u(t)$ solves the inhomogeneous problem*

$$P(\lambda)u = f,$$
$$(u, \partial_t u)|_{t=t_0} = (0,0).$$

*Then $u$ satisfies*

$$\|u\|_{e^{(n+\delta)t/q}L^p\left([t_0,\infty);{}^0W^{\frac{2}{q}-\gamma+\left(\frac{2}{q}-1\right)\left(\frac{n-1}{2}+\epsilon\right),q}(\,dk_t)\right)} \lesssim \|f\|_{e^{(n-\delta)t/q}L^{p'}\left([t_0,\infty);L^{q'}(\,dk_t)\right)},$$

*where $p$ and $p'$ (and $q$ and $q'$) are dual exponents, $p \neq 2$, $q \neq \infty$, and*

$$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2},$$
$$s = \frac{n}{2} - \frac{1}{p} - \frac{n}{q} + \epsilon.$$

*In particular, for $q < 2 + \frac{4}{n-1}$, $u$ satisfies*

$$\|u\|_{e^{(n+\delta)t/q}L^p([t_0,\infty);L^q(\,dk_t))} \lesssim \|f\|_{e^{(n-\delta)t/q}L^{p'}\left([t_0,\infty);L^{q'}(\,dk_t)\right)},$$

*where*

$$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2},$$

*and $p, q \neq \infty$.*

We require a variant of the Christ-Kiselev lemma ([CK01]), which we state now. Note that the proof given in [HTW06] remains valid when $X$ and $Y$ are replaced with smoothly varying families of Banach spaces $X(t)$ and $Y(t)$.

**Lemma 36** (Christ-Kiselev Lemma [CK01], see [HTW06] for this variant)**.** *Let $X(t)$ and $Y(t)$ be smoothly varying families of Banach spaces, and for all $s, t \in \mathbb{R}$, let $K(t,s) : X(s) \to Y(t)$ be an operator-valued kernel from $X(s)$ to $Y(t)$. Suppose we have the estimate*

$$\left\| \int_{s<t_0} K(t,s) f(s)\, ds \right\|_{L^q([t_0,\infty);Y(t))} \leq A \|f\|_{L^p(\mathbb{R};X(t))}$$

*for some $A > 0$ and $1 \leq p < q \leq \infty$, and all $t_0 \in \mathbb{R}$ and $f \in L^p((-\infty,t_0);X(t))$. Then we have*

$$\left\| \int_{s<t} K(t,s) f(s)\, ds \right\|_{L^q(\mathbb{R};Y(t))} \leq C_{p,q} A \|f\|_{L^p(\mathbb{R};X(t))}.$$

*Proof of Theorem 35.* The solution operator $E$ for the inhomogeneous problem is given by

$$(Ef)(t) = \int_{t_0}^{t} U_v(t,s) f(s)\, ds.$$

Consider first the operator $A$ given by

$$(AF)(t) = \int_{t_0}^{\infty} e^{-(n+\delta)t/q} U_v(t,s) e^{(n-\delta)s/q} F(s)\, ds.$$

The dispersive estimate in Section 7.3 shows that $A$ is bounded as an operator

$$L^{p'}\left([t_0,\infty), L^{q'}(\,dk_s)\right) \to L^{r}\left([t_0,\infty), {}^0W^{\frac{2}{q}-\gamma+\left(\frac{2}{q}-1\right)\left(\frac{n-1}{2}+\epsilon\right),q}(\,dk_t)\right),$$

where $\frac{1}{r}+1 = \frac{1}{p'} + \frac{n-1}{2}\left(\frac{1}{q'}-\frac{1}{q}\right)$. The Christ-Kiselev Lemma (Lemma 36) then shows that the operator $E_0$, given by

$$(E_0F)(t) = \int_{t_0}^{t} e^{-(n+\delta)t/q} U_v(t,s) e^{(n-\delta)s/q} F(s)\, ds,$$

is bounded as an operator between the same spaces.

Consider now the operator $E$, which is related to $E_0$ by

$$(Ef)(t) = e^{(n+\delta)t/q}\left(E_0\left(e^{-(n-\delta)s/q} f\right)\right)(t),$$

so that $E$ is bounded as an operator

$$e^{(n-\delta)t/q} L^{p'}\left([t_0,\infty); L^{q'}(\,dk_t)\right) \to e^{(n+\delta)t/q} L^{r}\left([t_0,\infty); {}^0W^{\frac{2}{q}-\gamma+\left(\frac{2}{q}-1\right)\left(\frac{n-1}{2}+\epsilon\right),q}(\,dk_t)\right),$$

where all exponents are as above. In particular, if we demand that $r=p$, then we must have that

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}.$$

The same argument as in the proof of Theorem 30 shows that the estimate holds for

$$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}.$$

□

We also obtain the analogous theorem for $\lambda > \frac{n^2}{4}$.

**Theorem 37.** *Suppose that $\lambda > \frac{n^2}{4}$, $\epsilon, \delta > 0$ and that $u(t)$ solves the inhomogeneous problem*

$$P(\lambda)u = f,$$
$$(u, \partial_t u)|_{t=t_0} = (0,0).$$

*Then $u$ satisfies*

$$\|u\|_{e^{\delta t}L^p\left([t_0,\infty); {}^0W^{\frac{2}{q}-\gamma+\left(\frac{2}{q}-1\right)\left(\frac{n-1}{2}+\epsilon\right),q}(\,dk_t)\right)} \lesssim \|f\|_{e^{-\delta t}L^{p'}\left([t_0,\infty); L^{q'}(\,dk_t)\right)},$$

*where $p$ and $p'$ (and $q$ and $q'$) are dual exponents, $p \neq 2$, $q \neq \infty$, and*

$$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2},$$
$$s = \frac{n}{2} - \frac{1}{p} - \frac{n}{q} + \epsilon.$$

*In particular, for* $q < 2 + \frac{4}{n-1}$*, u satisfies*

$$\|u\|_{e^{\delta t}L^p([t_0,\infty);L^q(\,dk_t))} \lesssim \|f\|_{e^{\delta t}L^{p'}\left([t_0,\infty);L^{q'}(\,dk_t)\right)},$$

*where*

$$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2},$$

*and* $p, q \neq \infty$*.*

*The function u also satisfies a uniform local estimate for* $q < 2 + \frac{4}{n-1}$*:*

$$\|u\|_{L^p([t_0,T];L^q(\,dk_t))} \leq C|T-t_0|^{\frac{n-1}{2}} \|f\|_{L^{p'}\left([t_0,T];L^{q'}(\,dk_t)\right)}.$$

*Proof.* The proof of the uniform local estimate uses that for $t, s \in [t_0, T]$, we may bound $U(t,s)$ as an operator

$$L^{q'}(\,dk_s) \to L^q(\,dk_t)$$

with bound $C|T-t_0|^{\frac{n-1}{2}}$. It also relies on the fact that we may ensure that the constants in the Christ-Kiselev lemma are uniform as we send $T \to \infty$.

The proofs of the other statements of the theorem are identical to those of Theorem 35. □

## 9. Applications to semilinear equations

**9.1. Global well-posedness for modified semilinear equations.** In this section we provide applications of the Strichartz estimates above to prove small-data results for modified semilinear Klein-Gordon equations. For general $\lambda \geq 0$, we consider the equation

$$P(\lambda)u = e^{-nt}F_k(u), \tag{21}$$

$$(u, \partial_t u)\,|_{t=t_0} = (0, \psi), \tag{22}$$

where $F_k$ satisfies

$$|F_k(u)| \lesssim |u|^k,$$
$$|u| \cdot |F_k'(u)| \sim |F_k(u)|.$$

Note that we only consider the "odd" problem due to the loss of decay for the "even" problem in Theorem 33.

For $\lambda > \frac{n^2}{4}$, we consider the equation

$$P(\lambda)u = e^{-\gamma t}F_k(u), \tag{23}$$
$$(u, \partial_t u)\,|_{t=t_0} = (0, \psi),$$

where $F_k$ is as above and $\gamma > 0$ is small.

The main result of this section is the following theorem.

**Theorem 38.** *Suppose that* $\lambda \geq 0$*, and* $1 < k < 1 + \frac{4}{n-1}$*. There is an* $\epsilon > 0$ *so that if* $\|\psi\|_{L^2(\,dk_{t_0})} < \epsilon$*, then there exists a unique solution*

$$u \in e^{(n+\delta)(t-t_0)/(k+1)} L^{k+1}\left([t_0,\infty); L^{k+1}(\,dk_t)\right),$$

*to equation* (21)*. Here* $\delta > 0$ *may be taken to be small.*

*If* $\lambda > \frac{n^2}{4}$ *and k is as above, then there is an* $\epsilon > 0$ *so that if* $\|\psi\|_{L^2(\,dk_{t_0})} < \epsilon$*, then there is a unique solution*

$$u \in e^{\delta(t-t_0)} L^{k+1}\left([t_0,\infty); L^{k+1}(\,dk_t)\right),$$

*to equation* (23)*.*

*Remark* 39. For $n = 3$, the bound on $k$ becomes $1 < k < 3$.

*Proof.* We prove only the first part of the theorem, as the second part is identical but uses the Strichartz estimates with better weights.

We first show that we may find $(p, q)$ so that the following conditions hold:

(1) The solution operators for the homogeneous "odd" problem gives a weighted $L^2 \to L^p L^q$ bound, i.e., so that $(p, q)$ are admissible Strichartz exponents such that the associated regularity exponent $s$ is no greater than 1.
(2) The solution operator for the inhomogeneous problem gives a weighted $L^{p'} L^{q'} \to L^p L^q$ bound, where $p'$ and $q'$ are the conjugate exponents to $p$ and $q$.
(3) We have inclusions $e^{-\delta t} L^p L^q \to L^{k+1} L^{k+1}$, and $e^{-\delta t} L^{\frac{k+1}{k}} L^{\frac{k+1}{k}} \to L^{p'} L^{q'}$.

If we take $q = k + 1$, then Theorem 35 implies that the second condition can be satisfied whenever

$$\frac{1}{p} \leq \frac{(n-1)(k-1)}{4(k+1)},$$

and

$$2 \leq q < 2 + \frac{4}{n-1},$$

which can be satisfied when $1 < k < 1 + \frac{4}{n-1}$.

By Theorem 30, the first condition is equivalent to requiring that

$$\frac{1}{p} \leq \frac{(n-1)(k-1)}{4(k+1)},$$

and that

$$\frac{1}{p} > \frac{(n-1)(k-1) - 4}{2(k+1)}.$$

These two inequalities can be satisfied whenever $1 < k < 1 + \frac{8}{n-3}$. Note that $1 + \frac{8}{n-3} > 1 + \frac{4}{n-1}$.

Finally, the third condition follows as long as $p \geq k+1$, which is possible as long as there is a gap between $\frac{1}{k+1}$ and $\frac{(n-1)(k-1)-4}{2(k+1)}$, which holds as long as $1 < k < 1 + \frac{6}{n-1}$.

We may thus find $p$ and $q$ which satisfy all three conditions. In particular, we may take $q = k + 1$ and

$$p = \max\left(k+1, \frac{4(k+1)}{(n-1)(k-1)}\right).$$

Now, given $p$ and $q$ as above, the solution operator of the homogeneous problem

$$P(\lambda) u = 0,$$
$$(u, \partial_t u)|_{t=t_0} = (0, \psi),$$

satisfies

$$\|u\|_{e^{(n+\delta)(t-t_0)/(k+1)} L^p \left([t_0, \infty]; L^{k+1}(\, dk_t)\right)} \leq C_{\epsilon,\delta} \|\psi\|_{L^2(\, dk_{t_0})}.$$

We now use the inclusion $e^{-\delta t} L^p ([t_0, \infty)) \to L^{k+1} ([t_0, \infty))$ to conclude that (for a slightly larger $\delta$)

$$\|u\|_{e^{(n+\delta)(t-t_0)/(k+1)} L^{k+1} \left([t_0, \infty]; L^{k+1}(\, dk_t)\right)} \leq C_{\epsilon,\delta} \|\psi\|_{L^2(\, dk_{t_0})}.$$

Using a similar inclusion, the solution of the inhomogeneous problem

$$P(\lambda) u = f, \tag{24}$$
$$(u, \partial_t u)|_{t=t_0} = (0, 0),$$

satisfies the estimate

$$\|u\|_{e^{(n+\delta)(t-t_0)/(k+1)}L^{k+1}\left([t_0,\infty);L^{k+1}(\,dk_t)\right)} \leq C_{\epsilon,\delta}\,\|f\|_{e^{(n-\delta)(t-t_0)/(k+1)}L^{\frac{k+1}{k}}\left([t_0,\infty);L^{\frac{k+1}{k}}(\,dk_t)\right)}. \tag{25}$$

We proceed by a contraction mapping argument. In other words, we wish to find a fixed point of the mapping

$$\mathcal{F}u(t) = \mathcal{S}(t)(0,\psi) + \mathcal{G}\left(e^{-nt}F_k(u)\right)(t),$$

where $\mathcal{S}(t)$ is the solution operator for the homogeneous problem and $\mathcal{G}$ is the solution operator for the inhomogeneous problem with zero initial data.

The assumptions on the nonlinearity imply that

$$|F_k(u) - F_k(v)| \lesssim |u-v|\,(|u|+|v|)^{k-1}$$

Let $Z = e^{(n+\delta)(t-t_0)/(k+1)}L^{k+1}\left([t_0,\infty);L^{k+1}(\,dk_t)\right)$. The main estimate used in the proof is

$$\left\|e^{-nt}\left(F_k(u)-F_k(v)\right)\right\|_{e^{(n-\delta)(t-t_0)/(k+1)}L^{\frac{k+1}{k}}\left([t_0,\infty);L^{\frac{k+1}{k}}(\,dk_t)\right)} \leq C\,\|u-v\|_Z\,\||u|+|v|\|_Z^{k-1}. \tag{26}$$

as long as $\delta$ (which depends on $k$) is small enough.

Let $u^{(0)}$ be the solution of the homogeneous problem with initial data $(0,\psi)$. For $m > 0$, let $u^{(m)}$ be the solution of the inhomogeneous problem with the same initial data and inhomogeneity $e^{-nt}F_k(u^{(m-1)}$. The estimates (24), (25), and (26) imply that if $\|\psi\|_{L^2(\,dk_{t_0})} < \epsilon$, and $\left\|u^{(m-1)}\right\|_Z < 2C\epsilon$, then

$$\left\|u^{(m)}\right\|_Z \leq C\epsilon + C'(2C\epsilon)^k,$$

where $Z = e^{(n+\delta)(t-t_0)/(k+1)}L^{k+1}\left([t_0,\infty);L^{k+1}(\,dk_t)\right)$. In particular, if $\epsilon$ is small enough, we may arrange that $\left\|u^{(m)}\right\|_Z < 2C\epsilon$ for all $m$.

We now consider $\mathcal{F}u^{(m)} - \mathcal{F}u^{(m-1)}$. By the estimate (25), we have that

$$\begin{aligned}\left\|u^{(m+1)}-u^{(m)}\right\|_Z &= \left\|\mathcal{F}u^{(m)}-\mathcal{F}u^{(m-1)}\right\|_Z \\ &= \left\|\mathcal{G}\left(e^{-nt}F_k(u^{(m)}) - e^{-nt}F_k(u^{(m-1)})\right)\right\|_Z \\ &\lesssim \left\|e^{-nt}F_k(u^{(m)}) - e^{-nt}F_k(u^{(m-1)})\right\|_{e^{(n-\delta)(t-t_0)/(k+1)}L^{\frac{k+1}{k}}\left([t_0,\infty);L^{\frac{k+1}{k}}(\,dk_t)\right)}.\end{aligned}$$

Using estimate (26), we obtain that

$$\left\|u^{(m+1)}-u^{(m)}\right\|_Z \leq C\left\|u^{(m)}-u^{(m-1)}\right\|_Z \left\|\left|u^{(m)}\right| + \left|u^{(m-1)}\right|\right\|_Z^{k-1}.$$

We now use that $\left\|u^{(m)}\right\|_Z < 2C\epsilon$ to conclude that

$$\left\|u^{(m+1)}-u^{(m)}\right\|_Z \leq \tilde{C}\epsilon\left\|u^{(m)}-u^{(m-1)}\right\|_Z.$$

Thus, if $\epsilon$ is small, the sequence $u^{(m)}$ converges in $Z$ to a fixed point $u$, which shows the existence of a solution.

Uniqueness follows in a similar manner. □

9.2. **Almost global existence for the semilinear equation.** In this section we consider the unmodified semilinear equation for $\lambda > \frac{n^2}{4}$:

$$P(\lambda)u = F_k(u), \qquad (27)$$
$$(u, \partial_t u)\,|_{t=t_0} = (0, \psi),$$

where $F_k$ satisfies

$$|F_k(u)| \lesssim |u|^k,$$
$$|u| \cdot |F_k'(u)| \sim |F_k(u)|\,.$$

We still consider only the "odd" problem due to the lack of decay for the "even" problem.

The main result of this section is the following theorem.

**Theorem 40.** *Suppose that $(X, g)$ is asymptotically de Sitter, $1 < k < 2 + \frac{4}{n-1}$, and $\lambda > \frac{n^2}{4}$. For any $T > t_0$, there is an $\epsilon > 0$ such that if $\|\psi\|_{L^2(\,dk_{t_0})} < \epsilon$, then there exists a unique solution*

$$u \in L^{k+1}\left([t_0, T]; L^q(\,dk_t)\right)$$

*to the problem* (27)*.*

*Proof.* After replacing the global weighted Strichartz estimates with the uniform local ones, the proof of this theorem is identical to the proof of Theorem 38. □

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