CONDITIONS OF FIXED SIGN FOR $N \times N$ OPERATOR MATRICES

I.V. ORLOV AND E.V. BOZHONOK

ABSTRACT. A positive definiteness criterion and, under the additional conditions, a non-negativity criterion for a self-adjoint continuous operator matrix, acting in product of an arbitrary number of real separable Hilbert spaces, are obtained. As application, both sufficient and necessary analytical conditions of functional extremum of several Hilbert variables are considered.

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1. INTRODUCTION

The conditions of fixed sign for the operator matrices acting in a product of Hilbert spaces play a significant role in operator theory and its numerous applications. Among the base results respective to 2×2 matrix, note the results by M.G. Krein and Ju.L. Shmulian ([1], [2]). In the various applied aspects, the conditions of fixed sign for 2×2 operator matrices arose in the works by V.P. Potapov [3], T.Ya. Azizov and I.S. Iohvidov [4], N.D. Kopachevskii and S.G. Krein ([5]–[7]), J. Mandel [8]. The specific direction, Schur analysis [9] using operator Schur complements, appeared later (see survey by A. Bultheel and K. Müller [10]), the paper of Y. Arlinskii [11]).

Among the recent works in this area, distinguish the results of S. Hassi, M. Malamud, H. de Snoo [12] and T.Ya. Azizov and V.A. Khatskevich [13].

However, the most of the works above is connected with 2×2 operator matrices. In the present work we are interested in a complete description of the positive definite and nonnegative operator matrices, including the suitable explicit conditions for 3×3 operator matrices.

In the paper, the self-adjoint linear continuous operator matrices acting in a product of an arbitrary number of the real separable Hilbert spaces are studied. In the first item, a positive definiteness criterion in terms of the first kind Schur operators (Theorem 2.4) is obtained and the precise number of the corresponding inequalities is calculated (Theorem 2.6). In the second item, under the additional conditions, a nonnegativity criterion in term of the second kind Schur operators (Theorem 3.3) is obtained. In the third item, applications of the results above to determination of the functional extrema in a product of several Hilbert spaces are investigated.

Throughout the paper, the case $n = 3$ is described explicitly (Examples 2.7 and 3.6, Theorem 4.5 and 4.8). In the conclusion, an example of determination of functional extremum in the case of $n = 3$ and non-commuting second partial derivatives is considered.

2. POSITIVE DEFINITENESS CRITERION FOR SELF-ADJOINT OPERATOR MATRIX IN PRODUCT OF n HILBERT SPACES

We are based on the well known criterion for 2×2 matrix by means of Schur complements [4].

Theorem 2.1. *Let H_1, H_2 be separable real Hilbert spaces, $B_2 = (B_{ij} : H_j \to H_i)^2_{i,j=1}$ is a linear self-adjoint continuous operator in $H_1 \times H_2$ ($B_{11} = B_{11}^*$, $B_{22} = B_{22}^*$, $B_{12} = B_{21}^*$). Then B_2 is positive definite ($B_2 \gg 0$) if and only if the following conditions are fulfilled:*

i) $B_{11} \gg 0$, $B_{22} \gg 0$;

ii) $\triangle_2^1(B_2) := B_{11} - B_{12} \cdot B_{22}^{-1} \cdot B_{21} \gg 0$, $\triangle_1^2(B_2) := B_{22} - B_{21} \cdot B_{11}^{-1} \cdot B_{12} \gg 0$.

Remark 2.2. It's easy to check the following relation between Schur complements:

$$B_{21}B_{11}^{-1}\triangle_2^1 = \triangle_1^2 B_{22}^{-1}B_{21} \quad \text{or} \quad B_{12}B_{22}^{-1}\triangle_1^2 = \triangle_2^1 B_{11}^{-1}B_{12}$$

Let's consider now a linear continuous operator B_n acting in product of n separable real Hilbert spaces $H = H_1 \times \ldots \times H_n$ and defined by operator matrix $(B_{ij})^n_{i,j=1}$. First, introduce *Schur operators of the first kind.*

Definition 2.3. Let's divide the matrix B_n to four units: B_n^{11} is main minor of the order $\left[\frac{n}{2}\right] \times \left[\frac{n}{2}\right]$, B_n^{22} is its adjacent minor of the order $\left(n - \left[\frac{n}{2}\right]\right) \times \left(n - \left[\frac{n}{2}\right]\right)$, B_n^{12} and B_n^{21} are corresponding rectangular blocks of the orders $\left[\frac{n}{2}\right] \times \left(n - \left[\frac{n}{2}\right]\right)$ and $\left(n - \left[\frac{n}{2}\right]\right) \times \left[\frac{n}{2}\right]$, respectively. (Here $[\,\cdot\,]$ denotes integral part of a number). Suppose that the block matrix $(B_n^{ij})^2_{i,j=1}$ satisfies necessary condition of positive definiteness *i)* from Theorem 2.1 and hence B_n^{11} B_n^{22} are continuously invertible.

Let's introduce now four types of *Schur operators of the first kind*:

$$\triangle_1^1(B_n) = B_n^{11}; \quad \triangle_2^1(B_n) = B_n^{11} - B_n^{12} \cdot (B_n^{22})^{-1} \cdot B_n^{21};$$

$$\triangle_2^2(B_n) = B_n^{22}; \quad \triangle_1^2(B_n) = B_n^{22} - B_n^{21} \cdot (B_n^{11})^{-1} \cdot B_n^{12}.$$

Note that the matrices $\triangle_j^i(B_n)$ have maximal order $\left(\left[\frac{n}{2}\right]+1\right) \times \left(\left[\frac{n}{2}\right]+1\right)$ in case of the odd n and $\left[\frac{n}{2}\right] \times \left[\frac{n}{2}\right]$ in case of the even n. Pass to the main result.

Theorem 2.4. *Let H_i ($i = \overline{1,n}$) be separable real Hilbert spaces, $H = H_1 \times \ldots \times H_n$, $B_n = (B_{ij})^n_{i,j=1}$ be a linear continuous self-adjoint operator in H, in the $B_{ij} \in \mathcal{L}(H_j, H_i)$; $(i, j = \overline{1,n})$. Then B_n is positive definite if and only if the following system of positive definiteness inequalities is fulfilled:*

$$\{\triangle_{j_m}^{i_m} \ldots \triangle_{j_2}^{i_2} \triangle_{j_1}^{i_1}(B_n) \gg 0\} \quad (i_l, j_l = 1, 2), \tag{1}$$

where

$$m = \begin{cases} k, & as & n = 2^k \\ k+1, & as & 2^k < n < 2^{k+1}. \end{cases} \tag{2}$$

Proof. Let's divide the matrix $B_n = (B_{ij})^n_{i,j=1}$ to four units, according to Definition 2.3, $B_n = (B_n^{ij})^2_{i,j=1}$. Let $\tilde{H}^1 = H_1 \times \ldots \times H_{[\frac{n}{2}]}$, $\tilde{H}^2 = H_{[\frac{n}{2}]+1} \times \ldots \times H_n$. Then it can be consider B_n as an operator matrix acting in $\tilde{H}^1 \times \tilde{H}^2$, in there $B_n^{ij} \in \mathcal{L}(\tilde{H}^j \to \tilde{H}^i)$: $i, j = 1, 2$. By virtue of Theorem 2.1, B_n is positive definite if and only if the following inequalities

$$1) B_n^{11} = \triangle_1^1(B_n) \gg 0; \quad B_n^{22} = \triangle_2^2(B_n) \gg 0;$$

$$2)\, \triangle_2^1(B_n) \gg 0; \quad \triangle_1^2(B_n) \gg 0$$

hold true. Again, dividing every from the operator matrices above to four units, according to Definition 2.3, and applying Theorem 2.1 for the corresponding operators acting in $\tilde{H}^1 \times \tilde{H}^1$ and $\tilde{H}^2 \times \tilde{H}^2$, respectively, it follows 16 inequalities of the form $\{\triangle_{j_2}^{i_2} \triangle_{j_1}^{i_1}(B_n) \gg 0\}$; $(i_l, j_l = 1, 2)$ for the spaces $\left(H_1 \times \ldots \times H_{\left[\frac{\left[\frac{n}{2}\right]}{2}\right]}\right)^2$, $\left(H_{\left[\frac{\left[\frac{n}{2}\right]}{2}\right]+1} \times \ldots \times H_{\left[\frac{n}{2}\right]}\right)^2$, $\left(H_{\left[\frac{n}{2}\right]+1} \times \ldots \times H_{\left[\frac{n-\left[\frac{n}{2}\right]}{2}\right]}\right)^2$ and $\left(H_{\left[\frac{n-\left[\frac{n}{2}\right]}{2}\right]+1} \times \ldots \times H_n\right)^2$ respectively. Prolonging by the construction and reducing the matrix sizes at least up to $2^{k+1-p} \times 2^{k+1-p}$ as $n < 2^{k+1}$ under p-th inductive step, it leads after m inductive steps to the system (1). $\qquad \square$

Remark 2.5. 1) It's easy to see that in case of the commuting operators B_{ij}, the system (1) can be reduced to the well known Sylvester condition of positive definiteness of n main minors of the matrix B_n.

2) Denoting by $[x]_+$ the "right integral part" of x (i.e. the nearest integer to x from the right), it can rewrite the condition (2) in the form

$$m = [\log_2 n]_+ .$$

Let's estimate the number of inequalities in the system (1).

Theorem 2.6. *The number of inequalities V_n in system* (1) *is*

$$V_n = 2^k \cdot (3n - 2^{k+1}) \quad as \quad 2^k \leq n \leq 2^{k+1} \quad (k \in \mathbb{N}_0,\, n \in \mathbb{N}). \tag{3}$$

Moreover,

$$n^2 \leq V_n \leq (n+1)^2. \tag{4}$$

Proof. From definition of the system (1), the following recursion relations

$$V_1 = 1; \ V_{2n} = 4V_n; \ V_{2n+1} = 2(V_n + V_{n+1}) \quad (n \in \mathbb{N})$$

easily follow. Consider three possible cases.

1) $2^k \leq n < 2^{k+1}$, $n = 2m$ $(m \in \mathbb{N})$. Then $2^{k-1} \leq m < 2^k$ and applying (3), it implies

$$V_n = 4V_m = 4 \cdot 2^{k-1} \cdot (3m - 2^k) = 2^k \cdot (3 \cdot 2m - 2^{k+1}) = 2^k \cdot (3n - 2^{k+1}).$$

2) $2^k \leq n < 2^{k+1}$, $n = 2m + 1$ $(m \in \mathbb{N}_{\not\leftarrow})$. Then $2^k \leq 2m + 1 < 2^{k+1}$, whence $2^{k-1} < m + 1 \leq 2^k$ and applying (3), it follows

$$V_n = 2(V_m + V_{m+1}) = 2^k \cdot (3n - 2^{k+1}).$$

3) $n = 2^{k+1}$ $(k \in \mathbb{N}_{\not\leftarrow})$. Then

$$V_n = V_{2^{k+1}} = 4V_{2^k} = 4 \cdot 2^{k-1} \cdot (3 \cdot 2^k - 2^k) = 2^k \cdot (3n - 2^{k+1}).$$

Thus, the equality (3) is obtained. In addition, $V_n = n^2$ as $n = 2^k$ and $V_n = (n+1)^2$ as $n = 2^{k+1}$. Whence, in view of increase V_n for $2^k \leq n \leq 2^{k+1}$, the inequality (4) follows. $\square$

Pass to the examples. First, consider the partial case $n = 3$.

Example 2.7. Introduce (in case of B_3 with invertible B_{ii} $(i = 1, 2, 3)$) generalized Schur complements

$$\triangle_{jk}^i = B_{ii} - B_{ij} \cdot B_{jj}^{-1} \cdot B_{jk} \quad (i, j, k = 1, 2, 3).$$

Then direct calculations show that the inequalities (1) take form of

$$B_{11} \gg 0; \quad B_{22} \gg 0; \quad B_{33} \gg 0;$$

$$\triangle_{12}^2 \gg 0; \quad \triangle_{32}^2 \gg 0; \quad \triangle_{13}^3 \gg 0; \quad \triangle_{23}^3 \gg 0;$$

$$\triangle_{12}^2 - \triangle_{13}^2 \cdot (\triangle_{13}^3)^{-1} \cdot \triangle_{12}^3 \gg 0; \quad \triangle_{13}^3 - \triangle_{12}^3 \cdot (\triangle_{12}^2)^{-1} \cdot \triangle_{13}^2 \gg 0; \tag{5}$$

$$B_{11} - (B_{12} \cdot B_{22}^- \cdot B_{21} + B_{12} \cdot B_{23}^- \cdot B_{31} + B_{13} \cdot B_{32}^- \cdot B_{21} + B_{13} \cdot B_{33}^- \cdot B_{31}) \gg 0.$$

Here we denote by B_{ij}^- corresponding elements of the inverse matrix to $(B_{ij})_{i,j=2}^3$. In this case,

$$B_{22}^- = B_{22}^{-1} \cdot (I_{H_2} + B_{23} \cdot (\triangle_{23}^3)^{-1} \cdot B_{32} \cdot B_{22}^{-1}) = (I_{H_2} + B_{22}^{-1} \cdot B_{23} \cdot (\triangle_{23}^3)^{-1} \cdot B_{32}) \cdot B_{22}^{-1};$$

$$B_{33}^- = B_{33}^{-1} \cdot (I_{H_3} + B_{32} \cdot (\triangle_{32}^2)^{-1} \cdot B_{23} \cdot B_{33}^{-1}) = (I_{H_3} + B_{33}^{-1} \cdot B_{32} \cdot (\triangle_{32}^2)^{-1} \cdot B_{23}) \cdot B_{33}^{-1};$$

$$B_{23}^- = -(\triangle_{32}^2)^{-1} \cdot B_{23} \cdot B_{33}^{-1} = -B_{22}^{-1} \cdot B_{23}(\triangle_{23}^3)^{-1};$$

$$B_{32}^- = -(\triangle_{23}^3)^{-1} \cdot B_{32} \cdot B_{22}^{-1} = -B_{33}^{-1} \cdot B_{32}(\triangle_{32}^2)^{-1}.$$

Note also that the expressions in the third row of (5) can be considered as the "second order Schur complements", $\triangle_{123}^{23}$ and $\triangle_{132}^{32}$, relatively.

Further, consider the case of "bidiagonal" operator matrix B_n. In this case general definiteness conditions (1) assume an essential simplification.

Example 2.8. Let $B_n = \{B_{ij}\}_{i,j=1}^n$ be operator matrix in H with nonzero principal diagonals and zero all other elements.

1) In case $n = 2m$,

$$B_n = \begin{pmatrix} B_{11} & 0 & \dots & 0 & 0 & \dots & 0 & B_{1n} \\ 0 & B_{22} & \dots & 0 & 0 & \dots & B_{2,n-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & B_{\frac{n}{2},\frac{n}{2}} & B_{\frac{n}{2},\frac{n}{2}+1} & \dots & 0 & 0 \\ 0 & 0 & \dots & B_{\frac{n}{2}+1,\frac{n}{2}} & B_{\frac{n}{2}+1,\frac{n}{2}+1} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ B_{n1} & 0 & \dots & 0 & 0 & \dots & 0 & B_{nn} \end{pmatrix}.$$

Then the units B_n^{ij} take form

$$B_n^{11} = \begin{pmatrix} B_{11} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & B_{\frac{n}{2},\frac{n}{2}} \end{pmatrix}, \quad B_n^{22} = \begin{pmatrix} B_{\frac{n}{2}+1,\frac{n}{2}+1} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & B_{nn} \end{pmatrix},$$

$$B_n^{12} = \begin{pmatrix} 0 & \dots & B_{1n} \\ \vdots & & \vdots \\ B_{\frac{n}{2},\frac{n}{2}+1} & \dots & 0 \end{pmatrix}, B_n^{21} = \begin{pmatrix} 0 & \dots & B_{\frac{n}{2}+1,\frac{n}{2}} \\ \vdots & & \vdots \\ B_{n1} & \dots & 0 \end{pmatrix}.$$

Applying Schur operators, we obtain the diagonal matrix

$$\triangle_1^1(B_n) = B_n^{11}, \quad diag\,\triangle_2^1(B_n) = \{B_{ii} - B_{i,n+1-i} \cdot B_{n+1-i,n+1-i}^{-1} \cdot B_{n+1-i,i}\}_{i=1}^{\frac{n}{2}}$$

$$\triangle_2^2(B_n) = B_n^{22}, \quad diag\,\triangle_1^2(B_n) = \{B_{ii} - B_{i,n+1-i} \cdot B_{n+1-i,n+1-i}^{-1} \cdot B_{n+1-i,i}\}_{i=\frac{n}{2}+1}^{n}$$

Since conditions (1)–(2) for a diagonal matrix take form $B_{ii} \gg 0$ then $B_n \gg 0$ if and only if the inequalities

$$B_{ii} \gg 0; \quad B_{ii} - B_{i,n+1-i} \cdot B_{n+1-i,n+1-i}^{-1} \cdot B_{n+1-i,i} \gg 0 \quad (i = \overline{1,n})$$

hold.

2) In case $n = 2m + 1$

$$B_n = \begin{pmatrix} B_{11} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & B_{1n} \\ 0 & B_{22} & \dots & 0 & 0 & 0 & \dots & B_{2,n-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & B_{\frac{n-1}{2},\frac{n-1}{2}} & 0 & B_{\frac{n-1}{2},\frac{n+3}{2}} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & B_{\frac{n+1}{2},\frac{n+1}{2}} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ B_{n1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & B_{nn} \end{pmatrix}.$$

The calculations, analogous with the previous case, lead to the following criterion: $B_n \gg 0$ if and only if the inequalities

$$B_{ii} \gg 0 \ (i = \overline{1,n}); \quad B_{ii} - B_{i,n+1-i} \cdot B_{n+1-i,n+1-i}^{-1} \cdot B_{n+1-i,i} \gg 0 \ (i = \overline{1,n}, i \neq \frac{n+1}{2})$$

hold.

Finally, Theorem 2.4 can be formulated as positive definiteness criterion for the quadratic forms acting in a product of Hilbert spaces.

Theorem 2.9. *Let H_i $(i = \overline{1,n})$ be separable real Hilbert spaces, $H = H_1 \times \dots \times H_n$, φ is continuous quadratic form on H associated with a self-adjoint linear continuous operator $B_n = (B_{ij})_{i,j=1}^n$, i.e. $\varphi(h) = \langle B_n h, h \rangle$. Then the form φ is positive definite on H if and only if the conditions (1)–(2) are fulfilled.*

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3. Nonnegativity conditions for self-adjoint operator matrix in product of n Hilbert spaces

First, consider a case of 2×2 operator matrix $B_2 = (B_{ij})_{i,j=1}^2$ acting in $H_1 \times H_2$. For the case of product of the two complex Hilbert spaces a general nonnegativity criterion for B_2 was obtained in ([12], cor. 2.2) in terms of linear relations. In this connection, self-adjointness of B_2 is necessary condition for $B_2 \gg 0$. We formulate below for real case a simplified sufficient condition of nonnegativity B_2 under simplifying assumptions of a "partial self-adjointness" of B_2 and continuous invertibility of B_{11}.

Theorem 3.1. *Let H_1, H_2 be real separable Hilbert spaces, $B_2 = (B_{ij})_{i,j=1}^2$ is continuous linear operator in $H_1 \times H_2$, where B_{11} is continuously invertible and self-adjoint and $B_{12} = B_{21}^*$. Then B_2 is nonnegative if and only if*

$$B_{11} \geq 0 \quad and \quad \triangle_1^2(B_2) = B_{22} - B_{21} \cdot (B_{11})^{-1} \cdot B_{12} \geq 0;$$

or, that is equivalent,

$$B_{22} \geq 0 \quad and \quad \triangle_2^1(B_2) = B_{11} - B_{12} \cdot (B_{22})^{-1} \cdot B_{21} \geq 0.$$

The proof follows from the verified identity

$$\langle B_2 h, h \rangle = \| B_{11}^{1/2} h_1 + B_{11}^{-1/2} B_{12} h_2 \|^2 + \langle \widetilde{\triangle_2^1} h_2, h_2 \rangle, \quad h = (h_1, h_2) \in H_1 \times H_2,$$

see, e.g. ([6], Vol. I, Ch. I, 1.3). Note also that, in accordance with ([4], Ch. II, Lemma 3.21), invertibility condition for B_{11} can be replaced by the considerably more general condition of invertibility of $B_{11}\big|_{\overline{ran\, B_{11}}}$.

Let's pass to general case. Once again, consider now a linear continuous operator B_n acting in product of n separable real Hilbert spaces $H = H_1 \times \ldots \times H_n$ and defined by operator matrix $(B_{ij})_{i,j=1}^n$. Introduce *Schur operators of the second kind*.

Definition 3.2. Let's divide the matrix B_n to four units: $\widetilde{B}_n^{11} = B_{11}$, $\widetilde{B}_n^{22}$ is its adjacent minor of the order $(n-1) \times (n-1)$, $\widetilde{B}_n^{12}, \widetilde{B}_n^{21}$ are corresponding row and column of the matrix B_n, having size $1 \times (n-1)$ and $(n-1) \times 1$, respectively. Suppose that $\widetilde{B}_n^{11} = B_{11}$ is continuously invertible. Let's introduce now two types of *Schur operators of the second kind*:

$$\widetilde{\triangle_1^1}(B_n) = \widetilde{B}_n^{11}; \quad \widetilde{\triangle_1^2}(B_n) = \widetilde{B}_n^{22} - \widetilde{B}_n^{21} \cdot (\widetilde{B}_n^{11})^{-1} \cdot \widetilde{B}_n^{12}.$$

Applying repeatedly second kind Schur operators to $B_n = (B_{ij})_{i,j=1}^n$ it is not difficult, by analogy with the proof of Theorem 2.1, to generalize Theorem 3.1 for the case of an arbitrary n.

Theorem 3.3. *Let H_i $(i = \overline{1,n})$ be separable real Hilbert spaces, $H = H_1 \times \ldots \times H_n$, $B_n = (B_{ij})_{i,j=1}^n$ be a linear continuous self-adjoint operator in H, in there $B_{ij} \in \mathcal{L}(H_j, H_i)$; $i, j = \overline{1,n}$. Suppose moreover that the all operators*

$$\widetilde{\triangle_1^1}(\widetilde{\triangle_1^2})^k(B_n) \quad (k = \overline{0, n-2}), \tag{6}$$

are continuously invertible. Then B_n is nonnegative if and only if the following system of nonnegativity inequalities is fulfilled:

$$\widetilde{\triangle_1^1}(\widetilde{\triangle_1^2})^k(B_n) \geq 0 \quad (k = \overline{0, n-2}); \quad (\widetilde{\triangle_1^2})^{n-1}(B_n) \geq 0. \tag{7}$$

Proof. Let's divide the matrix $B_n = (B_{ij})_{i,j=1}^n$ to four units, according to Definition 3.2, $B_n = (\widetilde{B}_n^{ij})_{i,j=1}^2$. Let $\tilde{H}^1 = H_1$, $\tilde{H}^2 = H_2 \times \ldots \times H_n$. Then it can consider B_n as an operator matrix acting in $\tilde{H}^1 \times \tilde{H}^2$, in there $\widetilde{B}_n^{ij} \in \mathcal{L}(\tilde{H}^j, \tilde{H}^i)$; $i, j = 1, 2$. Under assumptions

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of continuous invertibility of $\widetilde{B}_n^{11} = B_{11} = \widetilde{\triangle}_1^1(B_n)$ and self-adjointness of B_n, by virtue of Theorem 3.1, B_n is nonnegative if and only if the following inequalities

$$\widetilde{\triangle}_1^1(B_n) \geq 0, \qquad \widetilde{\triangle}_1^2(B_n) = \tilde{B}_n^{22} - \tilde{B}_n^{21} \cdot (\tilde{B}_n^{11})^{-1} \cdot \tilde{B}_n^{12} \geq 0$$

hold true. Again, dividing the operator matrix $\widetilde{\triangle}_1^2(B_n)$ of the size $(n-1) \times (n-1)$ according to Definition 3.2, it can apply Theorem 3.1 now to the operator $\widetilde{\triangle}_1^2(B_n)$. Here self-adjointness B_n implies one for $\widetilde{\triangle}_1^2(B_n)$ and continuous invertibility of $\widetilde{\triangle}_1^1(\widetilde{\triangle}_1^2(B_n))$ follows from (6). Hence, we obtain the following necessary and sufficient nonnegativity conditions for $\widetilde{\triangle}_1^2(B_n)$:

$$\widetilde{\triangle}_1^1\widetilde{\triangle}_1^2(B_n) \geq 0, \quad (\widetilde{\triangle}_1^2)^2(B_n) \geq 0.$$

Prolonging by induction the construction and reducing the matrix size up to $(n-p) \times (n-p)$ under the p-th inductive steps, it leads after n inductive steps to the system (7). $\qquad\square$

Remark 3.4. Note that general system of conditions in Theorem 3.3 consists of n nonnegativity inequalities (7) and $n-1$ invertibility conditions (6).

Finally, Theorem 3.3 can be formulated as nonnegativity condition for the quadratic forms acting in a product of Hilbert spaces.

Theorem 3.5. *Let H_i ($i = \overline{1,n}$) be separable real Hilbert spaces, $H = H_1 \times ... \times H_n$, φ is continuous quadratic form on H associated with a self-adjoint linear continuous operator $B_n = (B_{ij})_{i,j=1}^n$, i.e. $\varphi(h) = \langle B_n h, h \rangle$ and the conditions (6) are fulfilled. Then the form φ is nonnegative on H if and only if the inequalities (7) hold true.*

Finally, consider in this item also the partial case $n = 3$.

Example 3.6. Introduce (in case of B_3 with an invertible B_{jj}) generalized second order Schur complements

$$\triangle_j^{ik} = B_{ik} - B_{ij} \cdot B_{jj}^{-1} \cdot B_{jk} \quad (i,j,k = 1,2,3).$$

Then direct calculations show that the invertibility conditions (6) take form of invertibility of the operator

$$B_{11} \quad \text{and} \quad \triangle_1^{22} = B_{22} - B_{21} \cdot (B_{11})^{-1} \cdot B_{12},$$

and the nonnegativity conditions (7) take form of

$$B_{11} \gg 0; \quad \triangle_1^{22} \gg 0; \quad \text{and} \quad \triangle_1^{33} - \triangle_1^{32} \cdot (\triangle_1^{22})^{-1} \cdot \triangle_1^{23} \gg 0.$$

4. Application to functional extrema in product of Hilbert spaces

Remind a well known ([14], Ch. I, Th. 8.3.3) classical sufficient condition of a local extremum of functional acting in Hilbert space.

Theorem 4.1. *Let H be a real Hilbert space and a functional $\Phi : H \to \mathbb{R}$ is twice Fréchet differentiable at a point $y \in H$. Suppose that $\Phi'(y) = 0$ and $\Phi''(y) \gg 0$ ($\Phi''(y) \ll 0$, respectively). Then Φ has a strong local minimum (strong local maximum, respectively) at y.*

Since, in view of Young theorem ([14], Ch. I, Th. 5.1.1), bilinear form $\Phi''(y)$ is symmetric then it is associated with self-adjoint linear operator, namely Hessian $\mathcal{H}_n(\Phi) = (\partial_{ij}\Phi(y))_{i,j=1}^n$. Applying to $\mathcal{H}$ the positive definiteness criterion (Theorem 2.9) together with Theorem 4.1 immediately leads to the following sufficient extrema condition.

Theorem 4.2. *Let H_i ($i = \overline{1,n}$) be real separable Hilbert spaces, $H = H_1 \times \ldots \times H_n$, and a functional $\Phi : H \to \mathbb{R}$ is twice Fréchet differentiable at a point $y = (y_1, \ldots, y_n) \in H$. Suppose that*

i) $\partial_1\Phi(y) = \partial_2\Phi(y) = \ldots = \partial_n\Phi(y) = 0$;

ii) $\triangle_{j_m}^{i_m} \ldots \triangle_{j_2}^{i_2} \triangle_{j_1}^{i_1} \mathcal{H}_n(\Phi)(y) \gg 0$ for $i_l, j_l = 1, 2$ $(l = \overline{1,m})$, $m = [\log_2 n]_+$.

Then Φ has a strong local minimum at y.

Remind that by (3), the precise number of inequalities in the condition (ii) is $V_n = 2^k \cdot (3n - 2^{k+1})$ as $2^k \leq n \leq 2^{k+1}$ and $n^2 \leq V_n \leq (n+1)^2$. Let's distinguish the important in practice cases of two and three variables.

Theorem 4.3. *Let a functional $\Phi : H_1 \times H_2 \to \mathbb{R}$ be twice Fréchet differentiable at a point $(y_1, y_2) \in H_1 \times H_2$. Suppose that*

i) $\partial_1 \Phi(y) = 0$; $\partial_2 \Phi(y) = 0$;

ii) $\partial_{11} \Phi(y) \gg 0$; $\partial_{22} \Phi(y) \gg 0$;

iii) $\triangle_2^1 = (\partial_{11}\Phi - \partial_{12}\Phi \cdot \partial_{22}^{-1}\Phi \cdot \partial_{21}\Phi)(y) \gg 0$;
$\triangle_1^2 = (\partial_{22}\Phi - \partial_{21}\Phi \cdot \partial_{11}^{-1}\Phi \cdot \partial_{12}\Phi)(y) \gg 0$.

Then Φ has a strong local minimum at y.

Remark 4.4. Note, first of all that, in case of the commuting second partial derivatives $\partial_{ij}\Phi(y)$, conditions (ii)-(iii) take classical form

$$\partial_{11}\Phi(y) \gg 0; \quad (\partial_{11}\Phi \cdot \partial_{22}\Phi - \partial_{12}\Phi \cdot \partial_{21}\Phi)(y) \gg 0,$$

to within substitution $1 \leftrightarrow 2$.

Next, in case of a strong local maximum, the *all signs* in the inequalities from (ii)-(iii) must be replaced.

At last, in [15] the result of Theorem 4.3 was used to obtain sufficient conditions of the *compact extremum* (see def. in ([15], Def. 2.1)) of variational functional

$$\Phi(y) = \int_a^b f(x, y, y')dx \quad (y \in W_2^1([a; b], H)).$$

In the case of extremal $y(\cdot) \in W_2^2$ and $f \in C^2([a; b] \times H \times H)$ these sufficient conditions of a strong compact minimum take form

$$\partial_{11} f(x, y, y') \gg 0; \quad \partial_{22} f(x, y, y') \gg 0;$$

$$(\partial_{11} f - \partial_{12} f \cdot \partial_{22}^{-1} f \cdot \partial_{21} f)(x, y, y') \gg 0; \quad (\partial_{22} f - \partial_{21} f \cdot \partial_{11}^{-1} f \cdot \partial_{12} f)(x, y, y') \gg 0$$

for all $x \in [a; b]$. Let's emphasize that, in most eases (see [16]) the compact extremum of Φ is not local one.

In case of three variables, taking into account the result of Example 2.7, Theorem 4.2 takes form of

Theorem 4.5. *Let a functional $\Phi : H_1 \times H_2 \times H_3 \to \mathbb{R}$ be twice Fréchet differentiable at a point $(y_1, y_2, y_3) \in H_1 \times H_2 \times H_3$. Suppose that*

i) $\partial_1 \Phi(y) = 0$; $\partial_2 \Phi(y) = 0$; $\partial_3 \Phi(y) = 0$;

ii) $\partial_{11} \Phi(y) \gg 0$; $\partial_{22} \Phi(y) \gg 0$; $\partial_{33} \Phi(y) \gg 0$;

iii) $\triangle_{12}^2 = (\partial_{22}\Phi - \partial_{21}\Phi \cdot \partial_{11}^{-1}\Phi \cdot \partial_{12}\Phi)(y) \gg 0$; $\triangle_{32}^2 = (\partial_{22}\Phi - \partial_{23}\Phi \cdot \partial_{33}^{-1}\Phi \cdot \partial_{32}\Phi)(y) \gg 0$;
$\triangle_{13}^3 = (\partial_{33}\Phi - \partial_{31}\Phi \cdot \partial_{11}^{-1}\Phi \cdot \partial_{13}\Phi)(y) \gg 0$; $\triangle_{23}^3 = (\partial_{33}\Phi - \partial_{32}\Phi \cdot \partial_{22}^{-1}\Phi \cdot \partial_{23}\Phi)(y) \gg 0$;

iv) $\triangle_{123}^{23} = \triangle_{12}^2 - \triangle_{13}^3 \cdot (\triangle_{13}^3)^{-1} \cdot \triangle_{12}^3 \gg 0$; $\triangle_{132}^{32} = \triangle_{13}^3 - \triangle_{12}^3 \cdot (\triangle_{12}^2)^{-1} \cdot \triangle_{13}^2 \gg 0$;

v) $\partial_{11}\Phi(y) \gg (\partial_{12}\Phi \cdot \partial_{22}^-\Phi \cdot \partial_{21}\Phi + \partial_{13}\Phi \cdot \partial_{32}^-\Phi \cdot \partial_{21}\Phi + \partial_{12}\Phi \cdot \partial_{23}^-\Phi \cdot \partial_{31}\Phi + $
$+ \partial_{13}\Phi \cdot \partial_{33}^-\Phi \cdot \partial_{31}\Phi)(y)$.

Here we denote by $\partial_{ij}^-\Phi(y)$ corresponding elements of the inverse matrix to $(\partial_{ij}\Phi(y))_{i,j=2}^3$. In this case,

vi) $\partial_{22}^-\Phi(y) = (\partial_{22}^{-1}\Phi \cdot (I_{22} + \partial_{23}\Phi \cdot (\triangle_{23}^3)^{-1} \cdot \partial_{32}\Phi \cdot \partial_{22}^{-1}\Phi))(y)$;
$\partial_{33}^-\Phi(y) = (\partial_{33}^{-1}\Phi \cdot (I_{33} + \partial_{32}\Phi \cdot (\triangle_{32}^2)^{-1} \cdot \partial_{23}\Phi \cdot \partial_{33}^{-1}\Phi))(y)$;
$\partial_{23}^-\Phi(y) = -((\triangle_{32}^2)^{-1} \cdot \partial_{23}\Phi \cdot \partial_{33}^{-1}\Phi)(y)$;
$\partial_{32}^-\Phi(y) = -((\triangle_{23}^3)^{-1} \cdot \partial_{32}\Phi \cdot \partial_{22}^{-1}\Phi)(y)$.

Then Φ has a strong local minimum at y.

Pass to the necessary conditions. In analogous way, remind first a classical second order necessary condition of a local extremum of functional acting in Banach space ([14], Ch. I, Th. 8.2.1).

Theorem 4.6. *Let E be a real Banach space and a functional $\Phi : E \to \mathbb{R}$ is twice Fréchet differentiable at a point $y \in E$ and has a local minimum at y. Then not only $\Phi'(y) = 0$ but also $\Phi''(y) \geq 0$.*

Quite analogously with the preceding case, with the help of Theorems 3.3 and 4.6 we obtain the following second order necessary extremal condition.

Theorem 4.7. *Let H_i $(i = \overline{1, n})$ be real separable Hilbert spaces, $H = H_1 \times \ldots \times H_n$, and a functional $\Phi : H \to \mathbb{R}$ is twice Fréchet differentiable and has a local minimum at a point $y = (y_1, \ldots, y_n) \in H$. Suppose also that the all operators $\widetilde{\triangle}_1^1(\widetilde{\triangle}_1^2)^k \mathcal{H}_n(\Phi)$ $(k = \overline{0, n-2})$ are continuously invertible. Then not only $\partial_i \Phi(y) = 0$ $(i = \overline{1, n})$ but also the inequalities*

$$\widetilde{\triangle}_1^1(\widetilde{\triangle}_1^2)^k \mathcal{H}_n(\Phi) \geq 0 \ (k = \overline{0, n-2}); \quad (\widetilde{\triangle}_1^2)^{n-1} \mathcal{H}_n(\Phi) \geq 0$$

are valid.

In case of three variables Theorem 4.7, using Example 3.6, takes form of

Theorem 4.8. *Let a functional $\Phi : H_1 \times H_2 \times H_3 \to \mathbb{R}$ is twice Fréchet differentiable at a point $y \in H_1 \times H_2 \times H_3$ and has a local minimum at y. Suppose also that the operators $\partial_{11}\Phi(y)$ and $(\partial_{22}\Phi - \partial_{21}\Phi \cdot \partial_{11}^{-1}\Phi \cdot \partial_{12}\Phi)(y)$ are continuously invertible. Then not only $\partial_i \Phi(y) = 0$ $(i = 1, 2, 3)$ but also the inequalities*

i) $\partial_{11}\Phi(y) \geq 0$;

ii) $(\partial_{22}\Phi - \partial_{21}\Phi \cdot \partial_{11}^{-1}\Phi \cdot \partial_{12}\Phi)(y) \geq 0$;

iii) $(\partial_{33}\Phi - \partial_{31}\Phi \cdot \partial_{11}^{-1}\Phi \cdot \partial_{13}\Phi)(y) - (\partial_{32}\Phi - \partial_{31}\Phi \cdot \partial_{11}^{-1}\Phi \cdot \partial_{12}\Phi)(y) \times$
$\times (\partial_{22}\Phi - \partial_{21}\Phi \cdot \partial_{11}^{-1}\Phi \cdot \partial_{12}\Phi)^{-1}(y) \cdot (\partial_{23}\Phi - \partial_{21}\Phi \cdot \partial_{11}^{-1}\Phi \cdot \partial_{13}\Phi)(y) \geq 0$.

are valid.

Concluding this item, let's consider an example of finding functional extremum in case of three variables.

Example 4.9. Let $H_i = l_2$ $(i = 1, 2, 3)$ and a functional $\Phi : H_1 \times H_2 \times H_3 \to \mathbb{R}$ has form

$$\Phi(x, y, z) = \|x\|^2 + \|y\|^2 + \|z\|^2 + (x_1^2 + z_1^2 + x_1 x_2 + y_1 y_2 + z_1 z_2 + x_1 z_1).$$

Then $\nabla_x \Phi = (4x_1 + x_2 + z_1, 2x_2 + x_1, 2x_3, 2x_4, \ldots)$, $\nabla_y \Phi = (2y_1 + y_2, 2y_2 + y_1, 2y_3, 2y_4, \ldots)$, $\nabla_z \Phi = (4z_1 + z_2 + x_1, 2z_2 + z_1, 2z_3, 2z_4, \ldots)$, whence Φ has a critical point at zero and the second partial derivatives are

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial^2 \Phi}{\partial z^2} = \begin{pmatrix} 4 & 1 & 0 & 0 & \ldots & 0 & \ldots \\ 1 & 2 & 0 & 0 & \ldots & 0 & \ldots \\ 0 & 0 & 2 & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & 2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix},$$

$$\frac{\partial^2 \Phi}{\partial y^2} = \begin{pmatrix} 2 & 1 & 0 & 0 & \ldots & 0 & \ldots \\ 1 & 2 & 0 & 0 & \ldots & 0 & \ldots \\ 0 & 0 & 2 & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & 2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}, \quad \frac{\partial^2 \Phi}{\partial x \partial z} = \frac{\partial^2 \Phi}{\partial z \partial x} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & \ldots \\ 0 & 0 & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}, \quad (8)$$

$$\partial^2\Phi/\partial x\partial y = \partial^2\Phi/\partial y\partial x = \partial^2\Phi/\partial y\partial z = \partial^2\Phi/\partial z\partial x = 0.$$

Hence, the Hessian $\mathcal{H}_3(\Phi)$ satisfies conditions of Example 2.8 (2) and therefore we can restrict oneself by testing only five positive definiteness conditions. Note also that $(\partial^2\Phi/\partial x^2)$ and $(\partial^2\Phi/\partial y^2)$ not commute. So,

1) It follows immediately from (8) that

$$\frac{\partial^2\Phi}{\partial x^2} \gg 0, \quad \frac{\partial^2\Phi}{\partial y^2} \gg 0, \quad \frac{\partial^2\Phi}{\partial z^2} \gg 0.$$

2) The inverse matrix to $(\partial^2\Phi/\partial x^2)$ also can be easily calculated from (8):

$$\left(\frac{\partial^2\Phi}{\partial x^2}\right)^{-1} = \left(\frac{\partial^2\Phi}{\partial z^2}\right)^{-1} = \begin{pmatrix} 2/7 & -1/7 & 0 & 0 & \dots & 0 & \dots \\ -1/7 & 4/7 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & 1/2 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & 1/2 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}.$$

3) Hence, a direct calculation shows that the matrix

$$\frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial x\partial z} \cdot \left(\frac{\partial^2\Phi}{\partial z^2}\right)^{-1} \cdot \frac{\partial^2\Phi}{\partial z\partial x} = \frac{\partial^2\Phi}{\partial z^2} - \frac{\partial^2\Phi}{\partial z\partial x} \cdot \left(\frac{\partial^2\Phi}{\partial x^2}\right)^{-1} \cdot \frac{\partial^2\Phi}{\partial x\partial z} =$$

$$= \begin{pmatrix} 26/7 & 1 & 0 & 0 & \dots & 0 & \dots \\ 1 & 2 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}$$

is positive definite. So, Φ has a strong local minimum at zero.

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Department of Algebra and Functional Analysis

Taurida National V.I. Vernadskii University

Vernadskii ave., 4,

Simferopol 95007

Ukraine

E-mail: old@crimea.edu

katboz@mail.ru