Quantum Destruction of Spiral Order in Two Dimensional Frustrated Magnets

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We study the fate of $S = 1/2$ spiral ordered two dimensional quantum magnets when disordered by quantum fluctuations. The topological point defects of the spiral, the Z_2 vortices, play a crucial role in such disordering. Previous works established that a quantum spin liquid phase results when the spiral is disordered *without* proliferating the Z_2 vortices. Here we show that when the spiral is disordered by proliferating these vortices valence bond ordering occurs due to quantum Berry phase effects. We develop a general theory for this latter phase transition and apply it to a lattice model. This transition provides a new example of a Landau-forbidden deconfined quantum critical point.

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It is now understood that collinear Néel states in two dimensional spin-1/2 quantum antiferromagnets can give way to paramagnetic valence bond solids (VBS) through generic second order quantum phase transitions. The latter phase breaks lattice translation but not spin rotation symmetry like the former. The notion of *deconfined criticality* was introduced in the context of such *Landau-forbidden* phase transitions [1]. In contrast, despite good numerical and theoretical motivations, the possibility of a continuous transition between a spiral and a VBS paramagnet has remained unexplored. Early theoretical works discussed continuous phase transitions from a spiral to a Z_2 spin liquid [2]. These studies have given rise to speculations that collinear magnets mostly give way to confined paramagnets (VBS), while disordering spiral magnets naturally lead to deconfined spin liquids.

Quite generally one can view the nature of the transition out of the spiral from the perspective of its topological defects, the point-like Z_2 vortices in two spatial dimensions [3]. If the spiral is destroyed without proliferating these vortices a gapped spin liquid with fractionalized bosonic spinon excitations emerge and the resulting transition is well understood [2]. However the more conventional transition out of the spiral is driven by proliferating the Z_2 vortices. What is the nature of the resulting paramagnet and the associated transition? Here we argue that the quantum Berry phase effects lead to VBS order in such paramagnets, outline the general structure of the field theory of such a generic continuous transition and work out an example in context of a lattice model.

On the theoretical side there are several numerical studies of frustrated $S = 1/2$ quantum magnets. Most pertinent to this work are exact diagonalization calculations on "$J_1 - J_3$" magnets on a square lattice [4]. In general, consider spins on a *rectangular* lattice with first (J_1) and third neighbor (J_3) antiferromagnetic exchanges

$$\begin{aligned}\mathcal{H} = &\sum_{\mathbf{r}} \left(J_1\ \mathbf{S_r}\cdot\mathbf{S_{r+x}} + J_3\ \mathbf{S_r}\cdot\mathbf{S_{r+2x}}\right)\\ + &\ \lambda\sum_{\mathbf{r}}\left(J_1\ \mathbf{S_r}\cdot\mathbf{S_{r+y}} + J_3\ \mathbf{S_r}\cdot\mathbf{S_{r+2y}}\right). \end{aligned} \quad (1)$$

The couplings along y direction are λ (anisotropy factor) times those along the x direction ($0 \le \lambda \le 1$). The two tunable parameters in the Hamiltonian are λ and (J_3/J_1). The presence of the latter leads to frustration. Various limits of this model are well known. For $\lambda = 0$, one has decoupled spin chains with nearest and next nearest neighbour couplings, where, it is known that for $J_3/J_1 < 0.241$ there is power-law Néel order, while above this true long range VBS order is obtained [5]. On the other hand, $\lambda = 1$ represents an isotropic square lattice with nearest and third nearest neighbour interactions. Numerical results [4] suggest that this has three phases. For $J_3/J_1 \lesssim 0.3$ the usual collinear Neel state obtains while for $J_3/J_1 \gtrsim 0.7$ the ground state shows non-collinear spiral order. At intermediate values one gets a paramagnet which possibly breaks lattice symmetry, *i.e.* a VBS state. Imagine sitting in the spiral phase at $\lambda = 1$ (by choosing $J_3/J_1 > 0.7$). What happens if λ is decreased towards zero? Clearly decreasing λ increases quantum fluctuations so that the spiral order will be destroyed below some critical λ_c. For very small λ the VBS order of the decoupled chains will persist as columnar dimer order with a 2-fold degenerate ground state. Could this VBS state persist all the way up to λ_c ? If so could the resulting transition be second order? While the first question can only be answered by detailed numerical studies of this particular microscopic model in future we will formulate an answer to the second one in this paper.

A generic spiral phase (coplanar) is described by

$$\langle \mathbf{S_r}\rangle \sim \left(\mathbf{n_1}(\mathbf{r})\cos\left(\mathbf{Q}\cdot\mathbf{r}\right) + \mathbf{n_2}(\mathbf{r})\sin\left(\mathbf{Q}\cdot\mathbf{r}\right)\right) \neq 0, \quad (2)$$

where $\mathbf{Q}(\neq n\pi)$ is the ordering vector of the spiral and $\mathbf{n_1}(\mathbf{r})$, $\mathbf{n_2}(\mathbf{r})$ are two mutually orthogonal unit vectors (*i.e.* $\boldsymbol{n}_\alpha(\mathbf{r})\cdot\boldsymbol{n}_\beta(\mathbf{r}) = \delta_{\alpha\beta}$). The spiral order parameter is given by an $\mathbb{SO}(3)$ matrix, $\mathcal{R} \equiv [\mathbf{n_1},\ \mathbf{n_2},\ \mathbf{n_1}\times\mathbf{n_2}]$. The order parameter manifold is isomorphic to S^3/Z_2 which allows topologically stable point vortices, characterized by a Z_2 quantum number, in two spatial dimensions ($\Pi_1\left(S^3/Z_2\right) = \mathbb{Z}_2, \Pi_2\left(S^3/Z_2\right) = 0$) [3]. In the ordered phase the energy of a single vortex diverges logarithmically with the system size and free vortices are absent. However the vortices play a crucial role when the spiral order is destroyed by condensing them.

To this end we set up an effective description of the spiral and the proximate paramagnetic phases that captures easily the role of the Z_2 vortices. Hence we introduce the well-known redundant description of the order parameter in terms of the spinon variables. Specifically we write [2]

$$\mathbf{n}^+ = \mathbf{n_1} + \imath\mathbf{n_2} = \epsilon_{\alpha\beta} z_\beta \boldsymbol{\sigma}_{\alpha\gamma} z_\gamma, \tag{3}$$

where $\boldsymbol{\sigma}$s are the three Pauli matrices, $\epsilon_{\alpha\beta}$ is the two dimensional antisymmetric matrix ($\epsilon_{12} = -\epsilon_{21} = -1, \epsilon_{11} = \epsilon_{22} = 0$) and $\mathbf{z} = (z_1, z_2)$ is the uni-modular ($\mathbf{z}^\dagger \cdot \mathbf{z} = 1$) two component complex spinon field. The spinons are thus bosons that transform as spin-1/2 under spin rotations. This spinon parameterization is *two-to-one* and there is a discrete $\mathbb{Z}_2$ gauge symmetry corresponding to the change of sign of the $\mathbf{z}$ fields at each site ($\mathbf{z}(\mathbf{r}) \to -\mathbf{z}(\mathbf{r})$) which leaves the order parameter invariant. This reiterates the fact, in terms of the spinons, that the order parameter manifold is S^3/Z_2. It is now easy to see that, for the spinons, a vortex configuration in real space corresponds to the class of non-contractile paths in S^3/Z_2 [6] where the spinon wave function changes its sign on going around a vortex. This reveals the very important fact that the spinons and the Z_2 vortices (*visons* [7]) see each other as sources of π flux.

A fruitful description of the vortices is now achieved by introducing an Ising gauge field $\sigma_{ij} = \pm 1$, minimally coupled to the spinons, on the links of the direct lattice. The Z_2-vortices are then associated with the magnetic flux ($\mathcal{F}_\Box = \prod_\Box \sigma_{ij}$, where the product is taken over the links of the plaquette) of the Ising gauge field. $\mathcal{F}_\Box = -1(+1)$ indicates the presence(absence) of a vison inside the plaquette [8]. It is important to note [6] that visons are well defined excitations even in the paramagnetic phase. Here, the spinon fields fluctuate wildly and so does their corresponding paths in the order parameter space. However for spinons describing a closed loop around a vison the paths necessarily end at diametrically opposite points in order parameter manifold (S^3/Z_2).

The above classical picture must be augmented with the correct quantum Berry phase term. Semi-classical analysis [9] for the spiral phase shows that the non-trivial Berry phases are solely associated with the vortices and are given by

$$e^{\mathcal{S}_B} = \exp\left[\frac{i\pi}{2}\sum_{i,j=i+\tau}(1-\sigma_{ij})\right] = \prod_{i,j=i+\tau}\sigma_{ij}, \tag{4}$$

where $\sigma_{i,i+\tau}$ are the Z_2-gauge fields on the time-like links of the $(2+1)$D space-time lattice. A different study [10], starting from the spin disordered phase, recovers the same Berry phase term. Eqn. [4] is also, not surprisingly, the Z_2-Polyakov loop term obtained in the analysis of the *quantum dimer models* [11].

Various global symmetries and the $\mathbb{Z}_2$ gauge structure now fully determine the effective action which is invariant under the transformation group $(\mathbb{SU}(2)\times\mathbb{U}(1))_{global} \times (\mathbb{Z}_2)_{gauge}$. The minimal imaginary time Ginzburg-Landau action in $(2+1)$ space-time lattice, consistent with these symmetries, is

$$\mathcal{S} = \mathcal{S}_z + \mathcal{S}_B, \tag{5}$$

where

$$\mathcal{S}_z = -t_s\sum_{\langle ij\rangle}\sigma_{ij}\left(z_i^\dagger\cdot z_j + h.c\right) - r\sum_{\langle ij\rangle}\left(z_i^\dagger\cdot z_j - z_j^\dagger\cdot z_i\right)^2 \tag{6}$$

and $\mathcal{S}_B$ is the Berry phase contribution given by Eqn. [4]. Eqn. [5] may be derived more formally using Hubbard-Stronovich transformation [12]. Integrating out the higher energy spinons generate several terms allowed by the symmetry, the foremost being the Z_2-*Maxwell* term:

$$\mathcal{S}_\sigma = -\sum_P K_P \prod_\Box \sigma_{ij}, \tag{7}$$

where K_P depends on the plaquette orientation. Thus the effective low energy action is given by

$$\mathcal{S}_{eff} = \mathcal{S}_z + \mathcal{S}_\sigma + \mathcal{S}_B, \tag{8}$$

where $\mathcal{S}_z, \mathcal{S}_\sigma$ and $\mathcal{S}_B$ are given by Eqns. [4-7] (with appropriate high energy cut-off for the spinons). The spiral phase is obtained by condensing the spinons, *i.e.* $\langle z_\alpha\rangle \neq 0$ (hence $\langle \mathbf{n}\rangle \neq 0$). On the other hand, deep inside the paramagnet the spinon excitations are gapped and can be integrated out from Eqn. [8] to yield the effective action

$$\mathcal{S}' = \mathcal{S}_\sigma + \mathcal{S}_B. \tag{9}$$

This is the action for the *Odd Ising gauge theory* [11], which, in (2+1) dimensions, is dual to the fully frustrated transverse field Ising model on the dual lattice [7]

$$\mathcal{H}_{FFTFIM} = \sum_{ab}\tilde{K}_{ab}\rho_a^z\mu_{ab}^{ext}\rho_b^z - \Gamma\sum_{ab}\rho_a^x, \tag{10}$$

where ρ^α are dual Ising spins and μ_{ab}^{ext} (not to be confused with μ_{ab} introduced later) impose the constraint of maximal frustration by having $\prod_\Box \mu_{ab}^{ext} = -1$ over all space-like dual plaquettes and has its origin in the Berry phase term (Eqn. [4]). Detailed analysis [12] shows that ρ_a^z is the vison creation operator.

The dual Ising spins undergo an ordering transition and the ordered phase, of this model, breaks lattice translation symmetry due to the frustration [7]. The visons proliferate and condense in this phase. The VBS order parameter, a bilinear in the vison operators (hence gauge invariant), gains a non-zero expectation value and thus rendering the paramagnet dimerized [7, 12].

The momenta at which the visons condense may be easily obtained from a soft mode analysis of Eqn. [10][13].

For a fully frustrated transverse field quantum Ising model on the dual rectangular lattice there are two such modes: χ_0 and χ_π at wave-vectors $(0,0)$ and $(\pi,0)$ respectively. The low energy modes are linear combination of χ_0 and χ_π:

$$\Phi(\mathbf{r},t) = \Psi_0(\mathbf{r},t)\chi_0 + \Psi_\pi(\mathbf{r},t)\chi_\pi, \tag{11}$$

where $\Psi_0(\mathbf{r},t)$ and $\Psi_\pi(\mathbf{r},t)$ are the two complex amplitudes. The effective Ginzburg-Landau action for the soft vison modes can be constructed by considering transformations of the two amplitudes under various symmetries of the Hamiltonian. This is given by (we have used $\Psi = \Psi_0 + i\ \Psi_\pi = |\Psi|e^{i\phi}$)

$$\mathcal{S}_v = -t_v \sum_{\langle a,b\rangle} \cos(\phi_a - \phi_b) - \Gamma_v \sum_a \cos(4\phi_a). \tag{12}$$

The modes transform as *"XY spin"* under different symmetries of the Hamiltonian up to 4^{th} order.

At the critical point both visons and spinons are gapless and since they see each other as sources of π flux (mutual *semions*), there is a long range statistical interaction between them. Such interactions may be effectively implemented by introducing two Ising gauge fields $\sigma_{ij}(\mu_{ab})$, on the links of the direct(dual) lattice, coupled by an Ising Chern-Simmons term [7] :

$$\mathcal{S}_{CS} = \frac{i\pi}{4}\sum_{\langle ab\rangle}(1-\prod_{\Box}\sigma_{ij})(1-\mu_{ab}). \tag{13}$$

(This mutual Ising Chern-Simmons coupling is different from the $U(1)$ Chern-Simmons term used for similar problems in Ref. [14]). Thus the critical Ginzburg-Landau action is

$$\mathcal{S}_C = \mathcal{S}_v + \mathcal{S}_z + \mathcal{S}_{CS}. \tag{14}$$

The long range interaction between the spinons and the visons, as encoded in $\mathcal{S}_{CS}$, makes the analysis of this field theory difficult and a series of transformations are required to cast the theory in a useful form. To start with, we neglect the effect of the terms with coefficient r (Eqn. [6]) and Γ_v (Eqn. [12]) and consider their effects later. Making standard Villain approximation [7] in the vison action (Eqn. [12]) and introducing integer valued vison current field J_{ab} on the links of the dual lattice through Hubbard-Stratonovich decoupling we get

$$\mathcal{S}_v = \sum_{\langle ab\rangle}\frac{J_{ab}^2}{2t_v}, \tag{15}$$

where integrating over the vison fields (ϕ_a) and summing over μ_{ab} gives, respectively, the constraints:

$$\boldsymbol{\nabla}\cdot\mathbf{J} = 0, \quad (-1)^{J_{ab}} = \prod_{\Box}\sigma_{ij} \tag{16}$$

The first constraint is satisfied by defining an integer valued vector field $\mathbf{a}$ on the links of the direct lattice such that $\mathbf{J} = \boldsymbol{\nabla}\times\mathbf{a}$. The second constraint then becomes

$$\sigma_{ij} = e^{i\pi a_{ij}}. \tag{17}$$

We now write $a_{ij} = 2b_{ij} + s_{ij}$, where b_{ij} is an integer field and $s_{ij} = 0(1)$ for a_{ij} even(odd). Eqn. [17] then gives $\sigma_{ij} = 1 - 2s_{ij}$. The condition on a_{ij} to be an integer may be implemented by applying a *soft* potential:

$$V_{soft} = -g\sum_{ij}\cos(2\pi b_{ij}) = -g\sum_{ij}\sigma_{ij}\cos(\pi a_{ij}) \tag{18}$$

At this stage it is useful to break the 2 complex spinon fields, z_1, z_2 into a 4-component real vector field: $z_1 = \nu_1 + i\nu_2$; $z_2 = \nu_3 + i\nu_4$. Further, rescaling the gauge potential $a_{ij} \to a_{ij}/\pi$ and choosing the transverse gauge ($\boldsymbol{\nabla}\cdot\mathbf{a} = 0$) by defining a scalar field ζ on the direct lattice ($a_{ij} \to a_{ij} + \Delta\zeta_{ij}$) we have

$$\begin{aligned}\mathcal{S}_C = & -\sum_{ij}\sigma_{ij}\left[t_s\boldsymbol{\nu}_i\cdot\boldsymbol{\nu}_j + g\cos\left(a_{ij} + (\zeta_i - \zeta_j)\right)\right] \\ & + \frac{1}{2t_v\pi^2}\sum_{ij}(\nabla_{ij}\times a_{ij})^2,\end{aligned} \tag{19}$$

where $\boldsymbol{\nu} = (\nu_1,\nu_2,\nu_3,\nu_4)$. Summing over σ_{ij}s gives the critical action:

$$\mathcal{S}_C = -\sum_{ij}\left[t\left(e^{ia_{ij}}\boldsymbol{\chi}_i\cdot\boldsymbol{\chi}_j^* + h.c\right) - \frac{(\nabla_{ij}\times a_{ij})^2}{2t_v}\right], \tag{20}$$

where we have introduced: $\chi_{i\alpha} = \nu_{i\alpha}e^{i\zeta_i}$. The constraint over the χ fields being that they are uni-modular and parallel, *i.e.*, $\chi_\alpha^*\chi_\alpha = 1$ and $\chi_\alpha^*\chi_\beta - \chi_\beta^*\chi_\alpha = 0$. This is implemented by incorporating a soft potential:

$$V'_{soft} = \eta(1 - (\chi_i^*)^2(\chi_i)^2) \tag{21}$$

where $\eta > 0$. The continuum limit for this critical theory may now be written using a *soft-spin* description.

$$\begin{aligned}\mathcal{S}_{eff} = \int d^2x\, d\tau \quad & \Big[|\left(\partial_\mu - ia_\mu\right)\boldsymbol{\chi}|^2 + p|\boldsymbol{\chi}|^2 + u\left(|\boldsymbol{\chi}|^2\right)^2 \\ & -\eta(\boldsymbol{\chi}^*)^2(\boldsymbol{\chi})^2 + \frac{1}{e^2}\left(\nabla\times\mathbf{a}\right)^2\Big]\end{aligned} \tag{22}$$

So far we have neglected the effect of r coupling present in the microscopic model. Without this term the symmetry is enlarged from the original microscopic $\mathbb{SU}(2)\times\mathbb{U}(1)$ to $\mathbb{SO}(4)$. Hence on coarse graining terms allowed by microscopic symmetry should be generated to reduce this enlarged symmetry. Such a generic term can be found starting from the universal covering group $\mathbb{SU}(4)$ which is generated by fifteen 4×4 matrices, six of which generate the subgroup $\mathbb{SO}(4)(\equiv \mathbb{SU}(2)\times\mathbb{SU}(2))$. The two constituting $\mathbb{SU}(2)$ groups are mutually commuting and one $\mathbb{SU}(2)$ generates spin rotations. We can choose a generic term that breaks the second $\mathbb{SU}(2)$ to $\mathbb{U}(1)$, the latter being the generator of lattice translation. Considering transformation under various discrete symmetries (like lattice translation, inversion, reflection and time reversal) [12] we find that the lowest order term allowed by symmetry is $(\boldsymbol{\chi}^* \cdot \tau_{\mathbf{y}} \cdot \boldsymbol{\chi})^2$(the term $\boldsymbol{\chi}^* \cdot \tau_{\mathbf{y}} \cdot \boldsymbol{\chi}$ is forbidden by time reversal). So the final continuum field theory is

$$\mathcal{S}_{eff} = \int d^2x d\tau \quad \Big[|\left(\partial_\mu - ia_\mu\right)\boldsymbol{\chi}|^2 + p|\boldsymbol{\chi}|^2 + u\left(|\boldsymbol{\chi}|^2\right)^2 - \eta(\boldsymbol{\chi}^*)^2(\boldsymbol{\chi})^2 + \gamma\left(\boldsymbol{\chi}^*\tau_{\mathbf{y}}\boldsymbol{\chi}\right)^2 - \frac{1}{e^2}\left(\nabla\times\mathbf{a}\right)^2 \Big]. \tag{23}$$

Now consider the role of Γ_v, which introduces the 4-fold anisotropy. For $\Gamma_v = 0$, from Eqn.[12], the number operators conjugate to the ϕ fields are conserved. This is equivalent to the flux conservation of the $\mathbb{U}(1)$ gauge field a_μ. Finite Γ_v destroys this conservation. Remembering that $e^{i\phi}$ is a vison creation operator, we see that this term allows the simultaneous appearance or disappearance of 4 visons, *i.e.*, a *doubled instanton* operator [16].

The continuum theory given Eqn. [23] is an anisotropic version of the non-compact CP^3 model ($NCCP^3$) and belongs to the general family of $NCCP^{N-1}$ critical theories (N denotes the number of matter components). Anisotropic $NCCP^1$ describes the transition between collinear Néel and VBS phases in easy-plane spin-1/2 $2D$-antiferromagnets [1], while anisotropic $NCCP^2$ describes the continuous transition between spin-nematic and VBS in spin-1 $2D$-antiferromagnets [15].

In Eqn. [23], the condensation of the spinons lead to spiral ordering. Once they condense the gauge field dynamics is gapped through the Anderson-Higg's mechanism and the instantons are suppressed. On the other hand the instantons are relevant in the paramagnetic phase and their condensation lead to VBS order. Thus this field theory suggests that there can be a direct transition between spiral and the VBS [17].

This direct transition may be continuous only if the doubled instantons are irrelevant at the critical point. Presently, accurate estimates of the scaling dimension of this doubled instanton operator is missing. While these may be obtained numerically, here we make a crude estimate of this. Large-N, RPA treatment of the gauge fluctuations [18], suggest that the scaling dimensions of instanton of charge q is proportional to q^2N [15]. Recent numerical studies [19] on the isotropic $NCCP^1$ model find that a single instanton has a scaling dimension of 0.63. Combining these, we find that the scaling dimension of the doubled($q = 2$) instanton operator (Δ) is $\Delta = \frac{4}{2} \times 2^2 \times (0.63) \approx 5.04 > 3$. Hence this naive estimate suggests that doubled instantons are irrelevant at the critical point and so the $\mathbb{U}(1)$ gauge flux is conserved right at the critical point. This emerging $\mathbb{U}(1)$ symmetry, absent in the microscopic model, is typical to *deconfined quantum critical points*. An extensive characterization of such critical points is given in Ref. [1].

In this paper we have outlined the field theory for a direct second order quantum phase transition between a spiral state and a VBS in context of a concrete spin-1/2 lattice model in two spatial dimensions. This is potentially a new example of deconfined quantum criticality. Our theory can be extended to study other relevant cases. As an example, it would be interesting to study other two dimensional lattices, especially the triangular lattice, where a similar transition may occur. A recent attempt to understand the phase-diagram of triangular lattice antiferromagnets using a mutual $\mathbb{U}(1)$ Chern-Simmons theory [14] suffer from the limitation that within that approach, a continuous spiral-dimer transition is always fine tuned. An extension of our theory should overcome this limitation. It is also interesting to note that for analogous cases in 1 D spin chains the visons will always proliferate (since in $(1+1)$ D the Z_2 gauge theory is always in a confining phase) leading to dimerization.

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