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An Analytical Study of Strong Non Planer Shock Waves in Magnetogasdynamics

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Abstract

An analytical approach is used to derive a new exact solution of a problem of one-dimensional unsteady adiabatic flow of a plane and cylindrical strong shock wave propagating in a plasma whose density ahead of the shock front is assumed to vary as a power of the distance from the source of explosion. The plasma is assumed to be an ideal gas with infinite electrical conductivity permeated by a transverse magnetic field. A complete investigation is made for the cases of planer, and cylindrical flows in the presence of magnetic field. An analytical solution of the problem is obtained in terms of flow variables velocity, density and the pressure in the presence of the magnetic field which exhibits space time dependence. Also, the analytical expression for the total energy under the influence of transverse magnetic field is determined.

Mathematics Subject Classification: 76L05, 76W05

Keywords: Shock waves**,** Magnetogasdynamics**,** Rankine-Hugoniot condition, Analytical solution

1 Introduction

The occurrence of shock waves in a gaseous medium has drawn the attention of several investigators during past decades. The propagation of shock waves under the influence of strong magnetic field constitutes a problem of great interest to researchers in a variety of fields such as astrophysics, nuclear science, geo-physics and plasma physics. Korobeinikov [1], Greifinger and Cole [2] and Hunter [3] studied the problem of blast wave propagation in a homogeneous and inhomogeneous medium. The pioneering studies of this phenomenon were carried out by Taylor and Sedov [4] and their numerical solution based on self-similarity consideration were found in good agreement with experimental results. A number of analytical solutions for the blast wave propagation have been obtained by Rogers [5], Bach and Lee [6], Laumbach and Probstein [7], Sachdev [8], Poslavskii [9], Chisnell [10] and Murata [11]. Laumbach and Probstein [7] and Sachdev [8] used an approach, based on the shock propagation theory of Brinkley and Kirkwood [12], which permits a simple analytical solution to be obtained directly from the governing equations. Chisnell [10] provided an analytical solution of the problem of converging shock waves by the study of singular points of the differential equations.
A further contribution towards the determination of exact solution of gasdynamic equations, involving discontinuities, via Lie group transformation has been carried out by many authors e.g. Oliveri and Speciale [13-14], Radha and Sharma [15], Pandey et al. [16], Singh et al. [17] . Oliveri and Speciale [13-14] used substitution principle to obtain an exact solution for unsteady equation of perfect gas and ideal magnetogasdynamic equation.
Rogers [5] and Murata [11] obtained the closed form solution for spherical blast wave problem, when the density of the gas ahead of the shock front varies as a power of the distance from the origin. Singh et al. [17] used the method of Lie group transformation to obtain an approximate analytical solution to the system of first order quasi-linear partial differential equations that governs a one dimensional unsteady planer, cylindrically symmetric and spherically symmetric motion in a non-ideal gas, involving strong shock waves.
In the present paper, we have considered the problem of propagation of a one-dimensional unsteady non-planer flow of an inviscid ideal gas permeated by a transverse magnetic field with infinite electrical conductivity. It is assumed that mass density distribution in the medium follows a power law of the radial distance from the point of explosion. An analytical solution of the problem is obtained in terms of flow variables velocity, density and the pressure in the presence of the magnetic field. Also, the analytical expression for the total energy under the

influence of transverse magnetic field is determined. To our knowledge, such an analytic solution influenced by the transverse magnetic field, which exhibits space time dependence has not been discussed in the past.

2 Formulation of the Problem

Assuming the electrical conductivity to be infinite and the direction of the magnetic field orthogonal to the trajectories of the gas particles, the governing equations for a one-dimensional unsteady non-planer motion can be written as [1,18]

$$\rho,_t + u\rho,_r + \rho u,_r + (m-1)\rho u / r = 0, \qquad (1)$$

$$u,_t + uu,_r + \rho^{-1}(p,_r + h,_r) = 0, \qquad (2)$$

$$p,_t + up,_r - a^2(\rho,_t + u\rho,_r) = 0, \qquad (3)$$

$$h,_t + uh,_r + 2hu,_r + 2h(m-1)u / r = 0, \qquad (4)$$

where u is the gas velocity; ρ is the density; p is the pressure; γ is the constant specific heat ratio; t is the time; r is the single spatial co-ordinate being either axial in flows with planer geometry, or radial in cylindrically symmetric flows; $a^2 = \gamma p / \rho$ is the equilibrium speed of sound ; h is the magnetic pressure defined by $h = \mu H^2 / 2$ with μ as magnetic permeability and H is the transverse magnetic field; $m = 1$ and 2 correspond, respectively, to planer and cylindrical symmetry. A comma followed by a subscript r or t denotes partial differentiation unless stated otherwise. The system of equation (1)-(4) is supplemented with an equation of state $p = \rho RT$, where R is the gas constant and T is the temperature.

It is well known that a shock wave may be initiated in the flow region, and once it is formed, it will propagate by separating the portions of continuous region. At shock, the correct generalized solution satisfies the Rankine- Hugoniot jump conditions. Let $r = \chi(t)$ be the strong shock with the shock speed $W = d\chi/dt$ propagating into the medium characterized by,

$$\rho = \rho_0(r), \quad u = 0, \quad p = p_0(r), \quad h = h_0(r) \quad . \qquad (5)$$

Therefore, the boundary conditions at the shock front can be written as [1]

$$\rho = \frac{\gamma+1}{\gamma-1}\rho_0 \left\{1 + \frac{2}{\gamma-1}\left(\frac{a_0}{W}\right)^2\right\}, \qquad (6)$$

$$u = \frac{2}{\gamma+1}W\left\{1 - \frac{a_0}{W}\right\}, \qquad (7)$$

$$p = \frac{2}{\gamma+1}\rho_0 W^2\left\{1 - \frac{\gamma-1}{2}\left(\frac{a_0}{W}\right)^2\right\} - \frac{1}{2}\frac{(\gamma+1)^2}{(\gamma-1)^2}C_0\rho_0 W^2\left\{1 + \frac{2}{\gamma-1}\left(\frac{a_0}{W}\right)^2\right\}^2, \qquad (8)$$

$$h = \frac{1}{2}\frac{(\gamma+1)^2}{(\gamma-1)^2} C_0 \rho_0 W^2 \left\{1 + \frac{2}{\gamma-1}\left(\frac{a_0}{W}\right)^2\right\}^2 , \tag{9}$$

where a_0 is the sound speed of the undisturbed medium, $C_o = 2h_0 / \rho_0 W^2$ is the shock Cowling number and the suffix 0 denote evaluation of the flow parameters just ahead of the shock respectively.

Since the initial energy input E_0 of explosion is very large, the shocks speed $W >> a_0$ so that $a_0/W \to 0$ in the strong shock limit.

Therefore, the Rankine-Hugoniot jump conditions in the case of strong shock waves can be written as

$$\rho = \frac{\gamma+1}{\gamma-1}\rho_0 , \qquad u = \frac{2}{\gamma+1} W , \tag{10}$$

$$p = \frac{2}{\gamma+1}\rho_0 W^2 - \frac{1}{2}\frac{(\gamma+1)^2}{(\gamma-1)^2} C_0 \rho_0 W^2 , \qquad h = \frac{1}{2}\frac{(\gamma+1)^2}{(\gamma-1)^2} C_0 \rho_0 W^2 . \tag{11}$$

It is assumed that at time $t = 0$, an explosion takes place over a plane or along a line accompanied by release of a finite amount of energy E. A plane or cylindrical strong shock is instantaneously formed which begins to propagate outward into a perfectly conducting gas at rest. The density ρ_0 is assumed to vary as the inverse power of the radial distance from the source of explosion

$$\rho_0 = \rho_c \chi^{-\delta} , \tag{12}$$

where δ and ρ_c are constants.

The total energy E inside a blast wave is equal to the energy supplied by the explosive and thus constant. The total energy is given by the expression

$$E = 4\pi \int_0^W \left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1} + h\right) r^{(m-1)} dr , \tag{13}$$

which represents the sum of the kinetic and internal energies of the gas.

3. Analytical Solution with Shocks

With the help of equation (10), equation (11) can be written as

$$p = \frac{\gamma-1}{2}\rho u^2 - \frac{C_0}{8}\frac{(\gamma+1)^3}{(\gamma-1)}\rho u^2 , \qquad h = \frac{1}{8}\frac{(\gamma+1)^3}{(\gamma-1)} C_0 \rho u^2 . \tag{14}$$

After using equation (14), the governing equations (2), (3) and (4) can be transformed to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{2\rho}(\gamma-1)\left\{\frac{\partial \rho}{\partial r}u^2 + 2u\frac{\partial u}{\partial r}\right\} = 0 , \tag{15}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{2}(\gamma-1)u\left\{\frac{\partial u}{\partial r} + \frac{(m-1)u}{r}\right\} = 0 , \tag{16}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{2}u\left\{\frac{\partial u}{\partial r}+\frac{(m-1)u}{r}\right\}=0 \quad . \tag{17}$$

Using equations (15) and (16), and then integrating with respect to r, we get

$$\rho u r^{-(m-1)}=\xi(t) \quad , \tag{18}$$

where $\xi(t)$ is an arbitrary function of integration.

Using the solution (18), equation (1) reduces of the form

$$\frac{\partial u}{\partial t}-2(m-1)\frac{u}{r}-\frac{u}{\xi}\frac{d\xi}{dt}=0 \quad . \tag{19}$$

On solving equations (16) and (19), we have

$$u=-\eta\frac{r}{\xi}\frac{d\xi}{dt} \quad , \quad \text{where} \quad \eta=\frac{2}{(\gamma+1)}\left\{1+\frac{(m-1)(\gamma+3)}{(\gamma+1)}\right\}^{-1} \quad . \tag{20}$$

Plugging in equation (20) in equation (19) and then integrating, we obtained

$$\xi=\frac{\xi_0}{t^{\lambda}}\text{, where } \lambda=\left\{1+\frac{(m-1)(\gamma+3)}{(\gamma+1)}\right\}\left\{1+\frac{(m-1)(\gamma-1)}{(\gamma+1)}\right\}^{-1}, \tag{21}$$

and ξ_0 is arbitrary constant.

With the help of boundary condition (10), we can obtain the analytical expression of the distance χ as

$$\chi=t^{\left(\frac{\gamma+1}{2\omega}\right)} \text{ where } \omega=\frac{(m-1)(\gamma-1)+(\gamma+1)}{2}. \tag{22}$$

Using the Rankine-Hugoniot jump condition (10) and power law of the density (12) gives a value of the constant δ as

$$\delta=\left\{\frac{(m-1)(\gamma-3)-(\gamma+1)}{(\gamma+1)}\right\} \quad . \tag{23}$$

Consequently, with the help of equation (20), the analytical solutions of the flow variables and total energy in the presence of the magnetic field is given by

$$\left.\begin{aligned} &u=r\omega^{-1}t^{-1} \ , \\ &\rho=\xi_0\omega r^{m-2}t^{1-\lambda} \ , \\ &p=\left\{\frac{(\gamma-1)}{2}-\frac{C_0}{8}\frac{(\gamma+1)^3}{(\gamma-1)}\right\}\xi_0\omega^{-1}r^m t^{-(\lambda+1)} \ , \\ &h=\frac{C_0}{8}\frac{(\gamma+1)^3}{(\gamma-1)}\xi_0\omega^{-1}r^m t^{-(\lambda+1)} \ . \end{aligned}\right\} \tag{24}$$

In view of above solution (24), the analytical expression for the total energy is given by

$$E=\frac{2\pi}{m\omega}\xi_0\left\{1+\frac{C_0}{8}\frac{(\gamma+1)^3(\gamma-2)}{(\gamma+1)^2}\right\} \quad . \tag{25}$$

It may be noted here that in the absence of the magnetic field, the analytical solutions (24) and (25) obtained in this manner which exhibits space time dependence, is a well known solution to the blast wave problem under the consideration that the energy released in the blast wave is conserved, is carried out by various approaches {[1],[11]}.

4 Concluding Remarks

In the present investigation, a simple analytic method is used to obtain the solution of the problem of the propagation of a one-dimensional unsteady adiabatic non-planer shock wave through a perfectly conducting inviscid gas permeated by a transverse magnetic field. The density ahead of the shock front is assumed to vary according to power of the distance from the source of explosion. Here, it is assumed that the atmospheric pressure and magnetic pressure satisfy the related Rankine-Hugoniot conditions automatically. Then, the governing equations are integrated to provide the shock front as a function of time. An exact solution of the problem in form of power in the distance and time is obtained. It may be remarked that in the absence of the magnetic field, such an analytical solutions, are in close agreement with earlier results obtained {[1], [11]}.

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