

Lowest Weight Representations of Super Schrödinger Algebras in One Dimensional Space

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Abstract

Lowest weight modules, in particular, Verma modules over the $\mathcal{N} = 1, 2$ super Schrödinger algebras in $(1 + 1)$ dimensional spacetime are investigated. The reducibility of the Verma modules is analyzed via explicitly constructed singular vectors. The classification of the irreducible lowest weight modules is given for both massive and massless representations. A vector field realization of the $\mathcal{N} = 1, 2$ super Schrödinger algebras is also presented.

1 Introduction

Conformal symmetry and conformal supersymmetry are one of the fundamental concept in relativistic field theories. However, importance of conformal invariance in nonrelativistic physics has been recognized since early 70's (see for example [1–7] and references therein). Nonrelativistic counterpart of the conformal group is the Schrödinger group [1,2]. Recently, the Schrödinger group attracts very much attention in the context of nonrelativistic conformal field theory [7] and nonrelativistic version of AdS/CFT correspondence [8,9]. Supersymmetric extensions of the Schrödinger group and its Lie algebra have also been discussed in connection with various physical systems such as fermionic oscillator [10,11], spinning particles [12], nonrelativistic Chern-Simons matter [13–15], Dirac monopole and magnetic vortex [14] and so on. Some super Schrödinger algebras were constructed from the viewpoint of infinite dimensional Lie super algebra [16] or by embedding them to conformal superalgebras [17,18].

However, it seems that the representation theories of the Schrödinger group/algebra and their supersymmetric extensions are not studied well. There are a few works that were based on representation theoretic viewpoint. We mention the followings: Projective representations of the Schrödinger group in 3 spatial dimension were constructed in [19]. Irreducible representations of the Schrödinger algebras up to 3 spatial dimension were investigated in [20–22]. The author of [23] studied the highest weight representations of the $\mathcal{N} = 2$ super Schrödinger algebra in 2 spatial dimension ("exotic" algebra in the terminology of [14]).

In a series of paper, we try to classify irreducible representations of the super Schrödinger algebras. This is motivated by the physical importance of the superalgebras mentioned above. We believe that the super Schrödinger algebra has a lot of potential applications in both classical and quantum physics. In the present paper, the first one of the series, we study the simplest case, that is, the super Schrödinger algebra in (1+1) dimensional spacetime with $\mathcal{N} = 1, 2$ extensions. In general, the Schrödinger algebra of fixed spacetime dimension has some supersymmetric extension even for fixed value of \mathcal{N} . A systematic method to extend the Schrödinger algebra in $(1+n)$ dimensional spacetime to arbitrary \mathcal{N} has developed in [14]. We study the super Schrödinger algebras introduced in [14] throughout the present paper. We denote the centrally extended Schrödinger algebra in $(1+n)$ dimensional spacetime by $\mathfrak{s}(n)$. The $\mathcal{N} = 1, 2$ extension of $\mathfrak{s}(1)$ are denoted by $\mathfrak{s}(1/1)$ and $\mathfrak{s}(1/2)$, respectively. The superalgebra $\mathfrak{s}(1/2)$ corresponds to the one with $N_+ = 1$, $N_- = 0$ extension in [14].

The plan of this paper is as follows: We give the definitions of $\mathfrak{s}(1/1)$ and $\mathfrak{s}(1/2)$ in the next section. Two types of algebraic anti-automorphisms are also presented. In section 3 the lowest weight modules (Verma modules) of $\mathfrak{s}(1/1)$ are defined. To analyze the reducibility of the Verma modules we determine the singular vectors and give their explicit formula. Using the singular vectors the irreducible modules over $\mathfrak{s}(1/1)$ are classified for both massive and massless representations. The same analysis is repeated for $\mathfrak{s}(1/2)$ in section 4 and classification of the

irreducible modules over $\mathfrak{s}(1/2)$ is presented. In section 5 vector field realizations of $\mathfrak{s}(1/1)$ and $\mathfrak{s}(1/2)$ are introduced. Section 6 is devoted to concluding remarks.

2 $\mathcal{N} = 1, 2$ extensions of $\mathfrak{s}(1)$

The algebra $\mathfrak{s}(1)$ has six elements, i.e., time translation H , space translation P , Galilei boost G , dilatation D , conformal transformation K and central element M corresponding to the mass. The nonvanishing commutation relations of $\mathfrak{s}(1)$ are given by:

$$\begin{aligned} [H, D] &= 2H, & [H, K] &= D, & [D, K] &= 2K, & [P, G] &= M, \\ [H, G] &= P, & [D, G] &= G, & [P, D] &= P, & [P, K] &= G. \end{aligned} \quad (2.1)$$

$\mathcal{N} = 1$ extension of $\mathfrak{s}(1)$ is defined by adding three odd elements $\mathcal{Q}, \mathcal{S}, \mathcal{X}$ to $\mathfrak{s}(1)$. They enjoy the anti-commutation relations given below

$$\begin{aligned} \{\mathcal{Q}, \mathcal{Q}\} &= -2H, & \{\mathcal{S}, \mathcal{S}\} &= -2K, & \{\mathcal{X}, \mathcal{X}\} &= -M, \\ \{\mathcal{Q}, \mathcal{X}\} &= -P, & \{\mathcal{S}, \mathcal{X}\} &= -G, & \{\mathcal{Q}, \mathcal{S}\} &= -D, \end{aligned} \quad (2.2)$$

and satisfy the nontrivial commutation relations with $\mathfrak{s}(1)$:

$$\begin{aligned} [\mathcal{Q}, D] &= \mathcal{Q}, & [\mathcal{Q}, K] &= \mathcal{S}, & [D, \mathcal{S}] &= \mathcal{S}, & [H, \mathcal{S}] &= \mathcal{Q}, \\ [\mathcal{Q}, G] &= \mathcal{X}, & [P, \mathcal{S}] &= \mathcal{X}. \end{aligned} \quad (2.3)$$

The subset $\{H, D, K, T, Q\}$ forms a subalgebra of $\mathfrak{s}(1/1)$ isomorphic to $osp(1/2)$.

$\mathcal{N} = 2$ extension of $\mathfrak{s}(1)$ has six odd elements $\mathcal{Q}_i, \mathcal{S}_i, \mathcal{X}_i$ ($i = 1, 2$). In addition to these, there exist another even element, denoted by R_{12} , which commutes with all the generators of $\mathfrak{s}(1)$ so that $\mathcal{N} = 2$ extended $\mathfrak{s}(1)$ has 13 generators. Nontrivial relations of the generators newly added are listed below. The anti-commutators of odd elements:

$$\begin{aligned} \{\mathcal{Q}_j, \mathcal{Q}_k\} &= -2\delta_{jk}H, & \{\mathcal{S}_j, \mathcal{S}_k\} &= -2\delta_{jk}K, & \{\mathcal{X}_j, \mathcal{X}_k\} &= -\delta_{jk}M, \\ \{\mathcal{Q}_j, \mathcal{X}_k\} &= -\delta_{jk}P, & \{\mathcal{S}_j, \mathcal{X}_k\} &= -\delta_{jk}G, & \{\mathcal{Q}_j, \mathcal{S}_k\} &= -\delta_{jk}D + R_{jk}, \end{aligned} \quad (2.4)$$

and the commutators between odd and even elements:

$$\begin{aligned} [\mathcal{Q}_j, D] &= \mathcal{Q}_j, & [\mathcal{Q}_j, K] &= \mathcal{S}_j, & [D, \mathcal{S}_j] &= \mathcal{S}_j, & [H, \mathcal{S}_j] &= \mathcal{Q}_j, \\ [\mathcal{Q}_j, G] &= \mathcal{X}_j, & [P, \mathcal{S}_j] &= \mathcal{X}_j, & [\mathcal{Q}_j, R_{kl}] &= \delta_{j\ell}\mathcal{Q}_k - \delta_{jk}\mathcal{Q}_\ell, \\ [\mathcal{S}_j, R_{kl}] &= \delta_{j\ell}\mathcal{S}_k - \delta_{jk}\mathcal{S}_\ell, & [\mathcal{X}_j, R_{kl}] &= \delta_{j\ell}\mathcal{X}_k - \delta_{jk}\mathcal{X}_\ell, \end{aligned} \quad (2.5)$$

where $j, k \in \{1, 2\}$ and we impose $R_{ij} = -R_{ji}$.

The defining relations of $\mathfrak{s}(1/2)$ given above are not appropriate to study lowest or highest weight representations. We thus take a linear combination of the odd generators:

$$R = iR_{12}, \quad \mathcal{A}_\pm = \frac{1}{\sqrt{2}}(\mathcal{A}_1 \pm i\mathcal{A}_2), \quad (2.6)$$

where $\mathcal{A}_j = \mathcal{Q}_j, \mathcal{S}_j, \mathcal{X}_j$. Then the elements \mathcal{A}_\pm enjoy the fermionic property: $\mathcal{A}_\pm^2 = 0$. With the new generators, the counterpart of the relations (2.4) reads

$$\begin{aligned} \{\mathcal{Q}_\pm, \mathcal{Q}_\mp\} &= -2H, & \{\mathcal{S}_\pm, \mathcal{S}_\mp\} &= -2K, & \{\mathcal{X}_\pm, \mathcal{X}_\mp\} &= -M, \\ \{\mathcal{Q}_\pm, \mathcal{X}_\mp\} &= -P, & \{\mathcal{S}_\pm, \mathcal{X}_\mp\} &= -G, & \{\mathcal{Q}_\pm, \mathcal{S}_\mp\} &= -D \mp R, \end{aligned} \quad (2.7)$$

and that of (2.5) is given by

$$\begin{aligned} [\mathcal{Q}_\pm, D] &= \mathcal{Q}_\pm, & [\mathcal{Q}_\pm, K] &= \mathcal{S}_\pm, & [D, \mathcal{S}_\pm] &= \mathcal{S}_\pm, & [H, \mathcal{S}_\pm] &= \mathcal{Q}_\pm \\ [\mathcal{Q}_\pm, G] &= \mathcal{X}_\pm, & [P, \mathcal{S}_\pm] &= \mathcal{X}_\pm, & [R, \mathcal{A}_\pm] &= \pm \mathcal{A}_\pm. \end{aligned} \quad (2.8)$$

The subset $\{H, D, K, R, \mathcal{Q}_\pm, \mathcal{S}_\pm\}$ forms a subalgebra of $\mathfrak{s}(1/2)$ isomorphic to $osp(2/2)$.

One can define two types of adjoint operations (algebra anti-automorphism) [24] to the super Schrödinger algebras introduced above. The first one maps even elements as follows:

$$\omega_1(P) = (-1)^\epsilon G, \quad \omega_1(H) = K, \quad \omega_1(D) = D, \quad \omega_1(M) = M, \quad \omega_1(R) = R, \quad (2.9)$$

where $\epsilon \in \{0, 1\}$. The mapping of the odd elements of $\mathfrak{s}(1/1)$ is given by

$$\omega_1(\mathcal{Q}) = (-1)^\lambda \mathcal{S}, \quad \omega_1(\mathcal{X}) = (-1)^{\epsilon+\lambda} \mathcal{X}, \quad (2.10)$$

and of $\mathfrak{s}(1/2)$ by

$$\omega_1(\mathcal{Q}_a) = (-1)^\lambda \mathcal{S}_{-a}, \quad \omega_1(\mathcal{X}_a) = (-1)^{\epsilon+\lambda} \mathcal{X}_{-a}, \quad (2.11)$$

where $\lambda \in \{0, 1\}$ and $a = \pm$. The second one is defined by

$$\omega_2(P) = (-1)^\epsilon P, \quad \omega_2(G) = (-1)^\epsilon G, \quad \omega_2(X) = -X \quad (X = H, K, D, M), \quad \omega_2(R) = R, \quad (2.12)$$

together with for $\mathcal{N} = 1$

$$\omega_2(\mathcal{Q}) = i(-1)^\lambda \mathcal{Q}, \quad \omega_2(\mathcal{S}) = i(-1)^\lambda \mathcal{S}, \quad \omega_2(\mathcal{X}) = i(-1)^{\lambda+\epsilon+1} \mathcal{X}, \quad (2.13)$$

and for $\mathcal{N} = 2$

$$\omega_2(\mathcal{Q}_a) = i(-1)^\lambda \mathcal{Q}_{-a}, \quad \omega_2(\mathcal{S}_a) = i(-1)^\lambda \mathcal{S}_{-a}, \quad \omega_2(\mathcal{X}_a) = i(-1)^{\lambda+\epsilon+1} \mathcal{X}_{-a}. \quad (2.14)$$

Both adjoint operations are idempotent: $\omega_j^2 = id$.

On the contrary, there is no grade adjoint operation (another algebra anti-automorphism) [24] for $\mathfrak{s}(1/1)$ although $\mathfrak{s}(1/1)$ has $osp(1/2)$ as a subalgebra. To see this fact, we note that the grade adjoint operation σ preserves the parity of an element. For instance, $\sigma(\mathcal{Q})$ is a linear combination of \mathcal{Q}, \mathcal{S} and \mathcal{X} . We thus write the mapping in a matrix form:

$$\sigma \begin{pmatrix} \mathcal{Q} \\ \mathcal{S} \\ \mathcal{X} \end{pmatrix} = A \begin{pmatrix} \mathcal{Q} \\ \mathcal{S} \\ \mathcal{X} \end{pmatrix},$$

where A is a 3×3 complex matrix. By the definition of the grade adjoint operation $\sigma^2(\mathcal{A}) = -\mathcal{A}$ for an odd element \mathcal{A} . Therefore the matrix A has to satisfy $\bar{A}A = -1_3$. This yields a contradictory result: $|\det A|^2 = -1$.

The superalgebra $\mathfrak{s}(1/2)$ admits two types of grade adjoint operations. The first one is given by

$$\begin{aligned} \sigma_1(\mathcal{Q}_\pm) &= \pm(-1)^\epsilon \mathcal{S}_\mp, & \sigma_1(\mathcal{X}_\pm) &= \pm(-1)^\epsilon \mathcal{X}_\mp, & \sigma_1(K) &= H, & \sigma_1(P) &= G, \\ \sigma_1(X) &= X \quad (X = D, R, M), \end{aligned} \quad (2.15)$$

where $\epsilon \in \{0, 1\}$. The second one looks as follows:

$$\sigma_2(\mathcal{A}_\pm) = \pm(-1)^\epsilon i \mathcal{A}_\mp, \quad \sigma_2(R) = R, \quad \sigma_2(X) = -X \quad (X = H, K, D, M, P, G). \quad (2.16)$$

3 Lowest weight representations of $\mathfrak{s}(1/1)$

3.1 Verma modules and singular vectors

The superalgebra $\mathfrak{s}(1/1)$ is graded, in addition to the \mathbb{Z}_2 grading of Lie superalgebra, if we define

$$\begin{aligned} \deg K &= 2, & \deg G &= \deg \mathcal{S} = 1, & \deg D &= \deg M = \deg \mathcal{X} = 0, \\ \deg P &= \deg \mathcal{Q} = -1, & \deg H &= -2. \end{aligned} \quad (3.1)$$

The grading operator is D . The \mathbb{Z} grading defined above can be viewed as an analogue of the triangular decomposition of semisimple Lie algebra. Namely, one has the vector space decomposition defined by $\mathfrak{s}(1/1) = \mathfrak{s}(1/1)^+ \oplus \mathfrak{s}(1/1)^0 \oplus \mathfrak{s}(1/1)^-$ where $\mathfrak{s}(1/1)^+(\mathfrak{s}(1/1)^-)$ is spanned by the generators of positive (negative) degree and $\mathfrak{s}(1/1)^0$ is by zero degree. We remark that each subset $\mathfrak{s}(1/1)^\pm, \mathfrak{s}(1/1)^0$ forms an abelian subalgebra. We also remark that the adjoint operation ω_1 introduced in the previous section exchanges $\mathfrak{s}(1/1)^+$ and $\mathfrak{s}(1/1)^-$, while ω_2 preserves the triangular structure of $\mathfrak{s}(1/1)$.

The triangular decomposition enables us to define lowest weight modules of $\mathfrak{s}(1/1)$, in particular, Verma modules. The lowest weight vector v_0 is defined by

$$\begin{aligned} \mathcal{Q}v_0 &= Pv_0 = 0, \\ Dv_0 &= -dv_0, \quad Mv_0 = mv_0, \quad \mathcal{X}v_0 = \chi v_0, \end{aligned} \quad (3.2)$$

where $d \in \mathbb{R}$ is the conformal weight and the minus sign is for later convenience. The variable χ is of odd parity relating to the mass eigenvalue by the relation $m = 2\chi^2$. The Verma module V^d is defined as the lowest weight module with the lowest weight d : $V^d = U(\mathfrak{s}(1/1)^+) \otimes v_0$, where $U(\mathfrak{s}(1/1)^+)$ denotes the universal enveloping algebra of $\mathfrak{s}(1/1)^+$. More explicitly,

$$V^d = \{ G^k K^\ell v_0, G^k K^\ell \mathcal{S} v_0 \}, \quad (3.3)$$

where k, ℓ are non-negative integers. Note that the subset $\{ G^k K^\ell v_0 \}$ is a Verma module of the non-super Schrödinger algebra $\mathfrak{s}(1)$. It is an easy exercise to compute the action of $\mathfrak{s}(1/1)$ on the basis of V^d . Denoting $v_{k,\ell} = G^k K^\ell v_0$ and $\nu_{k,\ell} = G^k K^\ell \mathcal{S} v_0$, one finds the followings:

$$\begin{aligned}
K v_{k,\ell} &= v_{k,\ell+1}, & K \nu_{k,\ell} &= \nu_{k,\ell+1}, & G v_{k,\ell} &= v_{k+1,\ell}, & G \nu_{k,\ell} &= \nu_{k+1,\ell}, \\
D v_{k,\ell} &= (k + 2\ell - d) v_{k,\ell}, & D \nu_{k,\ell} &= (k + 2\ell + 1 - d) \nu_{k,\ell}, \\
M v_{k,\ell} &= m v_{k,\ell}, & M \nu_{k,\ell} &= m \nu_{k,\ell}, & \mathcal{X} v_{k,\ell} &= \chi v_{k,\ell}, & \mathcal{X} \nu_{k,\ell} &= \chi \nu_{k,\ell} - v_{k+1,\ell}, \\
P v_{k,\ell} &= \ell v_{k+1,\ell-1} + m k v_{k-1,\ell}, & P \nu_{k,\ell} &= \ell \nu_{k+1,\ell-1} + \chi v_{k,\ell} + m k \nu_{k-1,\ell}, \\
Q v_{k,\ell} &= \chi^k v_{k-1,\ell} + \ell \nu_{k,\ell-1}, & Q \nu_{k,\ell} &= \chi^k \nu_{k-1,\ell} + (d - \ell - k) \nu_{k,\ell}, \\
H v_{k,\ell} &= \ell(k + \ell - d - 1) v_{k,\ell-1} + \frac{1}{2} m k (k - 1) v_{k-2,\ell}, \\
H \nu_{k,\ell} &= \ell(k + \ell - d) \nu_{k,\ell-1} + \chi^k v_{k-1,\ell} + \frac{1}{2} m k (k - 1) \nu_{k-2,\ell}.
\end{aligned} \tag{3.4}$$

It follows that the Verma module V^d can be decomposed into homogeneous subspaces with respect to D :

$$V^d = \bigoplus_{n=0}^{\infty} V_n^d, \quad V_n^d = \text{lin.span.} \{ v_{k,\ell}, \nu_{k,\ell} \mid D v_{k,\ell} = n v_{k,\ell}, D \nu_{k,\ell} = n \nu_{k,\ell} \}. \tag{3.5}$$

To analyse the reducibility of V^d , one can use the singular vectors [20, 21, 23] although the superalgebra $\mathfrak{s}(1/1)$ is not semisimple. A singular vector v_s is defined as a homogeneous element of V^d such that $v_s \neq \mathbb{C} v_0$ and

$$\mathcal{Q} v_s = P v_s = 0. \tag{3.6}$$

We give all the possible singular vectors explicitly. To this end, the following proposition on non-super case is helpful [20, 21].

Proposition 1 *The singular vectors of the Verma module over $\mathfrak{s}(1)$ are given as follows:*

- i) For $m \neq 0$, a singular vector exists for $d = p - 3/2$ which is given by $(G^2 - 2mk)^p w_0$.
- ii) For $m = 0$, infinitely many singular vectors exist for each value of d which are given by $G^p w_0$.

In both cases, $p \in \mathbb{N}$ and w_0 denotes a lowest weight vector of $\mathfrak{s}(1)$ defined by $H w_0 = P w_0 = 0$.

Since a singular vector of $\mathfrak{s}(1/1)$ is a homogeneous element in V^d , it may have the form of

$$v_s = f(G, K) u_0, \quad u_0 = (\alpha G + \beta \mathcal{S}) v_0, \tag{3.7}$$

where $f(G, K)$ is a homogeneous polynomial in G, K and α, β are constant of parity even and odd, respectively. Assuming the degree of $f(G, K)$ is n , the general expression of $f(G, K)$ is

$$f(G, K) = \sum_{\ell} a_{\ell} G^{n-2\ell} K^{\ell}. \tag{3.8}$$

The action of P on v_s is computed as follows:

$$\begin{aligned} P v_s &= ([P, f(G, K)] + f(G, K)P)u_0 \\ &= \sum_{\ell} \{m(n - 2\ell)a_{\ell} + (\ell + 1)a_{\ell+1}\} G^{n-2\ell-1} K^{\ell} u_0 + f(G, K)P u_0. \end{aligned}$$

Since $P u_0 = (\alpha m + \beta \chi)v_0$, $P v_s = 0$ implies that $P u_0 = 0$. Clearly, $H u_0 = 0$ and $D u_0 = -(d - 1)u_0$. Thus the vector u_0 is a lowest weight vector of the non-super Schrödinger algebra $\mathfrak{s}(1)$ with the conformal weight $d - 1$.

One can determine the singular vectors of $\mathfrak{s}(1/1)$ applying Proposition 1. If $m \neq 0$, then $P u_0 = 0$ yields $\beta = -2\alpha\chi$. One see from Proposition 1 that if $d = p - 1/2$, then $v_s^p = (G^2 - 2mK)^p (G - 2\chi\mathcal{S})v_0$ is the only vector annihilated by P and H . It is verified by direct computation that $Q v_s^p = 0$. Thus v_s^p is a singular vector of $\mathfrak{s}(1/1)$ for $m \neq 0$. If $m = 0$, then χ becomes a Grassmann number ($\chi^2 = 0$) and $P u_0 = 0$ yields $\beta = 0$. In this case, there are infinite number of vectors $v_s^p = G^p v_0$ ($p \in \mathbb{N}$) for each value of d . It is straightforward to see that $\mathcal{Q} v_s^p = 0$. Therefore we have proved the following proposition.

Proposition 2 *The singular vectors of the Verma module V^d over $\mathfrak{s}(1/1)$ are given as follows:*

i) *For $m \neq 0$, a singular vector exists for $d = p - 1/2$ which is*

$$v_s^p = (G^2 - 2mK)^p (G - 2\chi\mathcal{S})v_0, \quad p \in \mathbb{Z}_{\geq 0} \quad (3.9)$$

ii) *For $m = 0$, infinitely many singular vector exist for each value of d which are*

$$v_s^p = G^p v_0, \quad p \in \mathbb{N} \quad (3.10)$$

We remark that the case of $m = 0$ corresponds to the super Schrödinger algebra without central extension. Proposition 2 tells that the representations of the centrally extended super Schrödinger algebra is very different from the unextended one as in the non-super algebra [20, 21].

3.2 Irreducible modules

Let us consider the reducibility of the Verma modules V^d . We start with $m \neq 0$. If $d \neq p - 1/2$, there is no singular vectors in V^d so that the module is irreducible. In the case of $d = p - 1/2$, there is one singular vector (3.9) in V^d . The submodule $I^d = U(\mathfrak{s}(1/1)^+) \otimes v_s^p$ is invariant under the action of $\mathfrak{s}(1/1)$. This means that the Verma module V^d with $d = p - 1/2$ is not irreducible. Furthermore, the submodule I^d is isomorphic to $V^{d'}$ with shifted weight $d' = d - 2p - 1 = -p - 3/2$. This Verma module $V^{d'}$ with the shifted weight does not have singular vectors since its weight has upper bound $d' < -3/2$, while Proposition 2 shows that the Verma module has a singular vector provided that its weight is $d > -1/2$.

We now consider the factor module V^d/I^d with $d = p - 1/2$. Its lowest weight vector is denoted by w_0^p . It satisfies the conditions:

$$Qw_0^p = Pw_0^p = (G^2 - 2mK)^p(G - 2\chi\mathcal{S})w_0^p = 0. \quad (3.11)$$

The vectors $G^k K^\ell w_0^p$, $G^k K^\ell \mathcal{S} w_0^p$ with k, ℓ non-negative integers span V^d/I^d . By the relation obtained from (3.11) one can get rid of all powers of G greater than $2p$:

$$G^{2p+1}w_0 = 2\chi G^{2p} \mathcal{S} w_0^p - \sum_{j=0}^{p-1} \binom{p}{j} (-2m)^{p-j} G^{2j} K^{p-j} (G - 2\chi\mathcal{S})w_0^p. \quad (3.12)$$

Hence the basis of V^d/I^d is given by

$$w_{k,\ell} = G^k K^\ell w_0^p, \quad \omega_{k,\ell} = G^k K^\ell \mathcal{S} w_0^p, \quad k \leq 2p \quad (3.13)$$

Search for the singular vectors in V^d/I^d is straightforward but requires lengthy computation. We omit the computation but the result is that there are no singular vectors in V^d/I^d . We thus conclude that V^d/I^d is irreducible.

Next we investigate the case of $m = 0$. All Verma modules in this case are reducible, since they contain infinitely many singular vectors. For a fixed value of d , the Verma module contains infinite submodules $I^p = U(\mathfrak{sl}(1/1)^+) \otimes v_s^p$ which are invariant under the action of $\mathfrak{sl}(1/1)$. Since the singular vector v_s^p ($p > 1$) is related to v_s^1 by the relation $v_s^p = G^{p-1}v_s^1$, the corresponding submodules satisfy the inclusion relation $I^p \subset I^1$. It follows that

$$V^d/I^1 \subset V^d/I^p.$$

Therefore it is sufficient to examine the factor module V^d/I^1 . We denote the lowest weight vector of V^d/I^1 by w_0 . It satisfies the conditions:

$$Qw_0 = Pw_0 = Gw_0 = 0. \quad (3.14)$$

The basis of V^d/I^1 is given by

$$w_\ell = K^\ell w_0, \quad \omega_\ell = K^\ell \mathcal{S} w_0.$$

It is easy to see the action of \mathcal{Q} and P on the basis:

$$\mathcal{Q}w_\ell = \ell\omega_{\ell-1}, \quad \mathcal{Q}\omega_\ell = (d - \ell)\omega_\ell, \quad Pw_\ell = P\omega_\ell = 0. \quad (3.15)$$

It follows that if $d \notin \mathbb{N}$ there is no singular vector in V^d/I^1 , therefore the module V^d/I^1 is irreducible. If $d = p \in \mathbb{N}$ there is one singular vector: $w_s^p = \omega_p$. It produces the submodule $\mathcal{I}^p = U(\mathfrak{sl}(1/1)^+) \otimes w_s^p$ invariant under the action of $\mathfrak{sl}(1/1)$. The lowest weight vector $|0\rangle$ of the factor module $(V^p/I^1)/\mathcal{I}^p$ satisfies the condition

$$Q|0\rangle = P|0\rangle = G|0\rangle = K^p \mathcal{S}|0\rangle = 0. \quad (3.16)$$

The basis of $(V^p/I^1)/\mathcal{I}^p$ is given by

$$K^\ell |0\rangle, \quad K^\ell \mathcal{S} |0\rangle \quad (\ell = 0, 1, \dots, p-1), \quad K^p |0\rangle.$$

It is clear that there is no singular vector in the factor module $(V^p/I^1)/\mathcal{I}^p$ thus the module is irreducible and is of dimension $2p+1$. The generators P, G, M and \mathcal{X} are represented trivially in V^d/I^1 and $(V^p/I^1)/\mathcal{I}^p$ if we set $\chi = 0$. One may identify V^d/I^1 and $(V^p/I^1)/\mathcal{I}^p$ as infinite and finite dimensional representations of the subalgebra $osp(1/2)$. Now we summarize our results in the following proposition.

Proposition 3 *The irreducible lowest weight modules over the Lie superalgebra $\mathfrak{s}(1/1)$ are classified as follows:*

i) $m \neq 0$

- V^d when $d \neq p - 1/2$, $p \in \mathbb{Z}_{\geq 0}$. $\dim V^d = \infty$
- V^d/I^d when $d = p - 1/2$, $p \in \mathbb{Z}_{\geq 0}$. $\dim V^d/I^d = \infty$

ii) $m = 0$

- V^d/I^1 when $d \notin \mathbb{N}$. $\dim V^d/I^1 = \infty$
- $(V^p/I^1)/\mathcal{I}^p$ when $p \in \mathbb{N}$. $\dim (V^p/I^1)/\mathcal{I}^p = 2p + 1$

The representations for $m = 0$ are also irreps of the subalgebra $osp(1/2)$.

4 Lowest weight representations of $\mathfrak{s}(1/2)$

4.1 Verma modules and singular vectors

The Lie superalgebra $\mathfrak{s}(1/2)$ has $\mathbb{Z} \times \mathbb{Z}$ grading structure if we define

$$\begin{aligned} \deg K &= (2, 0), & \deg G &= (1, 0), & \deg D &= \deg R = \deg M = (0, 0), \\ \deg H &= (-2, 0), & \deg P &= (-1, 0), \\ \deg \mathcal{S}_\pm &= (1, \pm 1), & \deg \mathcal{X}_\pm &= (0, \pm 1), & \deg \mathcal{Q}_\pm &= (-1, \pm 1). \end{aligned} \tag{4.1}$$

The grading operators are D and R . The grading enable us to introduce a triangular decomposition of $\mathfrak{s}(1/2)$. Reading the degree of an element from left to right if the first encountered non-zero entry is positive (negative), then we say that the element has positive (negative) degree. We define the following vector space decomposition according to the degree of elements:

$$\begin{aligned} \mathfrak{s}(1/2) &= \mathfrak{s}(1/2)^+ \oplus \mathfrak{s}(1/2)^0 \oplus \mathfrak{s}(1/2)^-, \\ \mathfrak{s}(1/2)^+ &= \{ K, G, \mathcal{S}_\pm, \mathcal{X}_+ \}, & \mathfrak{s}(1/2)^0 &= \{ D, R, M \}, \\ \mathfrak{s}(1/2)^- &= \{ H, P, \mathcal{Q}_\pm, \mathcal{X}_- \}. \end{aligned}$$

In contrast to $\mathfrak{s}(1/1)$, the subsets $\mathfrak{s}(1/2)^\pm$ form subalgebras but they are not abelian.

We define the lowest weight vector v_0 by

$$\begin{aligned} \mathcal{Q}_\pm v_0 &= P v_0 = \mathcal{X}_- v_0 = 0, \\ D v_0 &= -d v_0, \quad M v_0 = m v_0, \quad R v_0 = r v_0. \end{aligned} \quad (4.2)$$

The Verma modules over $\mathfrak{s}(1/2)$ are defined by

$$V^{d,r} = \{ G^k K^\ell \mathcal{S}_+^a \mathcal{S}_-^b \mathcal{X}_+^c v_0 \mid k, \ell \in \mathbb{Z}_{\geq 0}, a, b, c \in \{0, 1\} \}. \quad (4.3)$$

The action of the grading operators on the basis $v_{k,\ell,a,b,c} = G^k K^\ell \mathcal{S}_+^a \mathcal{S}_-^b \mathcal{X}_+^c v_0$ is easily computed:

$$D v_{k,\ell,a,b,c} = (k + 2\ell + a + b - d) v_{k,\ell,a,b,c}, \quad R v_{k,\ell,a,b,c} = (a - b + c + r) v_{k,\ell,a,b,c}. \quad (4.4)$$

It follows that the Verma module $V^{d,r}$ can be decomposed into homogeneous subspaces with respect to D and R :

$$\begin{aligned} V^{d,r} &= \bigoplus_{n_1, n_2=0}^{\infty} V_{n_1, n_2}^{d,r}, \\ V_{n_1, n_2}^{d,r} &= \text{lin.span.} \{ v_{k,\ell,a,b,c} \mid D v_{k,\ell,a,b,c} = n_1 v_{k,\ell,a,b,c}, R v_{k,\ell,a,b,c} = n_2 v_{k,\ell,a,b,c} \}. \end{aligned} \quad (4.5)$$

Since a singular vector of $\mathfrak{s}(1/2)$ is a homogeneous element in $V^{d,r}$ and $\mathcal{S}_\pm, \mathcal{X}_+$ are nilpotent, it may have the factorized form:

$$v_s = f(G, K) u_0, \quad (4.6)$$

where $f(G, K)$ is a homogeneous polynomial in G, K and u_0 is a homogeneous element of $V^{d,r}$ containing fermionic generators. We note that \mathcal{X}_- commutes with $f(G, K)$. This gives a necessary condition for u_0 :

$$\mathcal{X}_- u_0 = 0. \quad (4.7)$$

There are eight possible u_0 according to the values of a, b and c . It is seen from (4.4) that they are specified by the pair of integers $(a + b, a - b + c)$. We call this degree of u_0 . All possible u_0 are listed in Table 1. We examine the eight cases to seek possible singular vectors.

- i) $\text{deg } u_0 = (0, 0)$: Since $P v_0 = H v_0 = 0$, u_0 is the lowest weight vector of non-super $\mathfrak{s}(1)$. From Proposition 1 a candidate of the singular vector for $m \neq 0$ is given by $v_s = (G^2 - 2mK)^p v_0$ and for $m = 0$ by $v_s = G^p v_0$. It is however easy to see that $\mathcal{Q}_+ v_s \neq 0$ for both cases. Thus there is no singular vector.
- ii) $\text{deg } u_0 = (0, 1)$: One readily see that $\mathcal{X}_- u_0 = -m v_0$. Thus there is no singular vector for $m \neq 0$. Noting that $P u_0 = H u_0 = 0$, one can apply Proposition 1 for $m = 0$ and obtain $v_s = G^p u_0$ ($p \in \mathbb{Z}_{\geq 0}$) as candidates for singular vectors. It is easy to verify that $\mathcal{Q}_\pm v_s = 0$ so that v_s are indeed singular vectors.

	a	b	c	$(a + b, a - b + c)$	u_0
i)	0	0	0	(0, 0)	v_0
ii)	0	0	1	(0, 1)	\mathcal{X}_+v_0
iii)	0	1	0	(1, -1)	\mathcal{S}_-v_0
iv)	1	0	0	(1, 1)	$(\alpha\mathcal{S}_+ + \beta G\mathcal{X}_+)v_0$
v)	0	1	1	(1, 0)	$(\alpha\mathcal{S}_-\mathcal{X}_+ + \beta G)v_0$
vi)	1	0	1	(1, 2)	$\mathcal{S}_+\mathcal{X}_+v_0$
vii)	1	1	0	(2, 0)	see text
viii)	1	1	1	(2, 1)	see text

Table 1: List of u_0 and its degree

- iii) $\deg u_0 = (1, -1)$: One readily see that $\mathcal{X}_-u_0 = Pu_0 = Hu_0 = 0$. However, there is no singular vector in this case because of the same reason as the case i).
- iv) $\deg u_0 = (1, 1)$: The condition (4.7) yields $\mathcal{X}_-u_0 = -(\alpha + m\beta)Gv_0$. If $m \neq 0$, then $\alpha = -m\beta$ and $Pu_0 = Hu_0 = 0$. A candidate of the singular vector is given by $v_s = (G^2 - 2mK)^p u_0$, but one readily see that $\mathcal{Q}_-v_s \neq 0$. Thus there is no singular vector for $m \neq 0$. If $m = 0$, then the condition (4.7) yields $\alpha = 0$. Therefore u_0 is reduced to the one of the case ii).
- v) $\deg u_0 = (1, 0)$: The condition (4.7) yields $\mathcal{X}_-u_0 = \alpha m \mathcal{S}_-v_0$. If $m \neq 0$, then $\alpha = 0$ so that u_0 is reduced to the one of the case i). If $m = 0$ then α, β are arbitrary. Furthermore one verifies that $Pu_0 = Hu_0 = 0$. Setting $\beta = 0$, it is verified that $v_s = G^p \mathcal{S}_-\mathcal{X}_+v_0$ ($p \in \mathbb{Z}_{\geq 0}$) satisfy $\mathcal{Q}_{\pm}v_s = 0$ if $r = d - p - 1$. Therefore they are singular vectors.
- vi) $\deg u_0 = (1, 2)$: It is immediate to see that $\mathcal{X}_-u_0 = (m\mathcal{S}_+ - G\mathcal{X}_+)v_0 \neq 0$. Thus no singular vectors.
- vii) $\deg u_0 = (2, 0)$: The vector u_0 is given by

$$u_0 = (\alpha G\mathcal{S}_-\mathcal{X}_+ + \beta\mathcal{S}_+\mathcal{S}_- + \gamma G^2 + \delta K)v_0. \quad (4.8)$$

It follows that $\mathcal{X}_-u_0 = (\alpha m - \beta)G\mathcal{S}_-v_0$. If $m \neq 0$ then $\beta = \alpha m$. Furthermore both P and H does not annihilate u_0 :

$$Pu_0 = \{-2m(\alpha - \gamma) + \delta\}Gv_0, \quad Hu_0 = \{m(d - r - 1)\alpha + m\gamma - d\delta\}v_0. \quad (4.9)$$

The singular vectors may have the form of

$$v_s = \sum_{\ell} a_{\ell} G^{n-2\ell} K^{\ell} u_0, \quad n = k + 2\ell \quad (4.10)$$

The condition $\mathcal{Q}_\pm v_s = 0$ yields the following four recurrence relations:

$$\{(d - r - n + 2\ell)\alpha + k\gamma\}a_{\ell+1} + (n - 2\ell)\delta a_\ell = 0, \quad (4.11)$$

$$(\ell + 1)\gamma a_{\ell+1} + \{(-d + r + n - 2\ell)m\alpha + (\ell + 1)\delta\}a_\ell = 0, \quad (4.12)$$

$$(\ell + 1)a_{\ell+1} + (n - 2\ell)m a_\ell = 0, \quad (4.13)$$

$$(\ell + 1)\gamma a_{\ell+1} + \{(d + r - 2\ell - 1)m\alpha + (\ell + 1)\delta\}a_\ell = 0. \quad (4.14)$$

The condition $Pv_s = 0$ yields the recurrence relation below in addition to (4.13):

$$\{-2m\alpha + (n - 2\ell)m\gamma + (\ell + 2)\delta\}a_{\ell+1} + (n - 2\ell)m\delta a_\ell + (\ell + 2)\gamma a_{\ell+2} = 0. \quad (4.15)$$

In the derivation of (4.11)-(4.15) we used $\alpha \neq 0$, since if $\alpha = 0$ then u_0 is reduced to the one of the case i). Substitution of (4.13) into others and after some algebra we obtain

$$d = \frac{1}{2}(n + 1), \quad \gamma = \frac{d + r + 1}{2d + 1}\alpha, \quad \delta = 2m(\alpha - \gamma). \quad (4.16)$$

The relation (4.13) is identical to the non-super case [20]. By solving it we obtain for $n = 2p$ the expression $f(G, K) = (G^2 - 2mK)^p$. We thus obtain a singular vector for $d = p + 1/2$ ($p \in \mathbb{N}$) and arbitrary value of r . We remark that the u_0 with the condition (4.16) is annihilated by P and H but not by \mathcal{Q}_\pm .

If $m = 0$ we see that the condition (4.7) is equivalent to $\beta = 0$. The vector (4.8) with $\beta = 0$ is not annihilated by both P and H . We thus use the expression (4.10) again. It is then easy to see that $Pv_s = 0$ yields $\alpha = 0$. Thus our u_0 is reduced to the one on the case i).

viii) $\deg u_0 = (2, 1)$: The vector u_0 is given by

$$u_0 = (\alpha \mathcal{S}_+ \mathcal{S}_- \mathcal{X}_+ + \beta G \mathcal{S}_+ + \gamma G^2 \mathcal{X}_+ + \delta K \mathcal{X}_+)v_0. \quad (4.17)$$

It is immediate to verify that

$$\mathcal{X}_- u_0 = -\alpha(G \mathcal{S}_- \mathcal{X}_+ + m \mathcal{S}_+ \mathcal{S}_-)v_0 - (\beta + m\gamma)G^2 v_0 - m\delta K v_0.$$

It follows that if $m \neq 0$ then $\alpha = \delta = 0$ thus u_0 is reduced to the case iv). If $m = 0$ then $\alpha = \beta = 0$. Thus our u_0 is reduced to the case ii).

Now our investigation is summarized in the next proposition:

Proposition 4 *The singular vectors of the Verma module $V^{d,r}$ over $\mathfrak{s}(1/2)$ are given as follows:*

i) *For $m \neq 0$, a singular vector exist for $d = p + 1/2$ ($p \in \mathbb{N}$) and arbitrary value of r :*

$$\begin{aligned} v_s^p &= (G^2 - 2mK)^p u_0, \\ u_0 &= (G \mathcal{S}_- \mathcal{X}_+ + m \mathcal{S}_+ \mathcal{S}_- + 2mK)v_0 + \frac{d + r + 1}{2d + 1}(G^2 - 2mK)v_0. \end{aligned} \quad (4.18)$$

ii) For $m = 0$, infinitely many singular vector exist for each value of d and r :

$$v_s^p = G^p \mathcal{X}_+ v_0, \quad p \in \mathbb{Z}_{\geq 0}. \quad (4.19)$$

In addition to this, if $r = d - p - 1$ ($p \in \mathbb{Z}_{\geq 0}$) then infinitely many extra singular vectors exists:

$$\tilde{v}_s^p = G^p \mathcal{S}_- \mathcal{X}_+ v_0. \quad (4.20)$$

4.2 Irreducible modules

In this subsection we examine the reducibility of the Verma module $V^{d,r}$. We first consider the case of $m \neq 0$. If $d \neq p + 1/2$ for $p \in \mathbb{Z}_{\geq 0}$, there is no singular vector in $V^{d,r}$. Thus the Verma module is irreducible. For $d = p + 1/2$ there is a singular vector (4.18). The submodule $I^{d,r} = U(\mathfrak{g}(1/2)^+) \otimes v_s^p$ is invariant under the action of $\mathfrak{g}(1/2)$ and $I^{d,r} \simeq V^{d',r}$ with $d' = -p - 3/2$. From the same discussion as section 3.2 it can be seen that $I^{d,r}$ does not have singular vectors. Search for the singular vectors in the factor module $V^{d,r}/I^{d,r}$ can be carried out by the same way as section 3.2. After some work one may conclude that no singular vectors are found in $V^{d,r}/I^{d,r}$. Thus the module is irreducible.

Next we study the case of $m = 0$. Since all $V^{d,r}$ contain infinitely many singular vectors, they are reducible. One see from (4.19) and (4.20) that $v_s^p = G^p v_s^0$, $\tilde{v}_s^p = G^p \mathcal{S}_- v_s^0$. Hence, in order to find the irreducible modules, we need only consider the factor module

$$V^{d,r}/I^0, \quad I^0 = U(\mathfrak{g}(1/2)^+) \otimes v_s^0. \quad (4.21)$$

The lowest weight vector w_0 of the factor module $V^{d,r}/I^0$ satisfies the condition

$$\mathcal{Q}_{\pm} w_0 = \mathcal{X}_{\pm} w_0 = P w_0 = 0, \quad (4.22)$$

and the basis of $V^{d,r}/I^0$ is given by

$$G^k K^{\ell} \mathcal{S}_+^a \mathcal{S}_-^b w_0, \quad k, \ell \in \mathbb{Z}_{\geq 0}, \quad a, b \in \{0, 1\} \quad (4.23)$$

The singular vectors may have the same form as (4.6) provided that u_0 containing only \mathcal{S}_{\pm} as fermionic generators. All possible u_0 are listed in Table 2. Each u_0 is specified by the pair of integers $(a + b, a - b)$. The necessary condition for the singular vectors in this case is

$$\mathcal{X}_{\pm} u_0 = 0, \quad (4.24)$$

since \mathcal{X}_{\pm} commute with G and K . It is immediate to see that the cases ii) iii) do not satisfy the condition. For the u_0 of the case iv) one has $\mathcal{X}_{\pm} u_0 = \pm \alpha G \mathcal{S}_{\pm} w_0 = 0$ which means $\alpha = 0$. Thus the case iv) is reduced to the case i). Clearly, the vector u_0 of the case i) is a lowest weight vector of the non-super $\mathfrak{g}(1)$. Hence, from Proposition 1 a candidate of the singular vector is

	a	b	$(a + b, a - b)$	u_0
i)	0	0	(0,0)	w_0
ii)	0	1	(1,-1)	$\mathcal{S}_- w_0$
iii)	1	0	(1,1)	$\mathcal{S}_+ w_0$
iv)	1	1	(2,0)	$(\alpha \mathcal{S}_+ \mathcal{S}_- + \beta G^2 + \gamma K) w_0$

Table 2: List of u_0 for $m = 0$

given by $w_s^p = G^p w_0$ ($p \in \mathbb{N}$). It is also easy to see that these w_s^p are annihilated by \mathcal{Q}_\pm and P . Thus these w_s^p are singular vectors in $V^{d,r}/I^0$, namely, the module $V^{d,r}/I^0$ is reducible.

Because we have the relation $w_s^p = G^{p-1} w_s^1$, the factor module we need to investigate next is given by

$$L^{d,r} \equiv (V^{d,r}/I^0)/\mathcal{I}^1, \quad \mathcal{I}^1 = U(\mathfrak{g}(1/2)^+) \otimes w_s^1. \quad (4.25)$$

The lowest weight vector z_0 of $L^{d,r}$ satisfies the condition

$$\mathcal{Q}_\pm z_0 = \mathcal{X}_\pm z_0 = P z_0 = G z_0 = 0. \quad (4.26)$$

The basis of $L^{d,r}$ is

$$K^\ell \mathcal{S}_+^a \mathcal{S}_-^b z_0, \quad \ell \in \mathbb{Z}_{\geq 0}, \quad a, b \in \{0, 1\} \quad (4.27)$$

We remark that P, G, M and \mathcal{X}_\pm are represented trivially on the basis, that is, the module $L^{d,r}$ is also the one of the subalgebra $osp(2/2)$. A singular vector in $L^{d,r}$ may have the form

$$z_s^\ell = K^\ell u_0, \quad (4.28)$$

where u_0 is a homogeneous element of $L^{d,r}$ containing each \mathcal{S}_\pm at most once. Thus the vector u_0 is specified by the pair of integer $(a + b, a - b)$. This means that possible u_0 is identical to the ones given in Table 2 provided that w_0 is replaced with z_0 and that the terms having G^2 disappears from u_0 for the case iv). Because P, G, \mathcal{X}_\pm have a trivial representation, z_s^ℓ is a singular vector if $\mathcal{Q}_\pm z_s^\ell = 0$. We examine this condition for possible u_0 according to Table 2.

i) $u_0 = z_0$: For this case we have $\mathcal{Q}_\pm z_s^\ell = \ell K^{\ell-1} \mathcal{S}_\pm z_0$. Thus the condition requires $\ell = 0$ which is trivial. Thus there is no singular vector for this case.

ii) $u_0 = \mathcal{S}_- z_0$: One verifies easily that $\mathcal{Q}_- z_s^\ell = 0$ and

$$\mathcal{Q}_+ z_s^\ell = \{(d - r)k^\ell + \ell K^{\ell-1} \mathcal{S}_+ \mathcal{S}_-\} z_0.$$

It follows that $z_s^0 = \mathcal{S}_- z_0$ is a singular vector if $d = r$.

iii) $u = \mathcal{S}_+ z_0$: One readily see that $\mathcal{Q}_+ z_s^\ell = 0$ and

$$\mathcal{Q}_- z_s^\ell = \{(d + r)K^\ell + \ell K^{\ell-1} \mathcal{S}_- \mathcal{S}_+\} z_0.$$

It follows that $z_s^0 = \mathcal{S}_+ z_0$ is a singular vector if $d = -r$.

iv) $u = (\alpha \mathcal{S}_+ \mathcal{S}_- + \beta K) z_0$: The condition yields the following relations:

$$\begin{aligned} -(d-r)\alpha + (\ell+1)\beta &= 0, \\ (d+r-2\ell-2)\alpha + (\ell+1)\beta &= 0. \end{aligned}$$

It turns out by solving these relations ($\alpha \neq 0$) that if $d = \ell + 1$ we have a singular vector:

$$z_s^\ell = K^\ell \mathcal{S}_+ \mathcal{S}_- z_0 + \frac{d-r}{d} K^{\ell+1} z_0. \quad (4.29)$$

Therefore $L^{d,r}$ is irreducible if $r \neq \pm d$ or d is not a positive integer. On the other hand, if $r = \pm d$ or $d = p \in \mathbb{N}$, then $L^{d,r}$ is reducible. We thus need to investigate three factor modules:

$$\begin{aligned} \mathcal{L}_-^d &= L^{d,d}/U(\mathfrak{sl}(1/2)^+) \otimes \mathcal{S}_- z_0, \\ \mathcal{L}_+^d &= L^{d,-d}/U(\mathfrak{sl}(1/2)^+) \otimes \mathcal{S}_+ z_0, \\ \mathcal{L}^{p,r} &= L^{p,r}/U(\mathfrak{sl}(1/2)^+) \otimes z_s^{p-1}, \quad \text{with } z_s^{p-1} \text{ in (4.29)} \end{aligned}$$

We denote the lowest weight vector of both \mathcal{L}_\pm^d by $|0\rangle$ (this does not cause any confusion). Then it is annihilated by $\mathcal{Q}_\pm, \mathcal{X}_\pm, P, G$ and \mathcal{S}_+ for \mathcal{L}_+^d and by $\mathcal{Q}_\pm, \mathcal{X}_\pm, P, G$ and \mathcal{S}_- for \mathcal{L}_-^d . The bases of \mathcal{L}_+^d and \mathcal{L}_-^d are given by

$$K^\ell \mathcal{S}_-^a |0\rangle, \quad K^\ell \mathcal{S}_+^a |0\rangle, \quad \ell \in \mathbb{Z}_{\geq 0}, \quad a \in \{0, 1\}$$

respectively. It follows that \mathcal{L}_+^d and \mathcal{L}_-^d are isomorphic. The isomorphism is given by ω_2 defined in (2.12) (2.14) or σ_2 defined in (2.16). We thus examine \mathcal{L}_+^d and $\mathcal{L}^{p,r}$.

Search for the singular vector in \mathcal{L}_+^d is equivalent to find the basis vectors annihilated by \mathcal{Q}_\pm . The basis vectors with $a = 0$ are annihilated by \mathcal{Q}_+ and those with $a = 1$ are by \mathcal{Q}_- . While we have the followings:

$$\mathcal{Q}_- K^\ell |0\rangle = \ell K^{\ell-1} \mathcal{S}_- |0\rangle, \quad \mathcal{Q}_+ K^\ell \mathcal{S}_- |0\rangle = 2(d-\ell) K^\ell |0\rangle. \quad (4.30)$$

The first relation of (4.30) requires $\ell = 0$ for $a = 0$. This shows that there is no singular vector for $a = 0$. It follows from the second relation of (4.30) that if $d = \ell$ then $K^\ell \mathcal{S}_- |0\rangle$ is a singular vector. Hence, \mathcal{L}_+^d is irreducible if d is not a non-negative integer. For $d = \ell \in \mathbb{Z}_{\geq 0}$ we consider the factor module $\mathcal{L}_+^\ell / \mathcal{I}^\ell$ where $\mathcal{I}^\ell = U(\mathfrak{sl}(1/2)^+) \otimes K^\ell \mathcal{S}_- |0\rangle$. The basis of $\mathcal{L}_+^\ell / \mathcal{I}^\ell$ is given by

$$\mathcal{S}_-^a |\tilde{0}\rangle, \quad K \mathcal{S}_-^a |\tilde{0}\rangle, \quad K^2 \mathcal{S}_-^a |\tilde{0}\rangle, \quad \dots \quad K^{\ell-1} \mathcal{S}_-^a |\tilde{0}\rangle, \quad K^\ell |\tilde{0}\rangle, \quad a \in \{0, 1\}$$

where $|\tilde{0}\rangle$ is the lowest weight vector of $\mathcal{L}_+^\ell / \mathcal{I}^\ell$. It is easy to see that there is no singular vector in $\mathcal{L}_+^\ell / \mathcal{I}^\ell$. Therefore, $\mathcal{L}_+^\ell / \mathcal{I}^\ell$ is irreducible.

Next we study $\mathcal{L}^{p,r}$. We denote the lowest weight vector of $\mathcal{L}^{p,r}$ by $|0\rangle$. Then

$$\begin{aligned}\mathcal{Q}_\pm |0\rangle &= \mathcal{X}_\pm |0\rangle = P |0\rangle = G |0\rangle = 0, \\ K^p |0\rangle &= \frac{p}{p-r} K^{p-1} \mathcal{S}_+ \mathcal{S}_- |0\rangle.\end{aligned}$$

It follows that $K^p \mathcal{S}_\pm |0\rangle = K^p \mathcal{S}_+ \mathcal{S}_- |0\rangle = 0$. Thus the basis of $\mathcal{L}^{p,r}$ is given by

$$K^\ell \mathcal{S}_+^a \mathcal{S}_-^b |0\rangle, \quad \ell < p, \quad a, b \in \{0, 1\} \quad (4.31)$$

Singular vectors in $\mathcal{L}^{p,r}$ will be obtained by imposing the condition that the vectors are annihilated by \mathcal{Q}_\pm . If we carry this out according to the classification same as Table 2 then it turns out that the process is almost same as the case of $L^{d,r}$. We therefore mention the results and omit the detail. The case i) has no singular vectors. The case ii) has one singular vector $\mathcal{S}_- |0\rangle$ if $r = d$. Thus the analysis of this case gives the equivalent result to \mathcal{L}_-^d . The case iii) also has one singular vector $\mathcal{S}_+ |0\rangle$ if $r = -d$. This case is reduced to \mathcal{L}_+^d . The case i) does not have any singular vectors.

Summarizing the results obtained so far, we have proved the following proposition.

Proposition 5 *The irreducible lowest weight modules over the Lie superalgebra $\mathfrak{s}(1/2)$ are classified as follows:*

i) $m \neq 0$

- $V^{d,r}$ when $d \neq p + 1/2$, $p \in \mathbb{Z}_{\geq 0}$. $\dim V^{d,r} = \infty$.
- $V^{d,r}/I^{d,r}$ when $d = p + 1/2$, $p \in \mathbb{Z}_{\geq 0}$. $\dim V^{d,r}/I^{d,r} = \infty$.

For both cases, r takes an arbitrary value.

ii) $m = 0$

- $L^{d,r}$ when $r \neq \pm d$ or p is not a positive integer. $\dim L^{d,r} = \infty$.
- $\mathcal{L}_+^d \simeq \mathcal{L}_-^d$ when d is not a non-negative integer. $\dim \mathcal{L}_+^d = \infty$.
- $\mathcal{L}_+^\ell / \mathcal{I}^\ell$ when $\ell \in \mathbb{Z}_{\geq 0}$. $\dim \mathcal{L}_+^\ell / \mathcal{I}^\ell = 2\ell + 1$.

The representations for $m = 0$ are also irreps of the subalgebra $\mathfrak{osp}(2/2)$.

5 Vector field realization

In this section we present a vector field realization of $\mathfrak{s}(1/1)$ and $\mathfrak{s}(1/2)$ which is an extension of the standard vector field realization of the Schrödinger algebra $\mathfrak{s}(1)$. The standard vector field realization of $\mathfrak{s}(1)$ has one time and one space coordinates (t, x) . In order to realize $\mathfrak{s}(1/1)$

we further introduce two variables (θ, η) of parity odd. One of them is Grassmannian and the another is related to mass eigenvalue:

$$\{\theta, \theta\} = \{\theta, \eta\} = 0, \quad \{\eta, \eta\} = -m. \quad (5.1)$$

With the four variables (t, x, θ, η) the realization of $\mathfrak{s}(1/1)$ is given by

$$\begin{aligned} K &= t(t\partial_t + x\partial_x + \theta\partial_\theta) + \frac{m}{2}x^2 + x\theta\eta - td, \\ G &= t\partial_x + mx + \theta\eta, \quad D = 2t\partial_t + x\partial_x + \theta\partial_\theta - d, \\ H &= \partial_t, \quad P = \partial_x, \quad M = m, \quad \mathcal{Q} = -\theta\partial_t + \partial_\theta, \\ \mathcal{S} &= -\theta(t\partial_t + x\partial_x) + t\partial_\theta + x\eta + \theta d, \quad \mathcal{X} = -\theta\partial_x + \eta, \end{aligned} \quad (5.2)$$

where d is the conformal weight. To realize $\mathfrak{s}(1/2)$ we introduce three Grassmann variables (θ, ϕ, ρ) . The realization is given as follows:

$$\begin{aligned} K &= t(t\partial_t + x\partial_x + \theta\partial_\theta + \phi\partial_\phi) + \theta\phi\rho\partial_\rho - mx\theta\rho + \frac{m}{2}x^2 + x\phi\partial_\rho - td, \\ G &= t\partial_x + m(x - \theta\rho) + \phi\partial_\rho, \quad P = \partial_x, \\ D &= 2t\partial_t + x\partial_x + \theta\partial_\theta + \phi\partial_\phi - d, \\ H &= \partial_t, \quad R = -\theta\partial_\theta + \phi\partial_\phi + \rho\partial_\rho, \quad M = m, \\ \mathcal{Q}_+ &= -\phi\partial_t + \partial_\theta, \quad \mathcal{Q}_- = -\theta\partial_t + \partial_\phi \\ \mathcal{S}_+ &= \phi(-t\partial_t - x\partial_x - \theta\partial_\theta + \rho\partial_\rho) + t\partial_\theta - mx\rho + \phi d, \\ \mathcal{S}_- &= \theta(-t\partial_t - x\partial_x - \phi\partial_\phi - \rho\partial_\rho) + t\partial_\phi + x\partial_\rho + \theta d, \\ \mathcal{X}_+ &= -\phi\partial_x - m\rho, \quad \mathcal{X}_- = -\theta\partial_x + \partial_\rho. \end{aligned} \quad (5.3)$$

We have introduced two Clifford-like elements χ and η for $\mathfrak{s}(1/1)$. We remark that they are realized by a single Grassmann number ϕ as follows:

$$\chi = \sqrt{\frac{m}{2}}(\phi + \partial_\phi), \quad \eta = \sqrt{\frac{m}{2}}(\phi - \partial_\phi). \quad (5.4)$$

The vector field realizations given above together with singular vectors may give invariant partial differential equations of super Schrödinger algebras. This is a supersymmetric extension of the result for semisimple Lie algebras developed in [25,26]. The invariant equations obtained by this procedure for $\mathfrak{s}(n)$ have been obtained in [20,21,27,28].

6 Concluding remarks

We investigated the lowest weight modules (Verma modules) over $\mathfrak{s}(1/1)$ and $\mathfrak{s}(1/2)$. Explicit expression of the singular vectors were derived and reducibility of the Verma modules has been studied. This led us to classify irreducible modules over $\mathfrak{s}(1/1)$ and $\mathfrak{s}(1/2)$.

We comment on a bilinear form analogous to the Shapovalov form [29] of the semisimple Lie algebra. Let $\mathfrak{g} = \mathfrak{sl}(1/1)$ or $\mathfrak{sl}(1/2)$ and V be a Verma module over \mathfrak{g} . The lowest weight vector of V is denoted by v_0 as usual. We define the bilinear form $(\ , \) : V \otimes V \rightarrow \mathbb{C}$ by the relations:

$$(Xv_0, Yv_0) = (v_0, \omega_1(X)Yv_0), \quad (v_0, v_0) = 1, \quad X, Y \in U(\mathfrak{g}) \quad (6.1)$$

If $v_m, v_n \in V$ have different weight, then they are orthogonal with respect to this form:

$$(v_m, v_n) = 0. \quad (6.2)$$

To see this, suppose that the weights of v_m, v_n are m and n , respectively. Then

$$(Dv_m, v_n) = m(v_m, v_n).$$

The left hand side has alternate way of computation:

$$(Dv_m, v_n) = (v_m, \omega_1(D)v_n) = n(v_m, v_n).$$

Since $m \neq n$ we obtain (6.2). It follows that a singular vector of \mathfrak{g} is orthogonal to any other vectors in V . This is the same property as semisimple case. Thus one may analyze the reducibility of the Verma modules also via the bilinear form.

We provided a vector field realization of $\mathfrak{sl}(1/1)$ and $\mathfrak{sl}(1/2)$ in section 5. It will open a way to physical applications of the super Schrödinger algebras. One example we will do as a future work is a supersymmetric extension of the group theoretical approach to nonrelativistic holography discussed in [30]. Another important future work is the classification of irreducible modules for the super Schrödinger algebras of higher dimensional spacetime. Especially, the most physical $(1 + 3)$ dimensional spacetime is of importance. We have investigated $\mathcal{N} = 1, 2$ in the present paper. This is because physical applications in nonrelativistic setting are known for small values of \mathcal{N} . Of course, this does not mean the super Schrödinger algebras for large values of \mathcal{N} are useless. Analysis of irreducible representations for large \mathcal{N} is also an interesting problem.

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