SOME IDENTITIES OF BERNOULLI NUMBERS AND POLYNOMIALS ASSOCIATED WITH BERNSTEIN POLYNOMIALS

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Abstract. We investigate some interesting properties of the Bernstein polynomials related to the bosonic p-adic integrals on $\mathbb{Z}_p$.

1. Introduction

Let $C[0,1]$ be the set of continuous functions on $[0,1]$. Then the classical Bernstein polynomials of degree n for $f \in C[0,1]$ are defined by

$$\mathbb{B}_n(f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x), \quad 0 \le x \le 1 \tag{1.1}$$

where $\mathbb{B}_n(f)$ is called the Bernstein operator and

$$B_{k,n}(x) = \binom{n}{k} x^k (x-1)^{n-k} \tag{1.2}$$

are called the Bernstein basis polynomials (or the Bernstein polynomials of degree n) (see [10]). Recently, Acikgoz and Araci have studied the generating function for Bernstein polynomials (see [1, 2]). Their generating function for $B_{k,n}(x)$ is given by

$$F^{(k)}(t,x) = \frac{t^k e^{(1-x)t} x^k}{k!} = \sum_{n=0}^{\infty} B_{k,n}(x) \frac{t^n}{n!}, \tag{1.3}$$

where $k = 0, 1, \ldots$ and $x \in [0,1]$. Note that

$$B_{k,n}(x) = \begin{cases} \binom{n}{k} x^k (1-x)^{n-k} & \text{if } n \ge k \\ 0, & \text{if } n < k \end{cases}$$

for $n = 0, 1, \ldots$ (see [1, 2]).

The Bernstein polynomials can also be defined in many different ways. Thus, recently, many applications of these polynomials have been looked for by many authors. Some researchers have studied the Bernstein polynomials in the area of approximation theory (see [1, 2, 3, 7, 9, 10]). In recent years, Acikgoz and Araci [1, 2] have introduced several type Bernstein polynomials.

In the present paper, we introduce the Bernstein polynomials on the ring of p-adic integers $\mathbb{Z}_p$. We also investigate some interesting properties of the Bernstein polynomials related to the bosonic p-adic integrals on the ring of p-adic integers $\mathbb{Z}_p$.

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2. Bernstein polynomials related to the bosonic p-adic integrals on $\mathbb{Z}_p$

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-1}$. For $N \geq 1$, the bosonic distribution μ_1 on $\mathbb{Z}_p$

$$\mu(a + p^N \mathbb{Z}_p) = \frac{1}{p^N} \tag{2.1}$$

is known as the p-adic Haar distribution μ_{Haar}, where $a + p^N\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x-a|_p \leq p^{-N}\}$ (cf. [4]). We shall write $d\mu_1(x)$ to remind ourselves that x is the variable of integration. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. Then μ_1 yields the fermionic p-adic q-integral of a function $f \in UD(\mathbb{Z}_p)$:

$$I_1(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \tag{2.2}$$

(cf. [4]). Many interesting properties of (2.2) were studied by many authors (cf. [4, 8] and the references given there). For $n \in \mathbb{N}$, write $f_n(x) = f(x+n)$. We have

$$I_1(f_n) = I_1(f) + \sum_{l=0}^{n-1} f'(l). \tag{2.3}$$

This identity is to derives interesting relationships involving Bernoulli numbers and polynomials. Indeed, we note that

$$I_1((x+y)^n) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_1(y) = B_n(x), \tag{2.4}$$

where $B_n(x)$ are the Bernoulli polynomials (cf. [4]). From (1.2), we have

$$\int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_1(x) = \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} B_{n-j} \tag{2.5}$$

and

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_1(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x) d\mu_1(x) \\ &= \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{l=0}^{n-j} \binom{n-j}{l} (-1)^l B_l. \end{aligned} \tag{2.6}$$

By (2.5) and (2.6), we obtain the following proposition.

Proposition 2.1. *For $n \geq k$,*

$$\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} B_{n-j} = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{l=0}^{n-j} \binom{n-j}{l} (-1)^l B_l.$$

From (2.4), we note that

$$B_n(2) = (B(1)+1)^n - n = (B+1)^n = B_n, \quad n > 1 \tag{2.7}$$

with the usual convention of replacing B^n by B_n. Thus, we have

$$\begin{aligned}\int_{\mathbb{Z}_p} x^n d\mu_1(x) &= \int_{\mathbb{Z}_p} (x+2)^n d\mu_1(x) - n \\ &= (-1)^n \int_{\mathbb{Z}_p} (x-1)^n d\mu_1(x) - n \\ &= \int_{\mathbb{Z}_p} (1-x)^n d\mu_1(x) - n\end{aligned} \tag{2.8}$$

for $n > 1$, since $(-1)^n B_n(x) = B_n(1-x)$. Therefore we obtain the following theorem.

Theorem 2.2. *For $n > 1$,*

$$\int_{\mathbb{Z}_p} (1-x)^n d\mu_1(x) = \int_{\mathbb{Z}_p} x^n d\mu_1(x) + n.$$

And also we obtain

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{n-k,k}(x) d\mu_1(x) &= \int_{\mathbb{Z}_p} x^{n-k}(1-x)^k d\mu_1(x) \\ &= \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} (1-x)^{l+k} d\mu_1(x) \\ &= \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \left\{ \int_{\mathbb{Z}_p} x^{l+k} d\mu_1(x) + l + k \right\} \\ &= \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l (B_{l+k} + l + k).\end{aligned} \tag{2.9}$$

Therefore we obtain the following result.

Corollary 2.3. *For $k > 1$,*

$$\int_{\mathbb{Z}_p} B_{n-k,k}(x) d\mu_1(x) = \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l (B_{l+k} + l + k).$$

From the property of the Bernstein polynomials of degree n, we easily see that

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) d\mu_1(x) \\ &= \binom{n}{k}\binom{m}{k} \int_{\mathbb{Z}_p} x^{2k}(1-x)^{n+m-2k} d\mu_1(x) \\ &= \binom{n}{k}\binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l B_{2k+l}\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)B_{k,s}(x)d\mu_1(x) \\
&= \binom{n}{k}\binom{m}{k}\binom{s}{k}\int_{\mathbb{Z}_p} x^{3k}(1-x)^{n+m-3k}d\mu_1(x) \qquad (2.11)\\
&= \binom{n}{k}\binom{m}{k}\binom{s}{k}\sum_{l=0}^{n+m+s-3k}\binom{n+m+s-3k}{l}(-1)^l B_{3k+l}.
\end{aligned}$$

Continuing this process, we obtain the following theorem.

Theorem 2.4. *The multiplication of the sequence of Bernstein polynomials*

$$B_{k,n_1}(x), B_{k,n_2}(x), \ldots, B_{k,n_s}(x)$$

for $s \in \mathbb{N}$ with different degree under p-adic integral on $\mathbb{Z}_p$ can be given as

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)\cdots B_{k,n_s}(x)d\mu_1(x) \\
&= \binom{n_1}{k}\binom{n_2}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{n_1+n_2+\cdots+n_s-sk}\binom{n_1+n_2+\cdots+n_s-sk}{l}(-1)^l B_{sk+l}.
\end{aligned}$$

We put

$$B_{k,n}^m(x) = \underbrace{B_{k,n}(x)\times\cdots\times B_{k,n}(x)}_{m\text{-times}}.$$

Theorem 2.5. *The multiplication of*

$$B_{k,n_1}^{m_1}(x), B_{k,n_2}^{m_2}(x), \ldots, B_{k,n_s}^{m_s}(x)$$

Bernstein polynomials with different degrees $n_1, n_2, \cdots, n_s$ under p-adic integral on $\mathbb{Z}_p$ can be given as

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}^{m_1}(x)B_{k,n_2}^{m_2}(x)\cdots B_{k,n_s}^{m_s}(x)d\mu_1(x) \\
&= \binom{n_1}{k}^{m_1}\binom{n_2}{k}^{m_2}\cdots\binom{n_s}{k}^{m_s}\sum_{l=0}^{n_1m_1+n_2m_2+\cdots+n_sm_s-(m_1+\cdots+m_s)k}(-1)^l \\
&\quad\times\binom{n_1m_1+n_2m_2+\cdots+n_sm_s-(m_1+\cdots+m_s)k}{l}B_{(m_1+\cdots+m_s)k+l}.
\end{aligned}$$

Theorem 2.6. *The multiplication of*

$$B_{k_1,n_1}^{m_1}(x), B_{k_2,n_2}^{m_2}(x), \ldots, B_{k_s,n_s}^{m_s}(x)$$

Bernstein polynomials with different degrees $n_1, n_2, \cdots, n_s$ with different powers $m_1, m_2, \cdots, m_s$ under p-adic integral on $\mathbb{Z}_p$ can be given as

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k_1,n_1}^{m_1}(x)B_{k_2,n_2}^{m_2}(x)\cdots B_{k_s,n_s}^{m_s}(x)d\mu_1(x) \\
&= \binom{n_1}{k_1}^{m_1}\binom{n_2}{k_2}^{m_2}\cdots\binom{n_s}{k_s}^{m_s}\sum_{l=0}^{n_1m_1+n_2m_2+\cdots+n_sm_s-(k_1m_1+\cdots+k_sm_s)}(-1)^l \\
&\quad\times\binom{n_1m_1+n_2m_2+\cdots+n_sm_s-(k_1m_1+\cdots+k_sm_s)}{l}B_{k_1m_1+\cdots+k_sm_s+l}.
\end{aligned}$$

Problem. Find the Witt's formula for the Bernstein polynomials in p-adic number field.

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