TANGENT BUNDLES TO REGULAR BASIC SETS IN HYPERBOLIC DYNAMICS

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ABSTRACT. Given a locally maximal compact invariant hyperbolic set Λ for a C^1 flow or diffeomorphism on a Riemann manifold with C^1 unstable laminations, we construct an invariant continuous bundle of tangent vectors to local unstable manifolds that locally approximates Λ in a certain way.

1 INTRODUCTION

Let M be a C^1 complete (not necessarily compact) Riemann manifold M and let Λ be a basic set for a C^1 flow $\phi_t : M \longrightarrow M$ or a C^1 diffeomorphism $f : U \longrightarrow f(U) \subset M$, where U is an open neighbourhood of Λ in M. Let $E^u(x)$ and $E^s(x)$ ($x \in \Lambda$) be the tangent spaces to the local stable and unstable manifolds $W_\epsilon^s(x)$ and $W_\epsilon^u(x)$ of size $\epsilon > 0$, respectively (see Sect. 2). We say that ϕ_t has C^1 unstable laminations over Λ if the map $\Lambda \ni x \mapsto E^u(x)$ is C^1, i.e. at each $x \in \Lambda$ this map has a linearization that depends continuously on x.

Given $x \in \Lambda$, let $k_x \geq 1$ be the minimal integer such that there exist $\delta_x \in (0, \epsilon)$ and a k_x-dimensional C^1 submanifold S_x of $W_\epsilon^u(x)$ such that $\Lambda \cap W_{\delta_x}^u(x) \subset S_x$. We will then say that S_x is a u-submanifold of *minimal dimension at x* and *size δ_x*. In general S_x is not (locally) unique, however it turns out that, under certain regularity conditions, its tangent space $T_x S_x$ is uniquely determined, invariant and continuous:

Theorem 1. *Assume that Λ is a basic set for the flow ϕ_t (or the diffeomorphism f) and the local unstable laminations over Λ are C^1. Then there exists an integer $k = k^u \geq 1$ and a continuous $d\phi_t$-invariant (or df-invariant, respectively) distribution $E_\Lambda^u(x)$, $x \in \Lambda$, of k-dimensional linear spaces such that for any $x \in \Lambda$ we have $k_x = k$, $E_\Lambda^u(x) \subset E^u(x)$ and $T_x S_x = E_\Lambda^u(x)$ for any u-submanifold S_x of minimal dimension at x. Moreover, the distribution E_Λ^u is invariant under the linearizations of the local stable holonomy maps, as well.*

We refer the reader to Sect. 2 for the definition of the local stable holonomy maps.

In a similar way one defines s-submanifolds of minimal dimension at $x \in \Lambda$, and a result similar to the above holds for these.

Remark 1. It is easy to see that all vectors in $E^u(x)$ that are tangent to the basic set Λ belong to $E_\Lambda^u(x)$. However, in general the span of these tangent vectors could be a proper subspace of $E_\Lambda^u(x)$ – see Example 2 in Sect. 4 below.

The motivation for the introduction of the bundles $E_\Lambda^{u,s}$ comes from [St2] which deals with Dolgopyat type spectral estimates ([D]) for Axiom A flows on basic sets, where one has to take into account some fine geometric properties of basic sets. As it turns out there, in certain cases when the basic set Λ is 'relatively small', considering the tangent maps $d\phi_t$ over the whole bundle E^u (or E^s) does not say enough about the dynamics of the flow near Λ. So, it appears, restricting $d\phi_t$ over the smallest possible invariant subbbundle of E^u would be beneficial[1], and it is natural to believe that this might be the case in some other situations, as well, particularly when the dimension of the spaces $E^u(x)$ (or $E^s(x)$) is large.

In Sect. 3 below we prove Theorem 1 for flows; for diffeomorphisms the argument is very similar. Some basic definitions are given in Sect. 2, while Sect. 4 contains two examples.

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[1] Although in [St2] we found a way around the problem by different means.

2 PRELIMINARIES

Let M be a C^1 complete (not necessarily compact) Riemann manifold, and $\phi_t : M \longrightarrow M$ $(t \in \mathbb{R})$ a C^1 flow on M. A ϕ_t-invariant closed subset Λ of M is called *hyperbolic* if Λ contains no fixed points and there exist constants $C > 0$ and $0 < \lambda < 1$ and a $d\phi_t$-invariant decomposition $T_x M = E^0(x) \oplus E^u(x) \oplus E^s(x)$ of $T_x M$ $(x \in \Lambda)$ into a direct sum of non-zero linear subspaces, where $E^0(x)$ is the one-dimensional subspace determined by the direction of the flow at x, $\|d\phi_t(u)\| \leq C\,\lambda^t\,\|u\|$ for all $u \in E^s(x)$ and $t \geq 0$, and $\|d\phi_t(u)\| \leq C\,\lambda^{-t}\,\|u\|$ for all $u \in E^u(x)$ and $t \leq 0$. Here $\|\cdot\|$ is the *norm* on $T_x M$ determined by the Riemann metric on M.

A non-empty compact ϕ_t-invariant hyperbolic subset Λ of M which is not a single closed orbit is called a *basic set* for ϕ_t if ϕ_t is transitive on Λ and Λ is locally maximal, i.e. there exists an open neighbourhood V of Λ in M such that $\Lambda = \cap_{t \in \mathbb{R}} \phi_t(V)$.

From now on we will assume that Λ is a basic set for the flow ϕ_t. For $x \in \Lambda$ and a sufficiently small $\epsilon > 0$ let

$$W_\epsilon^s(x) = \{ y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \geq 0 \,,\, d(\phi_t(x), \phi_t(y)) \to_{t \to \infty} 0 \}\,,$$

$$W_\epsilon^u(x) = \{ y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \leq 0 \,,\, d(\phi_t(x), \phi_t(y)) \to_{t \to -\infty} 0 \}$$

be the (strong) *stable* and *unstable manifolds* of size ϵ. Then $E^u(x) = T_x W_\epsilon^u(x)$ and $E^s(x) = T_x W_\epsilon^s(x)$.

Assuming $\epsilon_1 \in (0, \epsilon_0]$ is sufficiently small, for any $x \in \Lambda$ and $y \in \Lambda \cap W_{\epsilon_1}^s(x)$ the local *holonomy map* $\mathcal{H}_x^y : \Lambda \cap W_{\epsilon_1}^u(x) \longrightarrow \Lambda \cap W_{\epsilon_0}^u(y)$ along stable laminations is well-defined and uniformly Hölder continuous (see e.g. [PSW]). Recall that the map $\mathcal{H}_x^y$ is defined as follows. Given $z \in \Lambda \cap W_{\epsilon_1}^u(x)$, there exist unique $z' \in W_{\epsilon_1}^s(z)$ and $y' \in W_{\epsilon_0}^u(y)$ such that $\phi_t(z') = y'$ for some $t \in \mathbb{R}$, $|t| \leq \epsilon_0$. Then we set $\mathcal{H}_x^y(z) = y'$. Under the additional condition that the unstable laminations are C^1, the maps $\mathcal{H}_x^y$ are C^1 as well (see e.g. Fact (2) on p. 647 in [Ha]). That is, for each $z \in \Lambda \cap W_{\epsilon_1}^u(x)$ the map $\mathcal{H}_x^y$ has a *linearization* $L_x^y(z) : E^u(z) \longrightarrow E^u(\mathcal{H}_x^y(z))$ at $z \in \Lambda \cap W_{\epsilon_1}^u(x)$ and $\|L_x^y(z)\| \leq C$ for some constant $C > 0$ independent of x, y and z.

3 PROOF OF THEOREM 1

So, assume that $\phi_t : M \longrightarrow M$ $(t \in \mathbb{R})$ is a C^1 flow on M and Λ is a basic set for ϕ_t.

By Whitney's Theorem (see e.g. [M]), the stable holonomy map $\mathcal{H}_x^y$ has a C^1 extension $\widetilde{\mathcal{H}}_x^y : W_{\epsilon_1}^u(x) \longrightarrow W_{\epsilon_0}^u(y)$ with $\|d\widetilde{\mathcal{H}}_x^y\| \leq C$ on $W_{\epsilon_1}^u(x)$. Since $\|(d\widetilde{\mathcal{H}}_x^y(x))^{-1}\| = \|(d\mathcal{H}_x^y(x))^{-1}\| = \|d\mathcal{H}_y^x(y)\| \leq C$, it follows from the Inverse Function Theorem that there exists $\epsilon_2 \in (0, \epsilon_1]$ such that

$$\widetilde{\mathcal{H}}_x^y : W_{\epsilon_2}^u(x) \longrightarrow \widetilde{\mathcal{H}}_x^y(W_{\epsilon_2}^u(x)) \subset W_{\epsilon_0}^u(y)$$

is a diffeomorphism for all $x \in \Lambda$ and all $y \in \Lambda \cap W_{\epsilon_1}^s(x)$.

Let k_x $(x \in \Lambda)$ be the integers introduced in Sect. 1. Set $k = \min_{x \in \Lambda} k_x$, and fix an arbitrary $x_0 \in \Lambda$ with $k_{x_0} = k$.

Lemma 1. *$k_x = k$ for every $x \in \Lambda$. Moreover, there exist $\delta > 0$ and for every $x \in \Lambda$ a k-dimensional C^1 submanifold S_x of $W_\delta^u(x)$ such that $\Lambda \cap W_\delta^u(x) \subset S_x$.*

Proof. First, notice that if S_x is a C^1 submanifold of $W_\delta^u(x)$ of minimal dimension such that $\Lambda \cap W_\delta^u(x) \subset S_x$ for some $x \in \Lambda$, then for any $y \in \Lambda \cap W_{\epsilon_1}^s(x)$, $\widetilde{\mathcal{H}}_x^y(S_x)$ is a C^1 submanifold of $W_{\epsilon_0}^u(y)$ with $\Lambda \cap W_{\delta/C}^u(y) \subset \widetilde{\mathcal{H}}_x^y(S_x)$, so $k_y \leq k_x$. Similarly, $k_x \leq k_y$, so $k_x = k_y$. Moreover, we can choose the same $\delta > 0$ for all $y \in \Lambda \cap W_{\epsilon_1}^s(x)$.

Let $S = S_{x_0}$ be a u-submanifold of $W^u_\delta(x_0)$ of minimal dimension such that $\Lambda \cap W^u_\delta(x_0) \subset S$ for some $\delta > 0$. Given $z \in \Lambda$, for every sufficiently large $t > 0$ we have $\phi_t(\Lambda \cap W^u_{\delta/2}(x_0)) \cap W^s_{\epsilon_1}(z) \neq \emptyset$ (see e.g. [KH]). Take $t > 0$ with this property so large that ϕ_t is expanding on $W^u_{\delta/2}(x_0)$, i.e. $d(\phi_t(p), \phi_t(q)) \geq d(x, y)$ for all $p, q \in W^u_{\delta/2}(x_0)$. Let $z' = \phi_t(x) \in W^s_{\epsilon_1}(z)$ for some $x \in \Lambda \cap W^u_{\delta/2}(x_0)$. Since $S_x = S \cap W^u_{\delta/2}(x) \subset S \cap W^u_\delta(x_0)$, S_x contains $\Lambda \cap W^u_{\delta/2}(x)$ and $\dim(S_x) = k \leq k_x$, it follows that $k_x = k$ and S_x is a u-submanifold of minimal dimension at x and size δ.

It now follows that $k_{z'} = k$ and, since ϕ_t is expanding on $W^u_{\delta/2}(x)$, there exists a u-submanifold of minimal dimension at z' of size $\delta/2$. The remark in the beginning of the proof now shows that $k_z = k_{z'} = k$, and there exists a u-submanifold of minimal dimension at z of size $\delta/(2C)$. ∎

Lemma 2. *For any $x \in \Lambda$ and any two u-submanifolds S and S' of minimal dimension at x we have $T_x S = T_x S'$.*

Proof. Let S and S' be u-submanifolds of minimal dimension k at x for some $x \in \Lambda$. Take $\delta > 0$ so small that $\Lambda \cap W^u_\delta(x) \subset S$ and $\Lambda \cap W^u_\delta(x) \subset S'$.

Assume that $T_x S \setminus T_x S' \neq \emptyset$ and fix an arbitrary $v \in T_x S \setminus T_x S'$. Then we must have $n = \dim(W^u_\delta(x)) > k$. Choosing a smaller $\delta > 0$ if necessary and using an appropriate submanifold chart for S' about x in $W^u_\delta(x)$ one constructs a C^1 submanifold S'' of $W^u_\delta(x)$ such that $\dim(S'') = n - 1$, $S' \subset S''$ and $v \notin T_x S''$. Then $T_x S + T_x S'' = E^u(x) = T_x(W^u_\delta(x))$, so the submanifolds S and S'' are transversal at x (see e.g. [GP]). Consequently, there exists $\delta' \in (0, \delta]$ so that $S''' = S \cap S'' \cap W^u_{\delta'}(x)$ is a submanifold of $W^u_{\delta'}(x)$ of dimension at most $k-1$. Since $\Lambda \cap W^u_{\delta'}(x) \subset S'''$, this is a contradiction with the minimality of $k = k_x$.

Thus we must have $T_x S \subset T_x S'$ and similarly $T_x S' \subset T_x S$. ∎

Proof of Theorem 1. For any $x \in \Lambda$ define $E^u_\Lambda(x) = T_x S$, where S is an arbitrary u-submanifold of minimal dimension at x. By Lemma 2, the definition is correct. Moreover, $E^u_\Lambda(x) \subset E^u(x)$ and $T_x S_x = E^u_\Lambda(x)$ for any u-submanifold S_x of minimal dimension at x. The latter implies $E^u_\Lambda(\phi_t(x)) = d\phi_t(x) \cdot E^u_\Lambda(x)$ for any $t \in \mathbb{R}$ and any $x \in \Lambda$.

It remains to show that the map $\Phi : \Lambda \ni x \mapsto E^u_\Lambda(x)$ is continuous. Fix for a moment $z \in \Lambda$ and let S_z be an u-submanifold of minimal dimension at z. Then for any $t \in \mathbb{R}$ with a sufficiently small $|t|$, $\phi_t(S_z)$ is an u-submanifold of minimal dimension at $z' = \phi_t(z)$, so $E^u_\Lambda(z') = T_{z'}(\phi_t(S_z)) = d\phi_t(z) \cdot (T_z S_z)$ depends continuously on $z' = \phi_t(z)$. That is, Φ is continuous along the orbit of z.

Next, choosing a sufficiently small $\delta > 0$, for $z' \in W^u_\delta(z) \cap \Lambda$ we have $z' \in S_z$, so $E^u_\Lambda(z') = T_{z'} S_z$, and therefore $\Phi(z') = E^u_\Lambda(z')$ depends continuously on $z' \in W^u_\delta(z) \cap \Lambda$.

Finally, for $z' \in W^s_{\epsilon_1}(z) \cap \Lambda$, $\widetilde{\mathcal{H}}^{z'}_z(S_z)$ is an u-submanifold of minimal dimension at z', so $E^u_\Lambda(z') = d\widetilde{\mathcal{H}}^{z'}_z(z) \cdot (E^u_\Lambda(z))$, and therefore $\Phi(z') = E^u_\Lambda(z')$ depends continuously on $z' \in W^s_{\epsilon_1}(z) \cap \Lambda$.

The above and the continuity of the local product near z implies that Φ is continuous at z. ∎

4 EXAMPLES

We use two very well known kind of flows to give: 1) a non-trivial example where $k^u < \dim(E^u)$, and 2) an example where the span of the tangent vectors in $E^u(z)$ to Λ at z form a proper subspace of $E^u_\Lambda(z)$ (for any $z \in \Lambda$). We use open billiard flows for the first example and geodesic flows on hyperbolic manifolds for the second. In higher dimensions both kind of flows should provide a great variety of non-wandering sets with complicated geometric (and metric) structures (see e.g [Ka]), however it appears they would not be easy to investigate. It is quite possible that one could use other models that are easier to describe to get similar examples.

There are various possible ways to define tangent vectors to a subset of a manifold. Here we use the most straightforward one. Given $z \in \Lambda$, let $\exp_z^u : E^u(z) \longrightarrow W_{\epsilon_0}^u(z)$ and $\exp_z^s : E^s(z) \longrightarrow W_{\epsilon_0}^s(z)$ be the corresponding *exponential maps*. A vector $b \in E^u(z) \setminus \{0\}$ is called *tangent to* Λ at z if there exist infinite sequences $\{v^{(m)}\} \subset E^u(z)$ and $\{t_m\} \subset \mathbb{R} \setminus \{0\}$ such that $\exp_z^u(t_m \, v^{(m)}) \in \Lambda \cap W_{\epsilon}^u(z)$ for all m, $v^{(m)} \to b$ and $t_m \to 0$ as $m \to \infty$. It is easy to see that a vector $b \in E^u(z) \setminus \{0\}$ is tangent to Λ at z iff there exists a C^1 curve $z(t)$, $0 \leq t \leq a$, in $W_{\epsilon}^u(z)$ for some $a > 0$ with $z(0) = z$, $\dot{z}(0) = b$, and $z(t) \in \Lambda$ for arbitrarily small $t > 0$. Denote by $\widehat{E}^u(z)$ the set of all non-zero tangent vectors $v \in E^u(z)$ to Λ at z. Tangent vectors to Λ in $E^s(z)$ and the set $\widehat{E}^s(z)$ are defined similarly. Clearly the sets $\widehat{E}^{u,s}(z)$ are non-empty and $d\phi_t$-invariant and $\widehat{E}^{u,s}(z) \subset E^{u,s}(z)$. However in general the span of $\widehat{E}^{u,s}(z)$ does not coincide with $E^{u,s}(z)$ – see Example 2 below.

Open Problem. Under the assumptions in Sect. 1, assume that $k_u < \dim(E^u(z))$ (or $k_s < \dim(E^s(z))$), $z \in \Lambda$. Does there always exist a C^1 submanifold M' of M of positive codimension and an open neighbourhood U of Λ in M such that $\phi_t(U \cap M') \subset M'$ for all $t \in \mathbb{R}$?

One would expect the answer to be negative, however at this stage we do not have examples to prove this. Example 1 below is non-trivial in a certain sense, and in fact might be good enough to answer the above question in the negative, however we do not go that far here.

4.1 OPEN BILLIARD FLOWS

Let K be a subset of $\mathbb{R}^n$ ($n \geq 2$) of the form $K = K_1 \cup K_2 \cup \ldots \cup K_{k_0}$, where K_i are compact strictly convex disjoint domains in $\mathbb{R}^n$ with C^r *boundaries* $\Gamma_i = \partial K_i$ ($r \geq 2$) and $k_0 \geq 3$. Set $\Omega = \overline{\mathbb{R}^n \setminus K}$ and $\Gamma = \partial K$. We assume that K satisfies the following *no-eclipse condition*: *for every pair K_i, K_j of different connected components of K the convex hull of $K_i \cup K_j$ has no common points with any other connected component of K*. With this condition, the *billiard flow* ϕ_t defined on the *cosphere bundle* $S^*(\Omega)$ in the standard way is called an *open billiard flow*. It has singularities, however its restriction to the *non-wandering set* Λ (the set of those $x \in S^*(\Omega)$ such that the trajectory $\{\phi_t(x) : t \in \mathbb{R}\}$ is bounded) has only simple discontinuities at reflection points[2]. Moreover, Λ is compact and ϕ_t is hyperbolic and transitive on Λ. Finally, it follows from [St3] that when the minimal distance between distinct connected components of K is relatively large compared to the maximal sectional curvature of ∂K the open billiard flow on Λ satisfies a certain pinching condition which implies that the (un)stable laminations over Λ are C^1.

In the following example we will use the natural symbolic coding of the open billiard. Let A be the $k_0 \times k_0$ matrix with entries $A(i,j) = 1$ if $i \neq j$ and $A(i,i) = 0$ for all i, and let Σ_A be the set of all sequences $\eta = (\eta_j)_{j=-\infty}^{\infty}$ of integer numbers $1 \leq \eta_j \leq k_0$ such that $\eta_j \neq \eta_{j+1}$ for all $j \in \mathbb{Z}$. Let $\mathrm{pr}_1 : S^*(\Omega) \longrightarrow \Omega$ be the natural *projection*. Given $\xi \in \Sigma_A$, let $(P_j(\xi))_{j=-\infty}^{\infty}$ be the successive reflection points of the unique billiard trajectory in the exterior of K such that $P_j(\xi) \in K_{\xi_j}$ for all $j \in \mathbb{Z}$ (see e.g. [St1]). Define the map $\Phi : \Sigma_A \longrightarrow \Lambda \cap S_{\partial K}^*(\Omega)$ by $\Phi(\xi) = (P_0(\xi), (P_1(\xi) - P_0(\xi))/\|P_1(\xi) - P_0(\xi)\|)$. Then Φ is a bijection such that $\Phi \circ \sigma = B \circ \Phi$, where $B : \Lambda \cap S_{\partial K}^*(\Omega) \longrightarrow \Lambda \cap S_{\partial K}^*(\Omega)$ is the *billiard ball map* from boundary to boundary, and σ is the shift map on Σ_A.

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[2]Notice that the natural projection of ϕ_t on the quotient space $S^*(\Omega)/ \sim$, where $\sim$ is the equivalence relation $(q,v) \sim (p,w)$ iff $q = p$ and $v = w$ or $q = p \in \partial K$ and v and w are symmetric with respect to $T_q(\partial K)$, is continuous. Moreover whenever both x and $\phi_t(x)$ are in the interior of $S^*(\Omega)$ and sufficiently close to Λ, the map $y \mapsto \phi_t(y)$ is smooth on a neighbourhood of x. It follows from well-known results of Sinai that Λ is a hyperbolic set for ϕ_t, and it is easily seen that Λ is the maximal compact ϕ_t-invariant subset of $S^*(\Omega)$. Moreover, it follows from the natural symbolic coding for the natural section of the flow that the periodic points are dense in Λ, and ϕ_t is transitive on Λ. Thus, Λ is a basic set for ϕ_t and the classical theory of hyperbolic flows applies (see e.g. Part 4 in [KH]).

Example 1. Assume that $n = 3$ and there exists a plane α such that each of the domains K_j is symmetric with respect to α. Setting $K' = K \cap \alpha$ and $\Omega' = \Omega \cap \alpha$, it is easy to observe that every billiard trajectory generated by a point in Λ is entirely contained in α. That is, $\Lambda = \Lambda'$, where Λ' is the non-wandering set for the open billiard flow in Ω'. Thus, $\dim(E_\Lambda^u(z)) = 1 < \dim(E^u(z)) = 2$ for any $z \in \Lambda$. This example is of course trivial, since Λ is contained in the flow-invariant submanifold $S^*(\Omega')$ of $S^*(\Omega)$.

However with a small local perturbation of the boundary ∂K of K we can get a non-trivial example. Choosing standard cartesian coordinates x, y, z in $\mathbb{R}^3$, we may assume that α is given by the equation $z = 0$, i.e. $\alpha = \mathbb{R}^2 \times \{0\}$. Let $\mathrm{pr}_1 : S^*(\mathbb{R}^3) \sim \mathbb{R}^3 \times \mathbb{S}^2 \longrightarrow \mathbb{R}^3$ be the natural projection, and let $C = \mathrm{pr}_1(\Lambda)$. We may choose the coordinates x, y in the plane $\alpha = \{z = 0\}$ so that the line $y = 0$ is tangent to K_1' and K_2' and K' is contained in the half-plane $y \geq 0$. Let $q_1 \in K_1'$ and $q_2 \in K_2'$ be such that $[q_1, q_2]$ is the shortest segment connecting K_1' and K_2'. Take a point $q_1' \in \partial K_1'$ close to q_1 and such that the y-coordinate of y_1' is smaller than that of q_1. Consider the *open arc* $\mathcal{A}$ on $\partial K_1'$ connecting q_1 and q_1'. It is clear that $\mathcal{A} \cap C = \emptyset$.

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a C^1 (we can make it even C^∞) diffeomorphism with $f(x) = x$ for all x outside a small open set U such that $q_1 \in \overline{U}$ and $U \cap \partial K' \subset \mathcal{A}$. Then for any $q \in C$ the tangent planes $T_q(\partial K)$ and $T_q(\partial \widetilde{K})$ coincide. We can choose f so that $\widetilde{K}_i = f(K_i) = K_i$ for $i > 1$, $\widetilde{K}_1 = f(K_1)$ is strictly convex, and $\tilde{\nu}(f(q)) \notin \alpha$ for $q \in \mathcal{A}$ arbitrarily close to q_1. Here $\tilde{\nu}$ is the outward unit normal field to $\partial \widetilde{K}$.

Notice that the non-wandering set $\widetilde{\Lambda}$ for the billiard flow $\tilde{\phi}_t$ in the closure $\widetilde{\Omega}$ of the exterior of $\widetilde{K}$ in $\mathbb{R}^3$ coincides with Λ. Indeed, let $\widetilde{\Phi} : \Sigma_A \longrightarrow \Lambda \cap S_{\partial \widetilde{K}}^*(\widetilde{\Omega})$ be the coding map for the billiard trajectories in $\widetilde{\Omega}$ and let $\tilde{z} = (\tilde{q}, \tilde{\xi}) \in S_{\partial \widetilde{K}}^*(\widetilde{\Omega})$ belong to $\widetilde{\Lambda}$. Then $\tilde{z} = \widetilde{\Phi}(\xi)$ for some $\xi \in \Sigma_A$, so $z = \Phi(\xi) \in \Lambda \cap S_{\partial K}^*(\Omega)$. Since any reflection point $P_i(\xi) \in C$, by the choice of f we have that $P_j(\xi)$ are the successive reflection points of a billiard trajectory in $\widetilde{\Omega}$ and this must be the trajectory determined by $\widetilde{\Phi}(\xi)$. Thus, $\tilde{z} = z$, so $\tilde{z} \in \Lambda$. This argument also shows that $\Lambda \subset \widetilde{\Lambda}$, so $\widetilde{\Lambda} = \Lambda \subset S^*(\alpha)$. Thus, $\dim(E_{\widetilde{\Lambda}}^u(z)) = 1 < \dim(E^u(z))$ for any $z \in \widetilde{\Lambda}$. However, it is clear from the construction that $S^*(\alpha \cap \Omega)$ is not invariant with respect to the billiard flow $\tilde{\phi}_t$. Moreover, it is not difficult to see that there is no two-dimensional submanifold $\tilde{\alpha}$ of $\widetilde{\Omega}$ such that $S^*(\tilde{\alpha})$ is $d\tilde{\phi}_t$-invariant and $\Lambda \subset S^*(\tilde{\alpha})$.

Indeed, assume such $\tilde{\alpha}$ exists; then $\tilde{\alpha}$ is a union of billiard trajectories and $C = \mathrm{pr}_1(\Lambda) \subset \tilde{\alpha}$. We will first show that $\tilde{\alpha} = \alpha$ outside a large disk D containing K'. Changing the coordinates x, y in the plane $\alpha = \{z = 0\}$, we may assume that the line $y = 0$ is tangent to K_2' and K_3' and K' is contained in the half-plane $y \geq 0$. Let $p_2 \in K_2'$ and $p_3 \in K_3'$ be such that $[p_2, p_3]$ is the shortest segment connecting K_2' and K_3'. Clearly $p_2, p_3 \in \tilde{\alpha}$. Since p_2 is not an isolated point in C, there are points in $C \cap \partial K_2'$ arbitrarily close to p_2. All of them are in $\tilde{\alpha}$, so the curve $\tilde{\alpha} \cap \partial K_2$ is tangent to $\partial K_2'$ at p_2. Similarly, the curve $\tilde{\alpha} \cap \partial K_3$ is tangent to $\partial K_3'$ at p_3. Take point $p_2' \in \tilde{\alpha} \cap \partial K_2$ close to p_2 and $p_3' \in \tilde{\alpha} \cap \partial K_3$ close to p_3 such that the y-coordinates of p_2' and p_3' are smaller than those of p_2 and p_3, respectively. Consider the arc C_2 on $\tilde{\alpha} \cap \partial K_2$ connecting p_2 and p_2' and the arc C_3 on $\tilde{\alpha} \cap \partial K_3$ connecting p_3 and p_3'. We have $(p_2, v_2) \in \Lambda$, where $v_2 = (p_3 - p_2)/\|p_3 - p_2\|$. For any $p \in C_2$ sufficiently close to p_2 and any $v \in S_p^*(\tilde{\alpha})$ sufficiently close to v_2, the billiard trajectory $\gamma(p, v)$ in $S^*(\widetilde{\Omega})$ issued from (p, v) is contained in $S^*(\tilde{\alpha})$, so its projection in $\mathbb{R}^3$ is contained in $\tilde{\alpha}$. Thus, the first reflection point of $\gamma(p, v)$ belongs to C_3. That is, for any $p \in C_2$ sufficiently close to p_2 and any $v \in S_p^*(\tilde{\alpha})$ sufficiently close to v_2, the straight line segment ray issued from p in the direction of v intersects the curve C_3. Similarly, for any $q \in C_3$ sufficiently close to p_3 and any $w \in S_q^*(\tilde{\alpha})$ sufficiently close to $-v_2$, the straight line segment ray issued from q in the direction of w intersects the curve C_2. Replacing the curves C_2 and C_3 by shorter ones (i.e. replacing the points p_2' and p_3' by points on C_2 and C_3 closer to p_2 and p_3, respectively), we now have that for

any $p \in C_2$ and any $q \in C_3$, the straight line segment $[p, q]$ lies in $\tilde{\alpha}$. This shows that C_2 and C_3 lie in the same plane, and it is clear from the choice of C_2 and C_3 that this plane is tangent to α at p_2, so it must coincide with α. Moreover, the union of the segments $[p, q]$ with $p \in C_2$ and any $q \in C_3$ is part of both α and $\tilde{\alpha}$.

From the latter one easily derives that if D is an open disk in α containing K', then $\alpha \setminus D \subset \tilde{\alpha}$. Since Λ is a nowhere dense subset of $S^*(\Omega)$ (and $S^*(\widetilde{\Omega})$, as well), every point $q \in \alpha \setminus K$ can be approximated arbitrarily well by points of the form $\mathrm{pr}_1(\phi_t(x, \xi))$, where $t > 0$ and $(x, \xi) \in S^*(\alpha \setminus D)$. This implies $\alpha \setminus K \subset \tilde{\alpha}$, and therefore $\alpha \setminus K = \tilde{\alpha}$, which is a contradiction with the definition of $\widetilde{K}$ and the perturbation f. Hence there does not exist a two-dimensional submanifold $\tilde{\alpha}$ of $\widetilde{\Omega}$ such that $S^*(\tilde{\alpha})$ is $d\tilde{\phi}_t$-invariant and $\Lambda \subset S^*(\tilde{\alpha})$.

4.2 GEODESIC FLOWS ON MANIFOLDS OF NEGATIVE CURVATURE

Let X be a complete (not necessarily compact) connected Riemann manifold of constant curvature $K = -1$ and dimension $\dim(X) = n + 1$, $n \geq 1$, and let $\phi_t : M = S^*(X) \longrightarrow M$ be the geodesic flow on the *unit cosphere bundle* of X. According to a classical result of Killing and Hopf any such X is a *hyperbolic manifold*, i.e. X is isometric to $\mathbb{H}^{n+1}/\Gamma$, where

$$\mathbb{H}^{n+1} = \{(x_1, \ldots, x_1) \in \mathbb{R}^{n+1} : x_1 > 0\}$$

is the upper half-space in $\mathbb{R}^{n+1}$ with the *Poincaré metric* $ds^2(x) = \frac{1}{x_1^2}(dx_1^2 + \ldots + dx_1^2)$ and Γ is a *Kleinian group*, i.e. a discrete group of isometries (Möbius transformations) acting freely and discontinuously on $\mathbb{H}^{n+1}$. See e.g. [Ratc] for basic information on hyperbolic manifolds. Given a hyperbolic manifold $X = \mathbb{H}^{n+1}/\Gamma$, the *limit set* $L(\Gamma)$ is defined as the set of accumulation points of all Γ orbits in $\overline{\partial \mathbb{H}^{n+1}}$, the topological closure of $\partial \mathbb{H}^{n+1} = \{0\} \times \mathbb{R}^n$ including ∞.

Throughout this sub-section we will assume that Γ is torsion-free and finitely generated (then Γ is called *geometrically finite*) and *non-elementary*, i.e. $L(\Gamma)$ is infinite (then $L(\Gamma)$ is a closed non-empty nowhere dense subset of $\partial \overline{\mathbb{H}}^{n+1}$ without isolated points; see e.g. Sect. 12.1 in [Ratc]). A geometrically finite Kleinian group with no parabolic elements is called *convex cocompact*. If X is compact, then Γ is called a *cocompact lattice*.

The *non-wandering set* Λ of $\varphi_t : M \longrightarrow M$ (also known as the *convex core* of $X = \mathbb{H}^{n+1}/\Gamma$) is the image in M of the set of all points of $S^*(\mathbb{H}^{n+1})$ generating geodesics with end points in $L(\Gamma)$. When Γ is convex cocompact, the non-wandering set Λ is compact.

Notice that the class of hyperbolic manifolds $X = \mathbb{H}^{n+1}/\Gamma$, with Γ a non-elementary convex cocompact Kleinian group, contains all classical and non-classical Schottky manifolds (cf. e.g. Sect. 12.1 in [Ratc]). In this case the (un)stable laminations of the geodesic flow over Λ are always C^1 (in fact C^∞).

Example 2. Consider the case $n = 2$ and let $\mathbb{H}^3/\Gamma$ be a hyperbolic manifold generated by a non-elementary convex cocompact Kleinian group Γ with an "Apolonian packing" limit set $L(\Gamma)$ (see e.g. [Su]). Then $L(\Gamma)$ is a subset of $\mathbb{R}^2$ such that for every $x \in L(\Gamma)$ the set of tangent vectors to $L(\Gamma)$ at x is one-dimensional, while $L(\Gamma) \cap U$ is not contained in an one-dimensional submanifold of $\mathbb{R}^2$ for any open neighbourhood U of x in $\mathbb{R}^2$. Given $z \in \Lambda$, the local unstable manifold $W_\epsilon^u(z)$ is given by the projection into $\mathbb{H}^3/\Gamma$ of the outward normal field to a (part of) a horosphere S in $\mathbb{H}^3$. The latter are just spheres in $\mathbb{H}^3$ tangent to $\partial \mathbb{H}^3$ or planes in the interior of $\mathbb{H}^3$ parallel to $\partial \mathbb{H}^3$. Any $z' \in \Lambda \cap W_\epsilon^u(z)$ can be identified with the point $x' \in L(\Gamma)$ which is just the limit at $+\infty$ of the geodesic determined by z'. It then follows that the set of tangent vectors in $E^u(z)$ to Λ at z is one-dimensional, while $\Lambda \cap U$ is not contained in an one-dimensional submanifold of $W_\epsilon^u(z)$ for any $\epsilon > 0$, i.e. $k^u = 2$.

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